Eureka Editor
The Archimedeans
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WA
United Kingdom

archim-eureka@srcf.net

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EUREKA

THE ARCHIMEDEANS' JOURNAL

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EUREKA

THE JOURNAL OF THE ARCHIMEDEANS

(The Cambridge University Mathematical Society: Junior Branch of the Mathematical Association)

Editor: *A. M. Macbeath (Clare)*

Committee: *A. R. Curtis (St. John's); A. J. Oldham (Clare); A. F. Ruston (Sidney Sussex); R. S. Scorer (Corpus)*

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Contributions and other communications should be addressed to:

The Editor, "Eureka,"

c/o The Mathematical Faculty Library,

The Arts School,

Bene't Street, Cambridge.

Editorial

THE grand response to our request for contributions, made only two months ago, has made possible the publication of this ninth number of EUREKA. We thank the researchers and more senior people who have taken time from more pressing business to write for us; but we hope to be forgiven if we pay special tribute to the undergraduate writers, who have not been deterred, even by the labour of Tripos preparation, from producing mathematics of their own.

Undergraduate contributions are the life-blood of EUREKA, and we look forward to the time when every Archimedean contributes his tit-bit, perhaps (like Dr. Todd's proofs) only a few lines long. So begin work now on your article for the next EUREKA; and, above all, don't be afraid that it will be too trivial. We try to offer novelty, rather than depth; and to steer clear of heavy mathematics. We want easy reading, relaxation for the mathematician; not more effort for his over-burdened brain.

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The Archimedeans

As the merit of a report lies in its terseness, I shall merely say that, in this last year, the Archimedeans have again taken up most of their pre-war activities. A life subscription has been introduced, and it has been suggested that the Society should have a tie.

We welcome the re-establishment of the Quintics and New Pythagoreans in their former strength and congratulate the Adams' Society on their 200th meeting and the T.M.S. on their unfailingly good lectures.

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MUSIC GROUP

The music group has, both this year and last, been as popular as ever. It has confined its activities largely to gramophone recitals, concentrating last year on the symphonies of Sibelius and so far this year on the works of the modern English composers.

Apart from this, there was a successful chamber music concert in Caius Parlour last year and it is hoped to have another soon.

I should like to conclude by thanking all those who, in any way, have helped to make for the smooth running of the group.

V. W. D. H.

PLAY-READING GROUP

After a hibernation of one year the Play-Reading Group has continued its policy of imbibing culture (and Namco) on Mondays in Trinity.

Attendances have not been so good as was hoped for, and any other Archimedeans interested would be welcome. My thanks are due to those whose regular attendance has prevented the cancellation of any meeting.

Plays have been read from such diverse authors as Shakespeare, Shaw, Wilde and Priestley. It is hoped to continue on similar lines this term; suggestions for future readings would be welcomed by

R. V. B.

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On Reading Encyclopaedias

By S. Lilley

It is always pleasant when one finds a source of amusement which, at the same time, has some self-educative or cultural value. It is pleasanter still when the source of amusement turns out to be an extremely cheap one. One which I find almost unfailing is the reading of *old* encyclopaedias. Modern encyclopaedias are works of reference and, to me at least, rather dull. But an encyclopaedia of 100 years ago is a very different affair: a mine of interesting information about its own period. I am not thinking chiefly of the information which its authors intended it to contain. Far more important is the insight they give us unintentionally by their selection of subjects, for instance, or the types of subjects they regarded as requiring peculiarly cautious and delicate treatment, or by the things they regarded as obvious. Their attitude to what was then recent history is also often revealing. In *some* respects, at least, one can learn more of the history of a period by browsing in the encyclopaedias of a few years later than by reading half a dozen modern history books.

Do not think that this is an occupation that has to be carried out in libraries. If it were it would lose much of its point. One must approach the encyclopaedia casually, turning to whatever pages whim or chance shall dictate. One starts by looking for an article that promises to be interesting; but as the pages turn one's eye is caught by something else, and the original objective is forgotten. In reading the article thus found by chance, one thinks of something else to be looked for. And so by a combination of free association and the random opening of the leaves, one is led from one thing to another. All this would, for me at least, be made impossible in a library by the ever-present consciousness that I ought to be studying systematically.

But luckily the library is not necessary. A little searching will produce an old encyclopaedia at a truly bargain price. For instance, I have just bought a copy of *The Penny Cyclopaedia* (1833–43) in 27 volumes—14,000 pages, nearly 10 million words. Total cost, £1.

The great virtue of an old encyclopaedia is the fact that whatever one sets out to find, one wanders to many other things. But as EUREKA is a mathematical journal I think I had better confine myself to mathematical topics. *The Penny Cyclopaedia* is quite a find in this connection, for its mathematical articles were written by Augustus De Morgan (1806–71), who (in the words of Rouse Ball) "was, perhaps, more deeply read in the philosophy and history of mathematics than any of his contemporaries." The articles are by no means mere popular presentations, but highly condensed expositions, together with discussions of the problems of rigorous demonstration, or the relative profitability of various lines of approach. Thus they give in a convenient form a picture of the mathematical development of the early 19th century through the eyes of one of the persons most qualified to present it.

We recall that the differential calculus was even then on a very insecure footing, and we read with great interest De Morgan's comparative discussion of six proposed methods of founding it. There are eight tightly packed quarto pages on Taylor and Taylor's theorem, containing a historical sketch, five "proofs," and then a discussion of its applications, which one seldom finds brought together in this way. Among these is the "method of derivations" of Argobast, now forgotten, I think, almost completely, yet perhaps worthy of attention from those who have to do complicated manipulations of series. One sees the weakness of the times in the opening of the article on convergence: "When a series of numbers proceeding without end has terms which diminish in such a manner that no number whatsoever of them added together will be as great as a certain given number, the series is called convergent." It turns out that he was thinking of series of positive terms (and in the atmosphere of the time the words "added together," without qualification, would often imply that), and he deals later with alternating series, and so on; but the fact that so deep a thinker as De Morgan could put it thus is very revealing as regards the progress of rigour up to the 1830's. By contrast, the article NEGATIVE AND IMPOSSIBLE (i.e. imaginary) QUANTITIES, on a subject which had only been cleared up in the previous decade or so, is a masterly exposition; though not up to modern standards for rigour, it would give many modern teachers useful hints on how to present the subject. In almost every article one learns something—there are caustic comments on the pre-analytical English school in the article GEOMETRY; the author (probably not De Morgan) of the article MECHANICS is under the impression that that subject has virtually completed its evolution; and so on. Another minor advantage is De Morgan's sense of humour, which keeps popping up; but anybody interested must search that out for himself.

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Some Fundamental Equations of Astronautics

By D. F. Lawden

The science of astronautics is concerned with the development of communications between the earth and the other planets and eventually other bodies in space and with the problems of interstellar navigation. I give below an elementary account of the fundamental problem of the science, viz. the motion of a rocket moving in a gravitational field.

