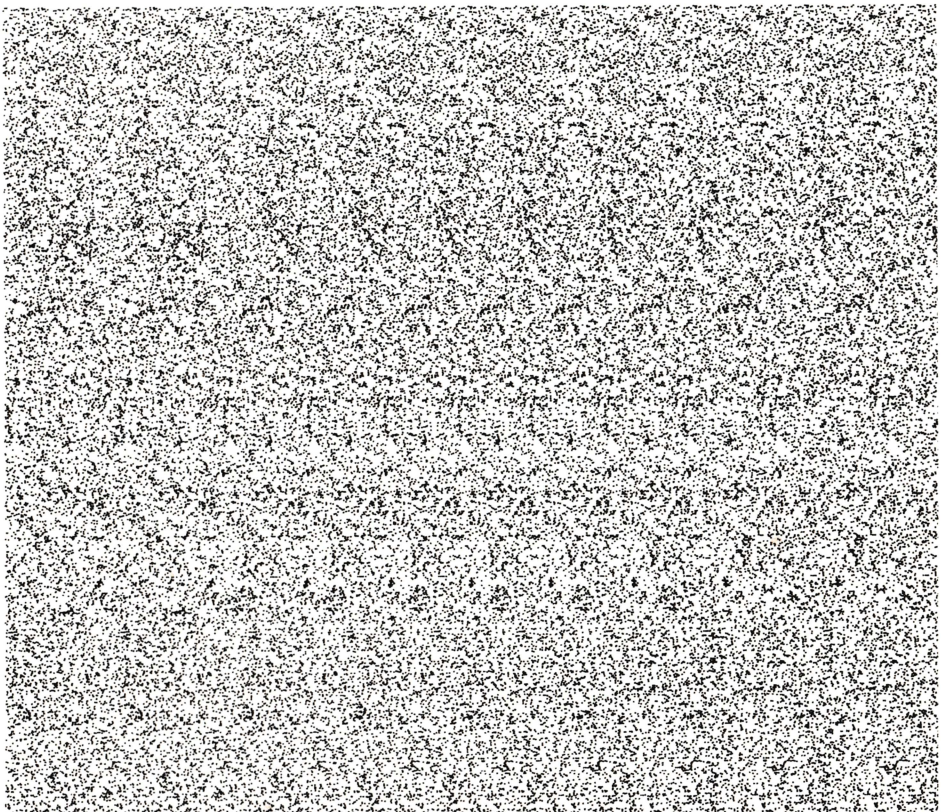
Eureka Editor
The Archimedeans
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WA
United Kingdom

archim-eureka@srcf.net

Published by The Archimedeans, the mathematics student society of the University of Cambridge

Thanks to the Betty & Gordon Moore Library, Cambridge

EUREKA



Number 52 March 1993

Eureka

Eureka is the journal of the Archimedeans, which is the Cambridge University Mathematical Society, and a Junior Branch of the Mathematical Association. *Eureka* is published approximately annually, but since it, like the Society, is run entirely by student volunteers, it is impossible to guarantee precise publication dates. The Society also publishes **QARCH**, a problems journal.

Subscriptions

A subscription account may be opened by sending at least ten pounds to the Business Manager at the address below; issues will be sent as they are published and you will be informed when your account is running low. It is also possible to subscribe to *Eureka* by standing order. Back numbers of some of the more recent issues are available, and photocopies of any issue may be made available by special arrangement. TEX source for issues since *Eureka* **49** may also be available, subject to copyright; contact the Business Manager for details.

Electronic Mail

The Society can receive electronic mail via JANET, ARPAnet and so on, and can read most electronic media. Our electronic mail addresses are given below. There may be long delays in dealing with messages of any kind in the University vacations (especially June to October), and you are asked not to send large amounts of text electronically without prior warning.

Addresses

Correspondence, clearly marked "Editor" or "Business Manager", should be sent to:

Eureka, The Arts School, Bene't Street, Cambridge CB2 3PY, England

Electronic mail from the UK should be sent to:

`archim@uk.ac.cambridge.phoenix`

From the rest of the World, the following should work:

`archim@phoenix.cambridge.ac.uk`

But if it doesn't, try whichever of these seems the more appropriate:

`archim%phx.cam.ac.uk@nss.cs.ucl.ac.uk`
`archim%uk.ac.cam.phx@ukacrl`

EUREKA

Number 52 Editor: Michael Greene March 1993

TABLE OF CONTENTS

ISSN 0071-2248

Committee of the Archimedeans, 1992–1993

President	Vin de Silva	Trinity
Vice-President	David Jones	Trinity
Secretary	Anton Cox	Clare
Junior Treasurer	Amy Mokady	Trinity
Registrar	Gareth Rees	Christ's
Publicity and Entertainments Manager	Michael Greene	Trinity
Chronicler	Patricia Smart	Girton
Business Manager	Stephen Frye	Clare
Senior Treasurer	Dr I. B. Leader	Peterhouse

ACKNOWLEDGEMENTS

Eureka affords its Editor the power to press-gang numerous people into helping with the less exciting tasks involved in its preparation. Above all, I would like to thank Colin Bell, who has brought a great deal of time, effort, and expertise to bear on the preparation of his own article and the diagrams throughout the journal; without his assistance, *Eureka* would have been next to impossible to complete. I am also deeply indebted to the other contributors, who have managed, between them, to submit a good selection of excellent articles; without them, the journal would not even exist.