For simplicity, we neglect the rotation and atmosphere of the body of origin and assume that the rocket moves along a radius from the centre of the body through the launching point.

Let m_0 be the initial mass of the rocket at time $t = t_0$.

m be the mass of the rocket at some subsequent time t.

m_1 be the mass of the rocket at "all burnt," i.e. when the fuel supply is cut off at $t = t_1$.

u be the velocity of the exhaust gases at time t, relative to a frame fixed in the body.

v be the velocity of the rocket at time t.

V be the potential of the gravitational field, so that $V = 0$ at infinity.

x be the distance of the rocket from the body's centre at time t.

The external force acting at time t is $-\dfrac{mdV}{dx}$. Hence, equating the impulse of this force during the interval $(t, t + \delta t)$ to the increase in momentum of the complete system, rocket and exhaust gases, we have

$$-m\frac{dV}{dx}\delta t = (m + \delta m)(v + \delta v) - u\delta m - mv$$
$$= v\delta m - u\delta m + m\delta v.$$

Hence

$$m\frac{dv}{dt} + (v - u)\frac{dm}{dt} = -m\frac{dV}{dx}.$$

Or, if U is the velocity of the exhaust gases relative to the rocket

$$m\frac{dv}{dt}+U\frac{dm}{dt}=-m\frac{dV}{dx} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

This is the equation of motion of a rocket. In free space, if U is constant, a first integral is immediate, viz.

$$v=U\log\frac{m_o}{m} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

giving the reduction in mass from m_o to m necessary to increase the velocity by v.

Later, we shall see that it is essential that the fuel be expended during the early part of the trajectory. In this case we shall have

$$\frac{dV}{dx}=g_o.$$

where g_o is the gravitational acceleration at the body's surface. Hence integrating (i) with respect to t

$$v=U\log\frac{m_o}{m}-g_o(t-t_o) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

If t_1-t_o is small we have for v_1 the final velocity

$$v_1=U\log\frac{m_o}{m_1}$$

In practice, using molecular fuels, exhaust velocities above 5 km/sec. are unlikely. Taking U to have this value it is easily calculated that the escape velocity from the earth of 7 miles/sec. is achieved if

$$m_o : m_1 = 9{\cdot}5.$$

If a molecular fuel is to be used to release a space ship from the earth's gravitational attraction, in view of the relatively small energy content of such a fuel it is essential to exercise the utmost economy, and we therefore seek the optimum conditions of release of the energy available. The two possibilities are, (*a*) to burn the fuel as rapidly as possible so that energy is not wasted raising the potential energy of the exhaust gases and (*b*) to burn fuel slowly, initially, until height has been gained, and then take advantage of the reduced escape velocity and increase the rate of burning. If an atomic fuel can be used the need for strict economy disappears. In fact, the development of such a fuel will usher in the space age as surely as the petrol engine ushered in the air age.

Suppose E to be the energy which has been introduced into the system at time t, from the fuel source. This will be partitioned between the rocket, the remaining fuel, and the exhaust gases which trail out along the trajectory.

Consider δE, the increase in E during the interval $(t, t + \delta t)$. It is given by

$$\delta E = \tfrac{1}{2}(m + \delta m)(v + \delta v)^2 + m\delta V - \tfrac{1}{2}\delta m.u^2 - \tfrac{1}{2}mv^2.$$

$$\therefore \frac{dE}{dt} = \tfrac{1}{2}(v^2 - u^2)\frac{dm}{dt} + mv\frac{dv}{dt} + mv\frac{dV}{dx}$$

and by (i) this reduces to

$$\frac{dE}{dt} = -\tfrac{1}{2}U^2\frac{dm}{dt} \quad \text{(iv)}$$

If the fuel is homogeneous the rate of energy release will be proportional to the rate of burning. Hence (iv) shows that in this case U is constant, as already assumed.

The total energy E_1 released into the system by time t_1 is

$$E_1 = \tfrac{1}{2}(m_o - m_1)U^2 \quad \text{(v)}$$

by integration of (iv).

If E_{R1} is the energy of the rocket when the fuel is expended

$$E_{R1} = \tfrac{1}{2}m_1v_1^2 + m_1V_1 \quad \text{(vi)}$$

If m_o, m_1, E_1 are given, then by (v) U is given, and we must select the rate of burning to make E_{R1} a maximum.

Now (i) may be written

$$v\frac{dv}{dx} + \frac{Uv}{m}\frac{dm}{dx} = -\frac{dV}{dx}.$$

Integrating with respect to x

$$\tfrac{1}{2}v_1^2 + V_1 + U\int_{m_0}^{m_1}\frac{v\,dm}{m} = V_o.$$

Hence, from (vi)

$$E_{R1} = m_1V_o + m_1U\int_{m_1}^{m_0}\frac{v\,dm}{m} \quad \text{(vii)}$$

But (i) may also be written

$$\frac{dv}{dm} + \frac{U}{m} = -\frac{dV}{dx}\frac{dt}{dm}.$$

Integrating with respect to m

$$v = U\log\frac{m_o}{m} - \int_{t_0}^{t}\frac{dV}{dx}dt.$$

Substituting in (vii)

$$E_{R1} = m_1V_o + \tfrac{1}{2}m_1U^2\left(\log\frac{m_o}{m_1}\right)^2 - m_1U\int_{m_1}^{mo}\frac{1}{m}\left\{\int^{t}\frac{dV}{dx}dt\right\}dm$$

$$= m_1V_o + \tfrac{1}{2}m_1U^2\left(\log\frac{m_o}{m_1}\right)^2 - m_1U\int_{t_0}^{t}\log\frac{m}{m_1}\frac{dV}{dx}dt \quad \text{(viii)}$$

by a single integration by parts.

The integral is essentially positive, and so

$$E_{R_1}^{MAX} = m_1 V_o + \tfrac{1}{2} m_1 U^2 \left(\log \frac{m_o}{m_1}\right)^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ix)}$$

this maximum being approached as the time of fuel consumption becomes small.

We observe from (ix) that the increment in the energy of the rocket is at most

$$\tfrac{1}{2} m_1 U^2 \left(\log \frac{m_o}{m_1}\right)^2.$$

By comparison with (v) we see that this is a fraction

$$\frac{\left(\log \dfrac{m_o}{m_1}\right)^2}{\dfrac{m_o}{m_1} - 1}$$

of the total energy E_1 injected. The maximum value this fraction can take is 0·65 at $m_o/m_1 = 4{\cdot}9$.

We have proved that the period of initial acceleration must be kept as short as possible if there is to be maximum transfer of energy from the fuel to the ship. In practice the acceleration cannot exceed a certain limiting value for two reasons, (*a*) the velocity of the rocket relative to the atmosphere must not become too large on account of the heating effect, and (*b*) the crew can not be subjected to accelerations above a certain value without danger of injury or temporary loss of consciousness.

Suppose, then, it is decided that an acceleration of ng_o must not be exceeded. To obtain the maximum efficiency of energy transfer we shall arrange to burn fuel so that this acceleration is maintained steady. Assuming that we may put $dV/dx = g_o$, we obtain from (i) by putting $dv/dt = ng_o$ and integrating with respect to t

$$m = m_1 \exp \left\{ \frac{(n+1)}{U} g_o (t_1 - t) \right\} \quad \ldots \quad \ldots \quad \ldots \quad \text{(x)}$$

An automatic fuel feed will ensure that this exponential law is obeyed.

(x) gives the duration of thrust, viz.

$$(t_1 - t_o) = \frac{U}{(n+1) g_o} \log \frac{m_o}{m_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(xi)}$$

Taking $m_o/m_1 = 10$, $U = 5$ km/sec. and $n = 4$, the duration of thrust is about 4 min.

From (viii) it appears that the actual energy transferred falls short of the maximum possible by

$$m_1 U \int_{t_0}^{t_1} \log \frac{m}{m_1} \cdot \frac{dV}{dx}\, dt = m_1 \int_{t_0}^{t_1} (n+1)\, g_o{}^2 (t_1 - t)\, dt \text{ by (x)}$$

$$= \tfrac{1}{2} m_1 (n+1) g_o{}^2 (t_1 - t_o)^2$$

$$= \frac{m_1 U^2}{2(n+1)} \left(\log \frac{m_o}{m_1} \right)^2 \text{ by (xi).}$$

Comparing this with (ix) we see that this energy deficiency is a fraction $1/(n+1)$ of the maximum possible energy increment. If then $n = 4$, the transfer is 80% efficient.

It may be remarked in conclusion that a complete discussion of a rocket's motion in a field of force can only be undertaken after a general system of dynamical equations, such as Lagrange's, has been modified to apply to a system of variable mass.

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The Coloured Cubes Problem

By F. de Carteblanche

THERE is a puzzle (sold commercially as the "Tantalizer") which consists of a set of 4 cubes whose faces are coloured each with one of four colours, say green, red, orange and white, or G, R, O, W for short. The problem is to stack these cubes in a vertical pile (thus forming a square prism) in such a way that each of the four vertical faces of this pile contains all four colours.

In order to make the discussion clearer we shall give names to the faces of the cubes in this way: the front, or nearest face will be named the x face; and opposite face, or back, will be named the x' face. On the right will be the y face, and on the left the y' face. The top will be the z face and the bottom the z' face. The problem then requires us to arrange the four cubes so that the x faces have all different colours, and the same for the x', y, and y' faces. If, for example, we had a set of cubes coloured and arranged in the following way, we would have a solution. (I am not sure if this is the same as the commercial version.)

Face	Cube number 1	2	3	4
x (front) ..	R	G	O	W
x' (back) ..	O	R	W	G
y (right) ..	W	O	G	R
y' (left) ..	O	R	W	G
z (top) ..	G	W	G	R
z' (bottom)	G	W	R	W

However, if one is only given the cubes, and not the solution, getting them into the correct position is none too easy. But here is a quick method of solving the problem which seems to indicate the principle on which the puzzle is based.

We make use of the diagram shown in Fig. 1, which is easily constructed from the given set of cubes.

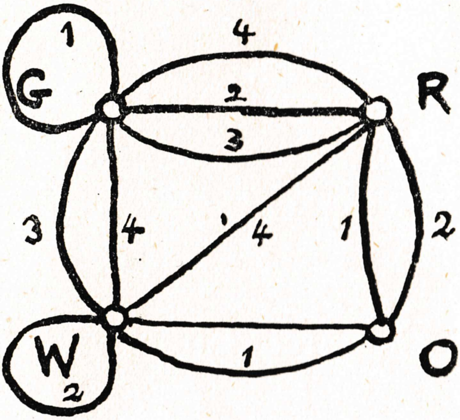

Fig. 1

Here the four points G, R, O, W represent the four colours. The line labelled 1 joining the points R and O represents the pair of opposite faces in cube 1, coloured R and O: and similarly every other pair of opposite faces has a corresponding line.

Now in the final arrangement all the faces x have different colours, and so also do all the opposite faces x'; so in taking all the pairs of opposite faces x, x' we must take each of the colours exactly twice. In the diagram (Fig. 1) this must correspond to a set of four lines with the properties

(i) there must be just one line with each of the labels 1, 2, 3, 4, (i.e. just one pair of faces from each cube);

(ii) there must be just two lines ending at each of the points G, R, O, W, except that we may allow instead one line having both its ends at the point.

Conversely, if we have a set of four lines with properties (i) and (ii), then we can arrange them so that the x faces have all different colours, and the same for the x' faces. And, in particular, if the four lines form a circuit like O (1) R, R (2) G, G (4) W, W (3) O, then there is only one possible arrangement, apart from an interchange of x and x'. For if in the first cube we place the R face in position x, and therefore the O face in position x', then as no two x faces can have the same colour, in the second cube the R face must have position x', and so the G must be x, and therefore in the fourth cube G must be x', and so on. Now, in order to fit in the y and y' faces to have all different colours, we must find a second set of four lines, distinct from the first, having properties (i) and (ii). But the only way of choosing two such sets of lines from Fig. 1 is to take the two circuits O (1) R, R (2) G, G (4) W, W (3) O and W (1) O, O (2) R, R (4) G, G (3) W, giving the solution we have already tabulated. This shows that it is an effectively unique solution, apart from a permutation of the order of the cubes, or of an interchange of x and x', or of y and y', or of x, x' and y, y'. The chance of obtaining the solution by a random arrangement of the cubes is only 1/41472.

The diagram of Fig. 1 is not the only possible one with 4 colours giving such a unique solution; in fact, there does not seem to be any reason why one should not use the same method to make a puzzle with n cubes, coloured in n different colours. But I leave that to your ingenuity.

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A Problem on Orchards

By G. C. Shephard

In this article* an attempt is made to generalise a problem which frequently occurs in mathematical puzzle-books,† and is, no doubt, familiar to the reader. Briefly, this is:

An eccentric farmer wishes to divide his orchard by n straight fences into enclosures, each of which must contain only one tree.

Given a plan of the orchard, show how this may be done.

The following is the generalisation for the case of seven "trees":

Given six general‡ points of a plane, to find the conditions which a seventh point must satisfy so that it will be possible to divide the plane by 3 straight lines into 7 regions, each of which will contain just one of the points.

Regarding all regions as open sets of points we shall adopt the convention that if a point lies on a line we may regard it as lying in either of the two regions to whose frontiers it belongs; and similarly, if a point lies on the intersection of two lines, we may regard it as lying in any one of the four regions to whose frontiers it belongs. We shall refer to this convention by the letter Γ.

(1) Select any one of the six given points, say A.

(2) Join A to each of the remaining five points.

(3) Select any two of these five lines, say AB, AC. (This may be done in ${}_5C_2 = 10$ ways). These two lines will divide the plane

* I am indebted to E. S. Page for reading the manuscript and making several valuable suggestions.

† E.g. H. E. Dudeney: *Puzzles and Curious Problems*, No. 220.