The text of the journal was typeset in TEX, and, completely unfamiliar with this program, I have found the macros used in the past few issues an enormous help. I would especially like to thank Anton Cox, David Jones, and Richard Tucker, for their help with the typing, and Mark Wainwright, for various nuggets of advice. Finally, I am indebted to the Department of Applied Mathematics and Theoretical Physics, for the use of their laser-printing facilities in the preparation of the hard-copy sent to the printers.

Editorial

Welcome to *Eureka* **52**, which follows hot on the heels (though less hot than originally intended) of the last issue. The delay is due not, perhaps surprisingly, to inactivity on the part of the Editor, but to the last-minute submission of one of the longer articles and to the well-known Hofstadter's Law:†

> It always takes longer than you think it will take,
> even if you take into account Hofstadter's Law.

Editors in the past have almost invariably bemoaned the shortage of articles submitted to them, and have taken great pleasure in describing the desperate methods to which they have had to resort to induce friends and acquaintances, who quickly become mortal enemies, to write something suitable. I am happy to say that I have had no such trouble. The wide range of articles in this issue have, almost without exception, been offered to me, and are all the more welcome for that. I wish my successors good luck in every respect, but in this above all others.

Michael T. Greene
March, 1993

Mathematical Verse

Reluctant to put a stop to the splendid tradition of verse in *Eureka*, I include the following from Dilip Sequeira, now stationed at a distant outpost of the Cambridge University Mathematical Limerick Laboratory (see *Eureka* **50**), Dilip Sequeira (no relation), and Michael Fryers, respectively:

If M's a complete metric space
And non-empty, it's always the case
That if f's a contraction
Then under its action
Just one point remains in its place.

To prove that a space is connected,
Just show that it can't be dissected
Into some open pair
Of non-empty sets where
There's no point in the two intersected.

Clerihew
Devised a new
Kind of poem.
What a lot we owe 'im.

† See Douglas R. Hofstadter's *Metamagical Themas*, p. 48.

The Archimedeans

Vin de Silva (Chronicler 1991–92)

The year 1991–92 will go down in the records as a mixed year for the fortunes of the society. In terms of social events and speaker meetings, there was a strong smell of success in the air. In other regards, sad to say, the day of reckoning and accounts-book cursing had come. After generations of secretive Junior Treasurers and overspending committees, the price that had to be paid turned out to be too much. There was no *Eureka* in 1991.

Once the smoke had cleared after the AGM in March 1991, and various frivolously appointed members of the new committee had done the decent thing and resigned, there was plenty left to enjoy. The traditional May Week entertainments flourished as never before. The highlight was the garden party, held on the Clare Memorial Court lawns on a bright and sunny summer's day (we have friends in the Met Office ...) and prodigiously well attended. Guest of Honour was the Archimedeans' Lecturer of the Year, Dr Raymond Lickorish, and music was provided by the Barbershop Subgroup, incomparably maestroed by Robert Hunt. Other events included a satisfyingly nocturnal punt trip to Grantchester, a ramble, and a croquet match against the Faculty.

There were many highly enjoyable speaker meetings. Right at the end of Lent 1991, we had a talk by Dr Mond of Warwick; Michaelmas 1991 saw Dr Shiu (Loughborough), Prof. Zeeman (Oxford), Prof. Hall (Manchester), Dr Roe (Oxford), and Prof. Lyne (Jodrell Bank) speak on a variety of subjects, a pleasing blend of pure and applied mathematics, not to mention astrophysics. The good work was kept up in Lent 1992 by Prof. Sciama (Oxford), Mr Paul Woodruff (Dulwich College), Dr Giblin (Liverpool), and the Master of Trinity, Prof. Sir Michael Atiyah. There were also three lunchtime speaker meetings: former presidents Messrs Stephen Turner and Alan Stacey talked on *What I Did in my Summer Holidays* and *How to Make Better Coffee* (respectively), and there was a talk on flexagons.

Several events were held jointly with our sister society, the Invariants of Oxford. Easter term saw the Archimedeans win the Varsity Croquet match, aided and abetted by the unique lie of the land on the Trinity lawn; in Michaelmas the Invariants organised an unusual and exciting Treasure Hunt on the streets of Oxford; in Lent, a large party of Oxonians descended on the Lecture Room Theatre, Great Court, Trinity, to take part in the annual Problems Drive. Appropriately, it was a half-Oxford half-Cambridge team that carried off the bottle of port this year.

The Subgroups carried on as usual, Othello and the PGR being the best attended. Musical Appreciation appeared briefly, the Barbershop sang, Landscape Gardening was advertised, and Mathematical Models almost had a meeting. Most bizarrely, a Cake-Cutting subgroup made several appearances, completely failing to get anywhere with the mathematical problem it had been set up to try and solve, but effective in making disappear large quantities of cake, Jaffa or otherwise.

The year ended with *Eureka* **51** in grave danger of being published, several versions of **QARCH 12** (with different cover dates) very nearly being distributed, and, miracle of miracles, our Junior Treasurer, Business Manager, and Secretary all standing for re-election at the AGM. It is with confidence that I predict a successful year for the Archimedeans during 1992–93.

Strictly Rectangular Representations

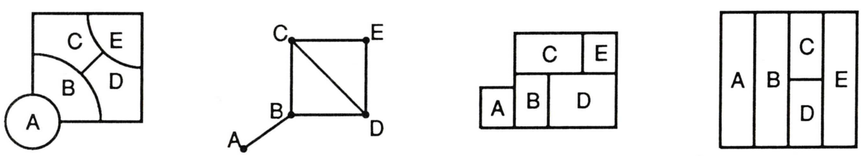
Colin Bell

My interest in this area was sparked by Adam Chalcraft, who, one hot sticky night in August, showed me a *Diplomacy*† map of an unusual form: all the regions (bar Spain) had been redrawn as rectangles, and, furthermore, the entire map had become a rectangle. He posed the following questions:

i) Given a map, is it possible to represent it with all the regions being rectangles? (I call this a *rectangular representation.*)
ii) If this is possible, is it further possible to represent the map such that the entire map is a rectangle (a *strictly rectangular representation*, or SRR)?

A *map* here is defined to be a finite connected planar graph all of whose regions are triangles, except for possibly the outside piece. In layman's terms, this says that the map is in one piece and has a finite number of countries, and only three countries meet at any point. Note that there is a potential notation clash between the regions of the graph and the regions of the map; we shall hence not use the term region at all, but shall stick to the graph-theoretic terminology and refer to the countries as *vertices.* A *representation* is a redrawing such that exactly those pairs of vertices (countries) bordering each other in the original map are connected by a line or *edge* in the new map.

For example, consider the map shown in Figure 1(a). It can be considered as the graph in 1(b). 1(c) is a rectangular representation of it, and 1(d) an SRR.



(a) (b) (c) (d)

Figure 1.

For general graphs, some things are obvious. It is clear, for example, that an internal vertex maps to an internal rectangle and a vertex on the boundary maps to a rectangle on the edge (since if you trace around the edge of the shape you get by putting all the rectangles together, you get the boundary of the graph). More usefully, we have the following conditions:

A If three vertices are all connected to each other, their three rectangles meet at a point, so if a representation of the graph is possible, there cannot be anything inside any triangle. Also, any internal vertex must have at least four neighbours since only one neighbour can touch any given side.

If we now restrict our attention to strictly rectangular representations we have some further obvious conditions:

† *Diplomacy* is a registered trade mark of Avalon Hill.

B Disregarding *isthmi* (points which, when removed, disconnect the graph) and *peninsulae* (vertices with only one neighbour), by a similar argument to before, points at a corner must have at least two neighbours, and points on the edge not at a corner must have three or more neighbours. Peninsulae must occupy an entire side. Also, a subgraph on the edge which does not include a corner must have three or more neighbours, and a subgraph occupying one corner must have two or more neighbours. So we may have a number of vertices or subgraphs which can only go in corners: this number must clearly be four or less. If there are any corners left after we've dealt with the forced ones, we can put whatever we like in them, subject to the other conditions.

C No point may occupy three or more corners (unless it is the only point). Also, if there are two vertices each occupying two corners, they cannot be neighbours, unless there are only two points in total.

D If we have an isthmus in the SRR, then its rectangle must go the entire width or height of the graph, for, if not, removing it would not disconnect the graph. Therefore we need it to have precisely two corners on one side and two on the other. In particular, an isthmus cannot occupy a corner itself, and if a point occupies two corners, then it cannot be next to an isthmus, unless that isthmus separates it from all the other points in the graph.

Now, if this were a detective story, I wouldn't tell you what the answer was until the end, and would do my best to hide it from you until then. But this isn't a detective story, so I'll tell you now: any graph satisfying condition A above has a rectangular representation, and any graph satisfying all four conditions above has an SRR. In other words, it's possible unless it's trivially impossible.

The first part follows from the second: the basic idea is that you take a maximal subgraph of the map which has an SRR (maximal here means that there's no subgraph strictly containing it which has an SRR) and then attach the rest of the graph on in bits. Any other chunk of graph can connect to at most two of the vertices already there, so find a representation of that, rotate and scale to fit, and attach. Since you can make it as small as you like, there's no problem with it colliding with other parts already laid down. There are a few details missing here, but they're not hard to fill in.

Let's start with some notation: we call the graph G. We assume that G satisfies A to D above, and want to construct a SRR for G. A *side* of the graph is defined to be those vertices which we'll map to one side of the final rectangle; for notational convenience we'll call the sides north, south, east, and west.

The basic idea of the proof, as you might expect, is inductive. We take our graph, cut it into two parts, find SRRs of each and stick them together. It's obvious that we can't divide it into two any old how; we therefore define a *partition line* to be a line down which we can. We denote such a line by $A = (a_1, a_2, \ldots a_n)$, where the a_i are vertices and each $a_i a_{i+1}$ is an edge. This is a path which connects one side of the graph to the opposite side (we shall always assume it goes from the west side to the east) which satisfies:

(side): Only a_1 is on the west side, and only a_n is on the east. (A path can be reduced to this form by deleting vertices on the ends.)