‡ I.e. no three of which are collinear.

into four sectors, and so the remaining three points will be divided by these lines into four groups. Three groupings are possible:

(*a*) 3 . . .

(*b*) 2 1 . .

(*c*) 1 1 1 .

(4) In cases (*b*) and (*c*), by Γ, we may regard each of A, B, C, as belonging to one of the sectors. Hence, assign A, B, and C, in such a way that the six points are divided by the lines AB, AC, into groups of 2, 2, 1, and 1. Denote this operation by Ω. Then Ω can be performed in certainly less than $2.2.4 = 16$ ways.

(5) Let the sectors which contain only one of the given points be labelled m_1 and m_2, and those containing two points M_1 and M_2. Now re-letter the points according to the following scheme:

$$P_1 \in m_1;\ P_2 \in m_2;\ Q_1, R_1, \in M_1;\ Q_2, R_2 \in M_2.$$

(6) Consider the region of the plane swept out by a varying line which always intersects Q_1R_1 between Q_1 and R_1 and always intersects the line Q_2R_2 between Q_2 and R_2. Call this region Λ.

(7) Define a region Δ $(A; B, C; \Omega; m_1)$ as follows:

In Case (*a*) $\Delta = 0$.

In Cases (*b*) and (*c*), using the notation of the algebra of sets of points,

if $P_1 \in \Lambda$, $\Delta = m_1$

if not, $\Delta = m_1 . \Lambda + \sigma$ where, if m_1 is divided into two disconnected parts by Λ, $\sigma =$ that part of $(m_1 - m_1 . \Lambda)$ in which P does not lie: otherwise, $\sigma = 0$ (see diagram 1). Since there are two sectors m, we will get two regions Δ for given A, B, C, and Ω.

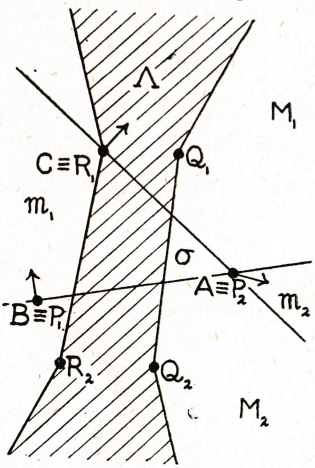

Fig. 1

Hence, by choosing different points in (1), different lines in (3), different methods of performing Ω, and one or other of the two sectors m, we shall get less than $6.10.16.2 = 1920$ regions Δ, say $\Delta_1 \Delta_2 \ldots \Delta_n$.

(8) Let $T = \Sigma\, \Delta_i$. (We may actually construct T in a finite number of steps.)

Then if, and only if, the seventh point G lies in T will it be possible to divide the seven points as required by three straight lines.

Diagram 2 shows T (the whole plane except for the shaded portion) for a certain array of 6 points, together with the division for one particular position of G. For some sets of 6 points, T may consist of the whole plane.*

The given condition is clearly sufficient. It will always be possible, if $G \epsilon \Delta\,(A; B, C; \Omega; m_1)$ as above defined, to draw a line

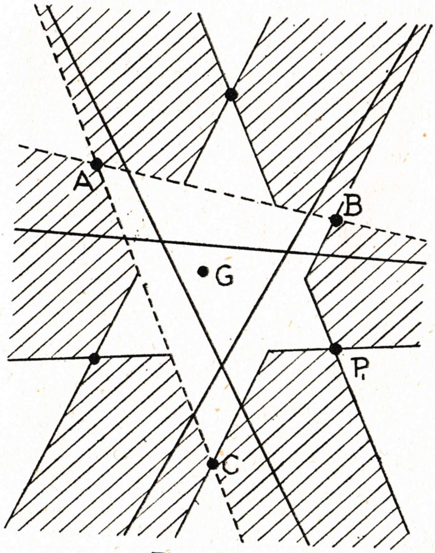

Fig. 2

lying in Λ which runs between P_1 and G. Then this line, together with AB, AC, will (after assigning A, B, C, to the appropriate sectors) be a solution.

It is also a necessary condition. For suppose that there is a solution in which the seven general points A, B, C, D, E, F, G, are divided in the required manner. Remove one of the three lines so that G and one other point (P_1) lie in one of the sectors formed by the remaining two lines. We may now alter the position of these lines (not passing over any of the points) so that their intersection

* E.g. when the points are arranged as the vertices and circumcentre of a regular pentagon.

coincides with one of the points (say A) and also each contains one other point, say B and C (see the dotted lines in diagram 2). Comparing this configuration with the above construction using A, B, C, and appropriate Ω, we see that G must lie in Δ (A; B, C; Ω; m) (and hence in T) for the division to be possible.

If we try to extend the above analysis to the case of more than seven points, or to points in space of n-dimensions, many difficulties appear, and I leave these problems to the reader should he be interested.

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Semi-Linear Functions

By A. F. Ruston

Our object is to investigate a real single-valued function $f(x)$ with the property:

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$$

($y=f(x)$ thus represents a generalisation of the straight line).

To simplify the argument we introduce a normalised function:

$$\phi(x)=\frac{f(px)-f(0)}{f(p)-f(0)}$$

where we choose p, assuming $f(x)$ not constant, so that $f(p)\neq f(0)$.

Clearly

$$\phi(0)=0,\ \phi(1)=1,\text{ and}$$

(1) $$\phi\left(\frac{x+y}{2}\right)=\frac{\phi(x)+\phi(y)}{2}$$

Hence $$\phi(x)=\tfrac{1}{2}[\phi(2x)+\phi(0)]=\tfrac{1}{2}\phi(2x).$$

Substituting in (1):

(2) $$\phi(x+y)=\phi(x)+\phi(y)$$

and, more generally, $\phi(nx)=n\phi(x)$ for an integer n.

So: $$q\phi\left(\frac{p}{q}x\right)=\phi(px)=p\phi(x),\text{ or:}$$

(3) $$\phi(rx)=r\phi(x)\text{ for rational } r.$$

If $\phi(x)$ is continuous at one point y, it is continuous at every point, for:

$$\phi(x+h)-\phi(x)=\phi(h)=\phi(y+h)-\phi(y).$$

Hence, since every real number is a limit of rationals, (3) holds for all real r, and so

$$\phi(x)=x\phi(1)=x.$$

A function satisfying (1) without being linear is thus continuous nowhere. (The existence of such functions follows from the Selection Axiom, by which it may be shown that there exists a "basis," i.e. a set of numbers such that every real number can be expressed uniquely as a sum of rational multiples of a finite number of these numbers. Provided 1 is a member of the basis, and this can always be arranged, we can take $\phi(x)$ as the sum of the coefficients in the unique expression of x in terms of the basis. This is non-linear, since all its values are rational; $\phi(0) = 0$, $\phi(1) = 1$ and equation (1) holds.)

Now consider such a function $\phi(x)$ in the interval $(0, h)$ where h is small. Let N be a large positive number. Since $\phi(x)$ is discontinuous at the origin and $\phi(0) = 0$, $\phi(x)$ does not tend to zero as x tends to zero.