(minimality): If a_i adjoins a_j, then $i = (j-1)$ or $(j+1)$. (It is clear that any path can be reduced to a path satisfying this property, simply by removing intermediate points.)

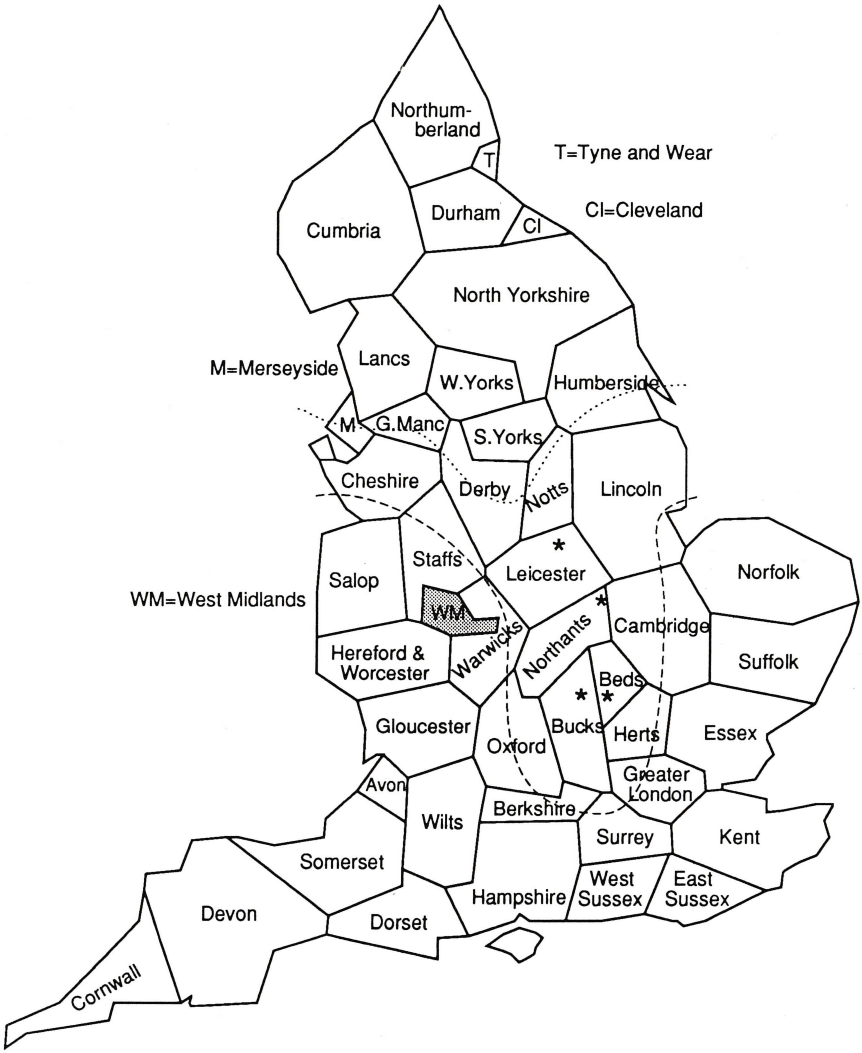
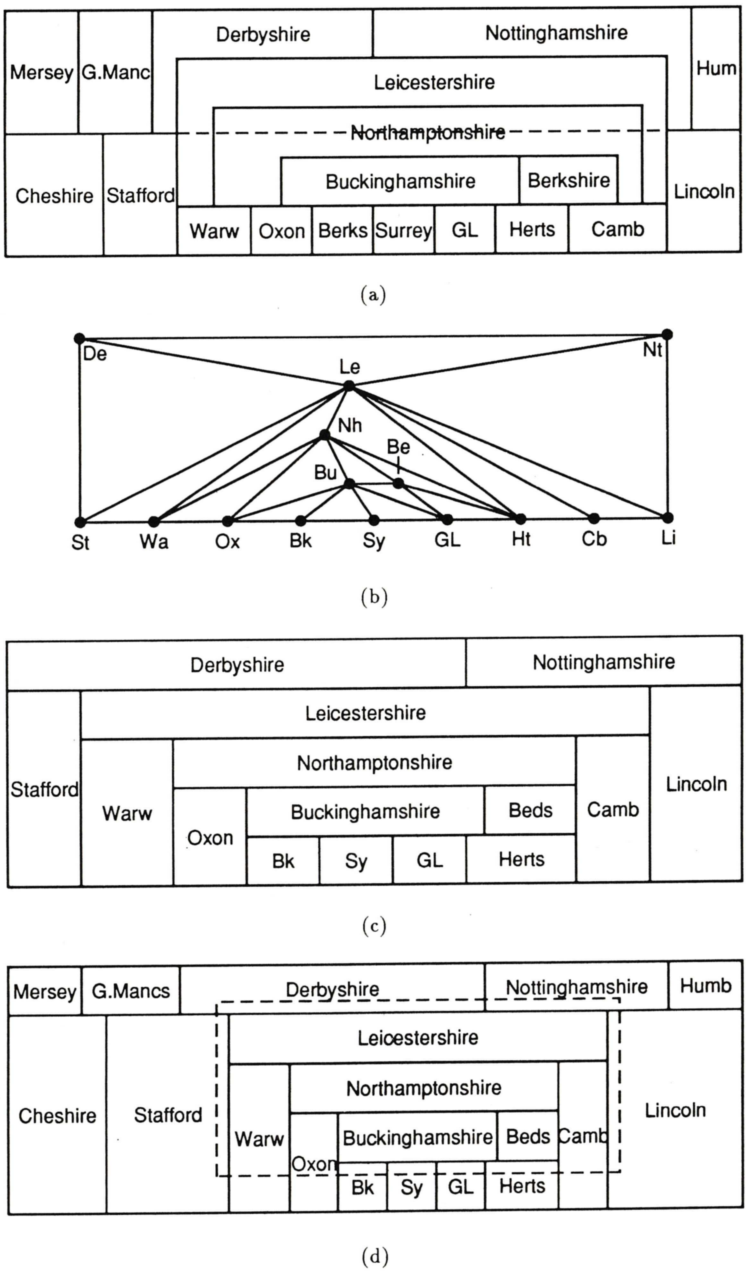