Hence there is a positive k such that $|\phi(x_0)| \geqslant k$ for some positive $x_0 < \frac{k}{N}$.

We now choose a rational r_0 such that $\frac{x_0}{h} < r_0 = \frac{k}{N}$

$$\text{so} \quad 0 < \frac{x_0}{r_0} < h$$

$$\text{and} \quad \left|\phi\left(\frac{x_0}{r_0}\right)\right| = \frac{1}{r_0}\,|\phi(x_0)| > \frac{N}{k}\,.\,k = N.$$

Hence $\phi(x)$ is unbounded in $(0, h)$ and so in any interval. (It follows that the function is not measurable.)

We can show more than this.

By (2), for $0 \leqslant h \leqslant b - a$, $\phi(a + h) + \phi(b - h) = \phi(a + b)$.

Thus $\phi(x)$ takes arbitrarily large positive and negative values in (a, b). We can, therefore, choose t, t' in (a, b) so that:

$$(4) \qquad \phi(t) < m, \ \phi(t') > M$$

where m, M are real numbers and $m < M$.

Let P be the point $(t, \phi(t))$, $Q(t', \phi(t'))$. (4) means that the line-segment PQ cuts the lines $y = m$, $y = M$, and so we can find a point $X(x, y)$ on PQ such that $PX : PQ = r$ (rational) and lying between those lines, i.e.

$$m < y < M$$

$$\begin{aligned} \text{and} \quad y &= r\phi(t) + (1 - r)\phi(t') \\ &= \phi(rt + (1 - r)t') = \phi(x). \end{aligned}$$

Thus we have found an x in the interval (a, b) such that $m < \phi(x) < M$.

Hence given any interval (a, b) and any interval (m, M), there is a number x in the first interval whose value $\phi(x)$ comes from the second. (This can be expressed by saying that the points $(x, \phi(x))$ are everywhere dense in the plane.)

To sum up, we have shown that, if a function has the property (1) then it is either linear or takes values arbitrarily near to any given value in any arbitrarily small interval, and that functions of the second type exist.

■ ■ ■

A Method of Describing Cassini's Ovals

By F. T. M. Smith

Cassini's Ovals are the curves given (in bi-polar co-ordinates) by

$$r_1 r_2 = \text{constant}$$

r_1, r_2 being the distances of a point on the curve from two fixed points, the *foci*. For different values of the constant we get a *confocal system* of Cassini's Ovals.

Familiar Instances of such confocal systems:

1. The lines of magnetic force due to like currents in two long parallel wires.
2. The cross-sections of the equipotential surfaces due to like electrostatic charges on two long parallel wires.
(In each case the wires pass through the foci.)

Method of Description: Given a fixed plane π, and a fixed circle γ cutting it orthogonally (in F_1 and F_2, say); let a sphere σ of fixed diameter a move so as to touch π (at P, say) and γ.

Then P describes a Cassini's oval, foci F_1 and F_2.

Proof: Invert with regard to the unit sphere centre P. Let dashes denote inverse points and loci. Then (π' being π, of course) σ' and π' are parallel planes, distant $\frac{1}{a}$ apart, while γ' is a circle touching σ' and meeting π' orthogonally at F'_1 and F'_2.

Hence $F'_1F'_2 = \frac{2}{a}$.

But by a familiar formula connecting the distance between two points with that between their inverses,

$$F_1F_2 = PF_1 \,.\, PF_2 \,.\, F'_1F'_2.$$

Denote F_1F_2 by $2d$, PF_1 by r_1, PF_2 by r_2.

Then $r_1r_2 = ad =$ constant.

Two Corollaries: (Note first that, S denoting the centre of the sphere σ, the loci of S and P are congruent.)

(1) If the axial sections of a torus are circles of radius k, then the sections by planes parallel to and at distance k from the axis are Cassini's Ovals.

For the locus of S is the intersection of the following two surfaces:

(i) The torus which S describes when σ touches γ.

(ii) The plane which S describes when σ touches π.

Note: If the distance is other than k, we get more general quartic curves known as "Spiriques de Perseus," cf. Teixeira, "Traité des Courbes Speciales Remarquables" where this result is deduced from the Cartesian Equations of the loci.

(2) If a sphere σ has a diameter equal to the distance between the centres C and D of two equal, fixed, coplanar circles, and touches them both, then the locus of its centre is a Cassini's oval.

For, under these conditions, σ evidently touches (e.g.) the plane through C perpendicular to CD. This may be taken as the plane π.

Remarks: This picture of the ovals as the tracks of balls going under semi-circular croquet-hoops, so to speak, makes obvious the various cases that can occur, viz.

(1) When $a < d$, the ball can get through the hoop, and the locus consists of two separate ovals, one round each focus. (Further, these obviously approximate to circles when the ball is very small.)

(2) When $a > d$, the ball cannot get through, and the locus is a single oval round both foci.

Two sub-cases: (2 . 1) $\frac{a}{2} < d$. The ball can go some way into the loop, so the oval is partly concave (hour-glass shaped).

(2 . 2) $\frac{a}{2} > d$. Oval entirely convex.

The critical case, $a = d$, separating (1) and (2), gives Bernouilli's lemniscate.

A Generalisation: By inversion (just as above) the following theorem is readily proved:

Given a fixed sphere of diameter b, and a fixed circle cutting it orthogonally in F_1 and F_2 ($F_1F_2 = 2d$), let a sphere of diameter a ($< b$) touch each. Then, P being the point of contact of the two spheres,

$$PF_1 \, . \, PF_2 = \frac{dab}{b \pm a} = \text{constant.}$$

Thus the locus of P on the fixed sphere might be termed a "spherical Cassini's oval."

The Editor has pointed out that if we revolve a plane Cassini's oval about the line joining its foci F_1 and F_2, the resulting surface will meet any sphere through F_1 and F_2 in such a curve.

Latin Squares

By H. A. Thurston

A LATIN square of order N consists of an arrangement of N objects, each occurring N times, in a square of N rows and N columns, in such a way that each object occurs just once in each row and just once in each column. Latin squares occur frequently in mathematical puzzle books, but they have also from time to time engaged the attention of serious mathematicians, including Euler. One of the most obvious problems which presents itself is that of finding out how many different latin squares of a given order exist. An answer has been given by McMahon.* The workings are too long to reproduce here, but the final result is that the number of distinct latin squares of order n is the coefficient of $(x_1x_2 \ldots x_n)^{2^n-1}$ in the expansion of S^n where S is the symmetric function $\Sigma x_1 x_2^3 \ldots x_n^{2^n-1}$.

Latin squares also occur in algebra. A *quasi-group* is a set of elements with a binary law of composition (i.e. to every pair of objects is assigned a third object of the set called the *product* of the pair) obeying the following postulates:—

(i) Division is always possible. I.e. for any a and b of the set there exists an x such that $ax = b$ and a y such that $ya = b$.