Northum-
berland
T
T=Tyne and Wear
Cumbria
Durham
Cl
Cl=Cleveland
North Yorkshire
Lancs
M=Merseyside
W.Yorks
Humberside
M
G.Manc
S.Yorks
Cheshire
Derby
Notts
Lincoln
Salop
Staffs
Leicester
Norfolk
WM=West Midlands
WM
Northants
Cambridge
Hereford &
Worcester
Warwicks
Suffolk
Beds
Gloucester
Bucks
Herts
Essex
Oxford
Avon
Greater
London
Berkshire
Wilts
Surrey
Kent
Somerset
West
Sussex
East
Sussex
Hampshire
Devon
Dorset
Cornwall

Figure 2.



Derbyshire
Nottinghamshire
Mersey
G.Manc
Hum
Leicestershire
Northamptonshire
Buckinghamshire
Berkshire
Cheshire
Stafford
Lincoln
Warw
Oxon
Berks
Surrey
GL
Herts
Camb
(a)
De
Nt
Le
Nh
Be
Bu
St
Wa
Ox
Bk
Sy
GL
Ht
Cb
Li
(b)
Derbyshire
Nottinghamshire
Leicestershire
Stafford
Northamptonshire
Lincoln
Warw
Buckinghamshire
Beds
Camb
Oxon
Bk
Sy
GL
Herts
(c)
Mersey
G.Mancs
Derbyshire
Nottinghamshire
Humb
Leicestershire
Northamptonshire
Cheshire
Stafford
Lincoln
Warw
Buckinghamshire
Beds
Camb
Oxon
Bk
Sy
GL
Herts
(d)

Figure 3.

(corner): If some a_i is a corner, then A is an entire side. If one of the vertices in the partition line has two corners, then the partition line must be that vertex alone.
(connectivity): Removing the vertices $\{a_i\}$ will disconnect the graph (except when the partition line is a side).

The corner and connectivity conditions are basically there to avoid annoying special cases later.

We now define three components of the graph. The first, G_1, is the partition line and everything on one side of it (say the south side). We then define the *opposition line*, B, to be the set of vertices on the north side of the partition line which border vertices in the partition line, reduced so that it satisfies the side and minimality conditions (opposition lines don't in general satisfy the corner or connectivity properties). We define G_2 to be the opposition line and everything to the north of it. Finally, the remaining vertices (the ones that were in B but were deleted) form the third component, which we call the residual set, or RS.

Confused? Well, let's have an example. We consider England partitioned into counties (for those foreign, used to the pre-1974 layout, or just geographically challenged, this is illustrated in Figure 2). The map is schematic, and has been adjusted in some places to show more clearly which pairs of counties are adjacent. It satisfies conditions A to D, with the exception of the West Midlands, which only has three neighbours. We hence delete it (some would say this should have been done years ago). The map thus obtained has four forced corners: two in Cornwall, one in Tyne and Wear, and one in Cleveland. A partition line (Cheshire to Lincolnshire by a circuitous route) has been marked by a dashed line. The opposition line (Merseyside to Humberside) is the dotted line, and the counties in the residual set are marked with a star.

To get an SRR for G, we take SRRs for G_1 and G_2, adjoin them, and then splice in the residual set. In this case, it is easy to check that G_1 (with corners Cornwall twice, Cheshire, and Lincolnshire) and G_2 (Tyne and Wear, Cleveland, Humberside, and Merseyside) satisfy the conditions; we'll prove this for general graphs later. So we take SRRs for G_1 and G_2 and place them together, compressing in the east-west direction so that counties which border across the divide join up correctly. This works fine, except in the case where we have a residual set. In this case we have to split the residual set into components, and for each component find an SRR (if possible) for the component and the vertices of G_1 and G_2 which border it (necessarily a subset of the partition and opposition lines), and then splice this in.

This is illustrated for our example in Figure 3 (using two-letter abbreviations which I hope are obvious); here the residual set only has one component. We take the graph in 3(b) and find its SRR in 3(c), which is fairly easy (take partition lines in order: De and Nt, St alone, Li, Le, Wa, Cb, Nh, Ox, and Bu and Be, and then you're done). It's then shown spliced in in 3(d). Note that we don't know that all the outer rectangles extend out north and south by the same amount; we'll find out how far they do when we consider the rest of the graph.