(ii) Division is always unique. I.e. there is only one such x and one such y.

If we have a finite quasi-group with N elements we can write out its "multiplication table" in the form of a square, each row and each column being labelled with one of the objects of the quasi-group, and the product of a and b being entered at the intersection of the row labelled a with the column labelled b.

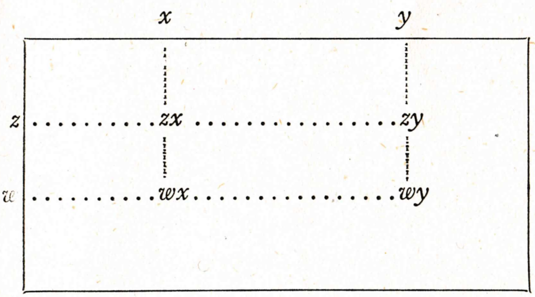
Then the first postulate means that every element occurs at least once in every row and column, while the second postulate means that no element can occur more than once in any row or column. Clearly either postulate implies the other (this is not so if the quasi-group is not finite). Thus, every finite quasi-group gives rise to a latin square. Conversely, from any latin square a quasi-group may be obtained merely by labelling the rows and columns.

In a quasi-group there may be an element f such that $fa = a$ for any a, i.e. there may be a row (labelled f) which is the same as the row of column labels. f is then called a left-identity. Similarly there may be a right-identity g ($ag = a$ for any a). If both exist they must be the same, for each must equal fg. A quasi-group with a right-and-left-identity is called a *loop*.

**Combinatory Analysis*, by P. McMahon. C.U.P.

A *group* is a quasi-group which obeys the associative law:—$(ab)c = a(bc)$ for any a, b and c. We do not require postulate (ii) explicitly as it can be deduced now from (i). We can prove easily that a group has an identity e, that for any a there exists an object, written a^{-1}, for which $aa^{-1} = a^{-1}a = e$, and that if $ax = b$ or $xa = b$, then $x = a^{-1}b$ or ba^{-1} respectively.

The multiplication table of a group is, therefore, just a latin square which obeys the associative law. The geometrical meaning of this is quite simple: choose, at the four corners of a rectangle, four entries of which one is e. It will be in, say, row a, and, therefore, in column a^{-1}. Let the other column involved be labelled b and the other row c. Then the element diagonally opposite e is cb, while the other two are ca and $a^{-1}b$, whose product, $(ca)\ (a^{-1}b) = c(a(a^{-1}b)) = c((aa^{-1})b)$ (using the associative law; in fact we can really write continued products without brackets where this law is in force) $= ceb = cb$. Conversely, any labelled latin square with an identity e and with the property that if four elements, a, b, c, e, are chosen at the corners of a rectangle, (b being in the same row as e and c in the same column), then $cb = a$, is the table of a group. To prove this we have to verify the associative law. Suppose the sides of the rectangle are the columns x, y and the rows z, w.



We are given that if $zy = e$, then $(wy)\ (zx) = wx$. In particular, if $w = x = e$, we have that $zy = e$ implies $yz = e$. Hence we can write $z = y^{-1}$ and we have that $(wy)\ (y^{-1}x) = wx$ for any w, y, x. Then $(pq)\ (q^{-1}p^{-1}) = pp^{-1} = e$. Therefore $(pq)^{-1} = q^{-1}p^{-1}$. Then $(y^{-1}x^{-1})\ (x(yz)) = y^{-1}(yz) = z = (xy)^{-1}((xy)z) = (y^{-1}x^{-1})((xy)z)$. Therefore $x(yz) = (xy)z$.

Another interesting problem which the reader may like to try is this: given a group table with the labels omitted, fill them in correctly (and find out to what extent the solution is unique.)

A fairly obvious generalisation of the latin square is the latin cube: an arrangement of N objects (each occurring N² times) in the form of a cube, each occurring just once in each rank, file and column. The extension to any number of dimensions is equally easy. The corresponding algebraic system would consist of a set of elements with an n-ary law of composition (to every ordered n-set of elements there is assigned an element called their product) with division possible and unique in each of the n positions. The multiplication table is then a latin hypercube in n-dimensional space.

We can prove for such systems some of the results well-known in group-theory. In particular we can define *normal subsystem* and *normal series* and prove the Hölder refinement theorem in the form: If an n-dimensional system has the two normal series $S \geqslant S_1 \ldots \geqslant S_r$ and $S \geqslant S_1' \ldots \geqslant S_q'$ *and S_r and S_q' have an element in common,* then the series have isomorphic refinements.

If we consider two-dimensional systems and confine our attention to loops, we can state the theorem exactly as in group-theory, dispensing with the italicised clause, as any two subloops of a loop must have the identity in common.

We can also define associativity. If we write $(a_1 a_2 \ldots a_n)$ for the product of $a_1, a_2, a_3, \ldots, a_n$ we can form continued products such as $(a_1 a_2 \ldots a_k(a_{k+1} \ldots a_{k+n}) \ldots a_{2n-1})$. If this equals $(a_1 a_2 \ldots a_{k-1}(a_k \ldots a_{k+n-1}) \ldots a_{2n-1})$ for any choice of the a_i, then all the n continued products of $a_1 \ldots a_{2n-1}$ (in that order) are equal. The system is then associative and all continued products can be written unambiguously without internal brackets. The simplest example of such a system is obtained by defining the product $(a_1 \ldots a_n)$ as the continued product of $a_1 \ldots a_n$ under the law of composition of a group. Then if e is the group identity $(ee \ldots eae \ldots e) = a$ whatever position the a is in. Conversely, if an associative system has an element e with this property, if we define a law of binary composition by $ab = (abee \ldots e)$ we will have a group and the system will be obtainable from it as a continued product. Not all associative systems have such an element.

Commutative systems are those in which a product is independent of the order of the elements forming it. There are many forms of partially commutative systems in which a product is invariant under some but not all permutations.

Again we can prove some of the standard results. For instance, there is a "unique factorisation" theorem similar to that for Abelian groups; a commutative associative system is uniquely expressible as the direct product of cyclic systems of prime-power order *provided that $n - 1$ is prime to the order of the system.*

As far as I know these latin hyper-cubes have no useful applications, but latin squares have occurred in a number of unexpected situations. One which may be mentioned is biological statistics: in investigating the yields of different kinds of seed, in order to equalise as far as possible the unevenness in quality of the ground, they are sown in a latin square formation.

Finally there is an interesting existence theorem for latin squares. Given r rows, each being a permutation of the same N objects with the property that in none of the incomplete columns formed does any object occur more than once, it is always possible to complete a latin square. The proof depends on a theorem that if a set of objects be divided in two ways into n subsets so that for all r it is impossible to choose r subsets of the second mode of division which are completely contained in $r - 1$ subsets of the first mode, then it is possible to choose n objects such that each subset of the first mode of subdivision contains one of them, and so does each subset of the second mode. (The proof has been given by P. Hall in Vol. 10 of the *Journal of the London Mathematical Society*.)

For each column of our incomplete latin square we can write down $N - r$ objects not occurring in it: the N *column residues*. These $N(N - r)$ objects can be divided into N subsets in another way, for there are $N - r$ of each of the N distinct objects. Since all the subsets are of the same size the conditions of the theorem are satisfied. Hence we can choose N objects such that one will be of each sort and one from each column residue. This we can take as our $(r + 1)^{th}$ row, and we can repeat the process until the square is complete. I do not know of any extension of the proof to more than two dimensions, and if any reader can devise one I should be interested to see it.

▪ ▪ ▪

A Three-colour Problem

By Blanche Descartes

By a *network* we mean a set of points P, Q, R, . . . certain pairs of which are joined by lines (not necessarily all in one plane). By a *circuit* in the network we mean a set of lines PQ, QR, RS, . . . VP joining cyclically the distinct points P, Q, . . . V. Find a network in which it is impossible to colour the points in three colours so that no two points of the same colour are joined, and which contains no circuit of less than six lines.

▪ ▪ ▪

House Numbers

By R. A. Rankin

Mr. Smith and his two friends, Mr. Brown and Mr. Robinson, all dwelt in Symmetry Street, a long street consisting of an equal number of semi-detached houses on each side. Robinson's house is the second last in the street and its number is a factor of the number of his Rolls Royce. The number of the Rolls Royce is equal to the sum of the cubes of the ages of Smith's two children, and also to the sum of the cubes of the ages of Brown's two children, all four ages being different. The number of Smith's house has no factors other than unity in common with the ages of either of Brown's children, and has no digit in common with the age of Brown's younger child. Brown and Smith are neighbours living under the same roof. One of their house numbers is equal to the sum of the squares of the number of Brown's Baby Austin and Robinson's telephone; when the other is divided by the second digit of the number of Robinson's Rolls Royce the result, which is not a square, is the sum of the squares of the ages of Robinson's two children.

What are the house numbers and the ages of the children in each of the three families?

▪ ▪ ▪

The Umbrella Problem

Six men, A, B, C, D, E, F, of negligible honesty, met on a perfectly rough day, each carrying a light inextensible umbrella. Each man brought his own umbrella, and took away—let us say "borrowed" —another's. The umbrella borrowed by A belonged to the borrower of B's umbrella. The owner of the umbrella borrowed by C borrowed the umbrella belonging to the borrower of D's umbrella. If the borrower of E's umbrella was not the owner of that borrowed by F, who borrowed A's umbrella?

▪ ▪ ▪

Fermat's Last Theorem

By H. A. Thurston

Five people make the following statements:—

(1) *Either* (*a*) 3's statement is false and 4's is true.
Or (*b*) 2's and 5's are both false.

(2) *Either* 4's statement is false *or* 3's is false.
Or 1's and 5's are both false.

(3) 2's statement is true *or* 4's and 5's are both true.
Moreover, *either* 5's is false *or* 4's is true.

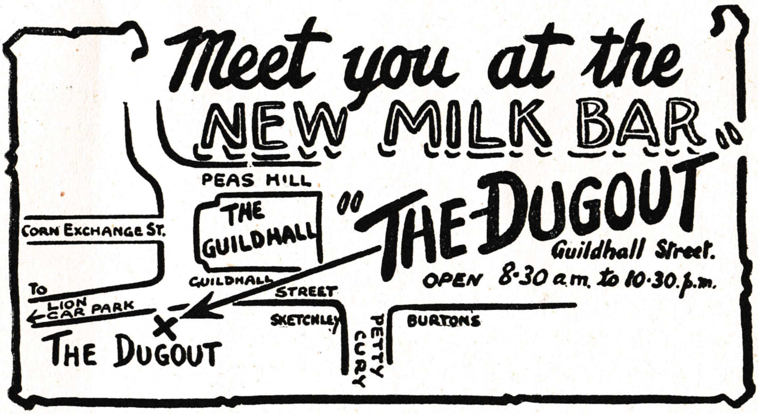
(4) 3's statement is false *or* 1's is true.

(5) Fermat's last theorem is true.

Which of these statements are true and which false? It will be found on trial that there is only one possibility. Thus, prove or disprove Fermat's last theorem.

COFFEE

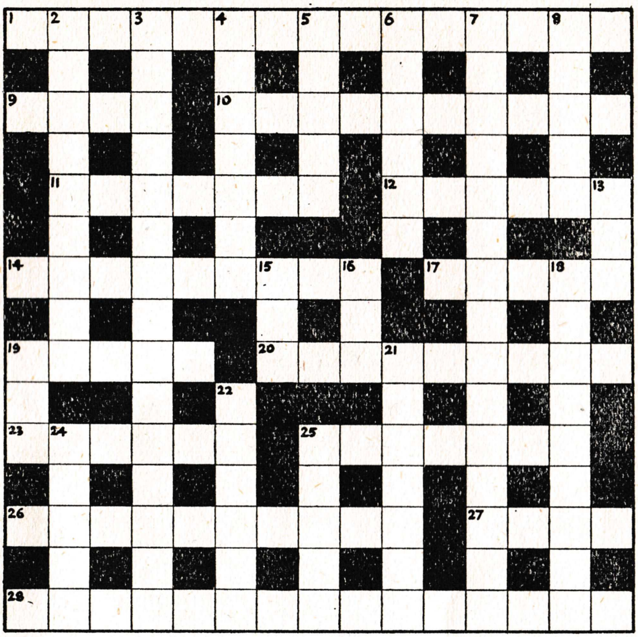
ICES



TEA

MILK-SHAKES

Crossword



ACROSS

1. Rash deed. (9, 6.)
9. Section of the whole body of common soldiers. (4.)
10. Everyone is restricted. (3, 7.)
11, 12. Logically deduced results will not do this. (7, 6.)
14. Parts for aircraft? (5, 4.)
17. A small matter in Jugoslavia. (5.)
19. Large alteration. (5.)
20. The half-dozen figures look very like the earth. (3, 6.)
23. Looking-glass creature. (6.)
25. Region in which we see America reduced to nothing. (7.)
26. Burn a prime (anag.). (6, 4.)
27. Spells doom for some fox. (4.)
28. Complete answer to the problem of high 9 in the Army. (7, 8.)

DOWN

2. The price of the average one is niggardly. (4, 5.)
3. x. (7, 8.)
4. The sailors have lost direction in their confusion. (7.)
5. Continuous in the interval selected. (5.)
6. Might be used to describe a tropical bear. (5.)
7. Ultimately a familiar mathematical process. (7, 2, 1, 5.)
8. Acid. (5.)
13. Sounds as though they are not all present. (3.)
15. See 19 down.
16. Over the boundary! (3.)
18. Vanishes too. (3, 2, 4.)
19, 15. Necessary and sufficient for straight lines? (6.)
21. There is nothing special about it. (7.)
22. (rev.). Path from A to B? (6.)
24. Sharp. (5.)
25. Small advertisements about the morning. (5.)