If we now consider the southern component (which was G_1), and take the same partition line, the opposition line consists only of Gloucestershire, Wiltshire, and Hampshire, leaving eight counties in two components of the residual set. Considering the western part, it's fairly obvious that the only thing that will work is a staircase arrangement; it turns out that the same thing happens in the east. The next part southwards of the SRR is shown in Figure 4. It's a fairly easy exercise to complete the representation for the whole of England; my version is in Figure 5.

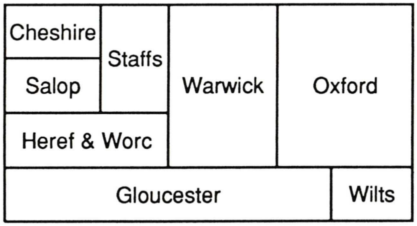


Figure 4.

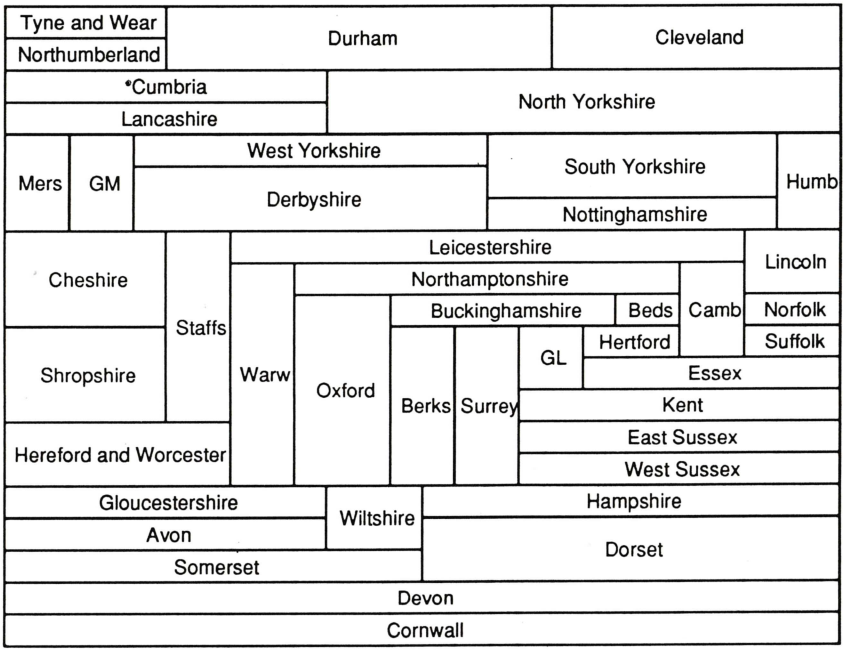


Figure 5.

Now we need to show that everything works in general: we first show that we can pick a partition line that works, and then that G_1, G_2, and the rectangles all have the right properties.

First of all, if we can find an isthmus, or two corners in the same place, we pick that as a partition line. We're going to have to have a partition line down there eventually anyway, and picking this line early simplifies the argument. Either of these is obviously a valid line, and the conditions are easy to check.

Secondly, if we have two opposite corners adjoining, we use a special construction to simplify the argument (this is the first special case considered at the end).

If neither of these conditions arises, we can always use the south side as the partition line, making G_1 exactly A, and G_2 a subgraph of the rest. It is easy to see this satisfies the conditions: only the south-west corner is on the west side, only the south-east corner is on the east side, and if this line fails to satisfy minimality then it has a subgraph on the edge with fewer than three neighbours; this is impossible, because it's a complete side and it doesn't disconnect the graph (we've got rid of the isthmi by assumption).

For a general partition line, we now show that G_1 and G_2 have SRRs, and also that the graphs we look at when considering the residual set have SRRs.

For G_2 (argument for G_1 is similar):

A This follows trivially from the same condition on G. Note that the internal vertices of G_2 haven't lost any neighbours from the split.

B The only vertices which have lost neighbours are those on the opposition line. If the opposition line consists of a single vertex, then this must occupy two corners of G_2, and so it has one neighbour (unless it's the only vertex); it has this from connectivity. If the opposition line consists of more than one vertex, then look at the endpoints: b_1 has b_2 as a neighbour, and must have a second neighbour, otherwise b_2 would be on the west side. The other endpoint is similar. For a point not on the end of the line, b_i, we know it has two neighbours, b_{i-1} and b_{i+1}. It must have a third neighbour since otherwise b_{i-1} and b_{i+1} are adjacent, which contradicts minimality.

C If we have three corners in the same place, then there are either two corners of G and an endpoint of the opposition line, or one corner of G and both endpoints of the opposition line. The first case is excluded by our choice of partition line (if this choice is not made then we have trouble if our partition line ends at a point one away from the point with two corners). In the latter case, the opposition line is an isthmus, and we know we can't have corners at isthmi. Two two-corner points adjoining means either a two-corner point next to an isthmus (which contradicts part of condition D which holds for G) or that the north side is of length 2 and the opposition line is of length 2 and they intersect. In this case there's no room left for anything else.

D Any isthmus either was there already, which means it goes from west to east in G and hence also in G_2, or has been created (note that we can't create new isthmi in G_1 because of connectivity). Hence, it must be on the opposition line, and therefore on the south side. There's no problem if it's also on the north side, so let's assume it's on one of the others (WLOG west). By reduction to the side condition, it's an endpoint of B (we may take it to be b_1). Now, if b_1 were the only point of B, it would have been an isthmus of G, and hence would go from west to east. In particular, it's now on three sides and doesn't disconnect G_2. So there are some other points of B, and they're all in the same component of $B \backslash b_1$. Consider the other components. Each one is connected and on the west side. They're either south of b_1 (which implies they're in the residual set), or north, which means the component has only one neighbour (namely b_1) since they aren't connected to any other part of G_2 and also miss G_1. So such an isthmus cannot exist. Now for the corner conditions: any pre-existing isthmus inherits them from G. A new isthmus has the old north-west and north-east corners on opposite sides, and the two endpoints of the opposition line on opposite sides, so there's no problem. If a point with two corners is next to the isthmus, this point is simultaneously an old corner and the endpoint of the opposition line, and hence there can't be anything else on that side of the isthmus.

Now for the bits of the residual set: the subgraphs we need to consider are bounded by pairs of points on the opposition line and their neighbours, namely b_j, b_{j+1}, and a_k to a_l, where $k = \max\{\,i \mid a_i \text{ borders } b_j\}$ and $l = \min\{\,i \mid a_i \text{ borders } b_{j+1}\}$, and all the vertices bounded by those. (If you don't believe this, draw a picture and convince yourself.) We need this graph, with corners at b_j, b_{j+1}, a_k, and a_l, to have an SRR.

A Same argument as for G_1.
B Same argument as for G_1.
C The four corners are distinct.
D There can't be any isthmi.

There is one potential snag: if the graph we have to consider here is the same as G, then the induction falls over. This can only happen if A is one side of the graph (we may assume south) and B is the north. Then all the sides apart from the south must have precisely two vertices. If the south one doesn't, take the north side as a partition line and we're done. If the south side has precisely two vertices, we need a special construction, which we'll consider as special case 2.

SPECIAL CASE 1. NW and SE adjoin, where NW and SE are the countries in the north-west and south-east corners respectively. Split as in Figure 6(a). The location of the new corners is obvious; we need to make sure the conditions are satisfied. WLOG take the north-east subgraph:

A and B Derived from the corresponding conditions on G; note that only the vertices on the west and south sides of this subgraph have lost any neighbours (one each, except for the south-west corner).

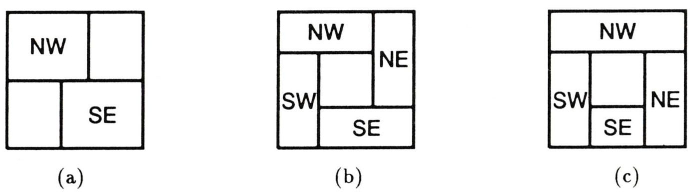


Figure 6.

C Any newly created isthmus will force there to be two subgraphs of the north-east subgraph with only one neighbour. Considering them as part of G, they will either become internal subgraphs with a maximum of three neighbours (which violates condition A for G), or external subgraphs with a maximum of two neighbours (if either connects to both NW and SE, then we can't have an isthmus since all regions are triangles).

D Any point occupying three or more corners must *either* be the sole neighbour of both NW and SE, and hence either an isthmus or the sole point in the north-east subgraph, *or* the original north-east corner and the sole neighbour of (WLOG) NW, in which case the remaining vertices of the north-east subgraph form a subgraph with at most two neighbours, not at a corner. Two points with two corners apiece means WLOG that NW has a unique neighbour and SE adjoins the old north-east corner. In this case, if they adjoin, the remaining vertices form a subgraph (of G) with a maximum of three neighbours internally or two externally.

SPECIAL CASE 2. All four sides have precisely two vertices. Split as in Figure 6(b)—then the truth of the required conditions can be derived by pretending the split is as in 6(c) and applying the first special case twice.

Notes and further ideas

1. It is possible to have a representation of a graph where four regions do indeed meet at a point (if you insert an extra region in at the point, get a representation and then compress the rectangle corresponding to the extra region to a point). But if you have two or more of these points, it isn't clear that you can do this for both simultaneously without crushing one of the rectangles you want to keep.

2. You have considerably more flexibility over how to organise the rectangles than you might think at first, and this is in part the reason why so many graphs have representations. This leads to further possibilities: for example, it is not hard to redraw the map of the counties of England so that the three Ridings of Yorkshire form one rectangle—try it and see!

3. After proving this result, I posted it to the newsgroup `sci.math` on USENET (a worldwide discussion board) wondering if it was known. With information provided by Douglas Wood I tracked down a paper of Carl Thomassen in which he proved the result for two-connected (i.e., isthmus-free) graphs; his proof is essentially the same as mine.

4. Is there a natural extension of these ideas to higher dimensions? If so, I would imagine there would be a very much more complicated condition on the graphs that work, since every graph has a 3D representation, whereas only a restricted set have a 2D representation.

5. What happens if we allow rectangles of zero width or height? Brief inspection shows that this means we can effectively deal with a single vertex inside a triangle, and also gets round some of the corner restrictions. But how far does this idea work? It should be noted that reducing all the rectangles to points doesn't work, since now all the rectangles meet each other.