The solutions to this crossword will be printed in our next issue.

Book Reviews

The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coordinates. By E. A. MAXWELL. (Cambridge University Press.) 12/6.

This introduction to plane projective geometry contains many welcome features, outstanding among which is the clear and well-written account in Chapter II on (1-1) algebraic correspondences. These are of fundamental use in geometry, and they are dealt with as early in the book as possible. It is emphasised that the correspondence must be algebraic and (equally important) that the parameter must be algebraic and unique. Once this chapter has been mastered, a book like Duporcq's will be much easier to assimilate.

One suspects, however, that Chapter I, on "Coordinates and the Straight Line," was either more hastily written or less often revised than the rest. In the proof that a line is defined by any two of its points, it is shown that if C, D are points of the line AB, then any point of CD is a point of AB; but it is not shown (and this is more difficult to do) that any point of AB is a point of CD. On page 3 the reader is told that "different systems of coordinates may be chosen"; but this statement is not elaborated, and the student may imagine that it refers to polar or bipolar coordinates. It is small wonder, then, that the author omits to prove his definition of a line invariant under change of coordinate system.

Chapter III on cross-ratios and harmonic ranges is a natural continuation of Chapter II. Chapter IV treats the conic as given by a quite general quadratic parametrisation. Chapters V to X deal with many of the usual properties of conics, including a chapter on the quadrilateral and quadrangle and a chapter on (1-1) correspondence of points on a conic. The illustrations are well chosen and often include well-known theorems.

A caveat on page 87 warns "the reader to treat . . . degenerate cases with respect." The reader should be told *why* he must treat them with respect, and where, precisely, e.g. the proof on page 80 breaks down in these cases.

The last two chapters, on applications to Euclidean geometry, illustrate the power and scope of projective methods; but the idea of infinity is a little awkwardly handled, both here and in the introduction. We read on page 182 that: "the points on the line $z = 0$ have infinite coordinates in the Cartesian plane." But the points on $z = 0$ are not points of the Cartesian plane. They have no coordinates in it, finite or infinite. Infinity, as used in geometry, should be independent of magnitude or limiting processes. We can have a line at infinity equally well if our coordinates are elements of a finite field. In showing the connection with physical space (which is in any case only approximate) limiting processes may be mentioned; but this is an explanation of why the term "infinity" is used, and not a definition. It should be made clear that the points of $z = 0$ are points in the algebraic sense of ordered number-triples, so that any property derived by algebra is equally valid for those points; but they do not belong to the Cartesian plane. Indeed, there are other ways of completing the Cartesian plane, such as the single point at infinity used in the Theory of Functions.

Despite these criticisms, the book is in some ways an improvement on existing text-books. Its special merit is the emphasis laid on methods likely to be of value in more advanced work. There is a large number of examples, and the book should be very useful to the Part I candidate; the second-year man, too, may find much to interest him; but for scholarship purposes its matter is rather difficult and concisely presented.

[NOTE—on page 94, line 4 is either a misprint or a slip of the pen:

$$``\frac{x}{f} + \frac{y}{g} + \frac{z}{h}"$$

should be: $``\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0,"$]

A. M. M.

Newton at the Mint. By Sir JOHN CRAIG, C.B., LL.D. Pp. 128. (Cambridge University Press.) Price 7/6.

This book, first published in September, 1946, deals with an aspect of Newton's work not usually met with in his biographies. Accustomed as we are to think of him as the greatest mathematician of his age, it is salutory for us to meet him in the role of public servant, first as Warden and then as Master of the Mint. From 1696 until his death in 1727 we see him striving to improve the working of the Mint and to protect the coinage from the activities of coiners and clippers. At the same time he had continual trivial worries: for example, he had to request "That the Gunner of the Tower do order the Guns in such a manner that upon firing they may do least harm to the glass windows of the Mint." In all, the picture emerges of a man of wide intellect and general good sense who left the Mint in a much more efficient state than he found it, and I thoroughly recommend the book to the general reader.

V. W. D. H.

Fundamental Theory. By the late Sir A. S. EDDINGTON.

There is fairly widespread agreement with the idea that a connection must exist between atomicity and the properties of the universe as a whole. In this book Eddington continues his exploration of this suggestion. The work falls outside the usual domain of theoretical physics, and for this reason its value cannot be immediately assessed. Indeed, it seems clear that much research must be done before a comprehensive judgment on this work becomes available. In the meantime it may be noted that many physicists are in disagreement with Eddington over a number of specific issues.

In the first place there is a philosophic disagreement concerning the nature of the unsolved problems of physics. For example, Eddington attaches great significance to the value of the fine structure constant, but the numerical value of this constant seems to be of far less importance than the circumstance that present electromagnetic theory allows several units of charge to exist simultaneously, whereas only one unit of charge is in fact observed. In this sense charge appears to be on a similar footing to Planck's constant and to the velocity of light. The special theory of relativity explained the deep rooted significance of the velocity of light, while the commutation relations of quantum theory establish the unique rôle played by Planck's constant. These cases suggest that an improved electromagnetic theory should lead to a deeper understanding of the nature of charge—and of why, for example, it has not been found experimentally possible to subdivide the electron.

Second, there is direct disagreement on a number of technical questions. In particular, Eddington will have little support among physicists in his refusal to admit the Stoner-Anderson formula for the pressure of relativistically degenerate electrons. The reviewer is strongly opposed to Eddington's statement that this formula "continues to work devastation in astronomy." A whole range of astrophysical phenomena are readily explicable in terms of the Stoner-Anderson formula, whereas Eddington's formula fails to supply any satisfactory solution to these problems. There would indeed be "devastation" if Eddington's formula were adopted.

Third, it may reasonably be doubted whether a system of cosmology that fails to satisfy astronomical requirements can be adequate to provide a satisfactory explanation of the structure of the laws of physics. The three main results that cosmology must eventually provide are the red shift of the spectra of the extra-galactic nebulæ, the condensation of the nebulæ themselves, and the axial rotations of the nebulæ. So far attention has been almost entirely confined to the first of these requirements. A "cosmical constant" has been introduced in order to obtain an expanding universe with an age at least as great as the measured age of the Earth. Eddington attempts to place the introduction of the cosmical constant beyond criticism by asserting that it is one of the main constants of nature. This argument is open to doubt, however, because it can be shown that if the cosmical constant is adjusted to give the observed rate of expansion, then nebular condensation is prevented except during the earliest phases of the expansion—a most serious difficulty in a universe that seems to be undergoing a vigorous dynamical evolution (quite apart from the expansion). Indeed, the crucial feature of cosmology is the *combination* of extensive condensation with the red shifts.

Although, for the reasons set out above, the reviewer feels unable to accept a number of Eddington's conclusions, he recognises that these specific points do not constitute an objection to the work as a whole. It would indeed be surprising if immediate agreement were possible on so controversial a subject. A final judgment on this highly original and pioneering attempt on the most difficult problems of physics must be left to the future.

F. H.

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