Reference

Thomassen C. (1984). Section 7 of "Plane Representations of Graphs", pp. 69–84, *Progress in Graph Theory*, edited by Bondy, J. Adrian; Murty U. S. R., Academic Press 1984.

A Logical Puzzle

Anton Cox

While walking through Cambridge one foggy night, you come across the committee of a certain well-known society, transporting its ill-gotten gains (in sacks) to their secret hideout. Unfortunately—it being so foggy—you cannot see how many sacks each of them is carrying. A friendly passing porter tells you that the product of the number of sacks each is carrying is 120, and (very helpfully!) that each carries an integer number of sacks.

You know from experience that no-one can carry more than ten sacks. You are also aware that the committee is divided into two groups: those who always tell the truth, and those who always lie ...

Within each group, the earlier a person's name is printed in the programme card, the more sacks that person can carry. This order is:

A: President
B: Vice-President
C: Secretary
D: Junior Treasurer
E: Chronicler
F: Registrar
G: Pub. and Ents.
H: Business Manager

(Thus if A and D are in the same group,
A carries at least as much as D,
but if they are in different groups,
either may carry more than the other.)

You come across the party as they are being accosted by a pair of Proctors, anxious to learn the distribution of the sacks. As you arrive the conversation continues ...

A: D is lying—she has 2 sacks.
B: G is lying—he has sacks equal to half the number of liars.
C: A speaks the truth—F has 2 sacks.
D: B is lying—he has an even number of sacks.
E: H tells the truth—C has 1 sack.
F: E is telling the truth—she has more than 1 sack.
G: A lies—he has 2 sacks more than there are liars.
H: F is telling the truth—the number of liars is odd.

The Proctors (who know all that you do so far) have powerful torches; one of them can see what A, C and E are each carrying, while the other can see what A, B and C are each carrying. Each separately is unable to deduce who carries what (and as they never confer, they will never know).

Determine the distribution of sacks—and the identity of the liars!

The Gromov Distance or Isometry up to ϵ

Roger Hendrickx

The Gromov distance is a measure for the resemblance between metric spaces. Roughly speaking, the Gromov distance between two metric spaces is smaller than ϵ if there exists an isometry up to ϵ between the spaces.

No general methods for calculating Gromov distances are known.

In this article a few theorems are proved which can be useful in computing Gromov distances. An algorithm to determine the Gromov distance between two finite metric spaces is given.

DEFINITION. If $R \subseteq X \times Y$, then we define

$$p_1 R = \{x \in X \mid \exists\, y \in Y \text{ with } (x, y) \in R\}$$

and

$$p_2 R = \{y \in Y \mid \exists\, x \in X \text{ with } (x, y) \in R\}.$$

DEFINITION. Suppose (X, d) and (Y, e), also written X_d and Y_e, are metric spaces, and $\epsilon \in \mathbf{R}^+$. A binary relation $R \subseteq X \times Y$ with $p_1 R = X$ and $p_2 R = Y$ is called an *ϵ-approximation between X and Y* if and only if

$$xRy \text{ and } x'Ry' \;\Rightarrow\; |d(x, x') - e(y, y')| \leqslant \epsilon.$$

DEFINITION. If we set

$$\epsilon = \inf \{\, \epsilon' \in \mathbf{R}^+ \mid \text{there is an } \epsilon'\text{-approximation between } X_d \text{ and } Y_e\},$$

then we call ϵ the *Gromov distance between X_d and Y_e* (Paulin, 1988), and denote it by $d_G(X_d, Y_e)$.

We show with an example that the Gromov distance is not a metric over the class of all metric spaces, by proving:

PROPOSITION 1. *Two non-isometric spaces can give Gromov distance zero* (Gromov, 1981).

PROOF. Let d be the usual metric on $\mathbf{R}$, and set

$$X = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}, \quad Y = X \cup \{0\}.$$

Then X_d and Y_d are not isometric, for only in Y do we have two points a distance 1 apart. Choose some $N \in \{1, 2, 3, \ldots\}$. If we define $f : X \to Y$ by

$$f(1/n) = \begin{cases} 1/n & n < N \\ 0 & n = N \\ 1/(n-1) & n > N \end{cases}$$

then f is clearly bijective. For $x \in X$ we have $|x - f(x)| \leqslant \frac{1}{N}$, because either $f(x) = x$, or x and $f(x)$ lie in $\left[0, \frac{1}{N}\right]$. Thus for all x and y in X we have:

$$\begin{aligned}\left|d(x,y) - d\big(f(x), f(y)\big)\right| &= \big||x-y| - |f(x) - f(y)|\big| \\ &\leqslant |x - f(x)| + |y - f(y)| \\ &\leqslant \frac{2}{N}.\end{aligned}$$

Thus $d_G(X_d, Y_d) \leqslant \frac{2}{N}$ for all N, so $d_G(X_d, Y_d) = 0$ as required. □

However, over the class of all metric spaces, we *do* have

i) $d_G(X_d, X_d) = 0$ for all X_d,
ii) $d_G(X_d, Y_e) = d_G(Y_e, X_d)$, and
iii) $d_G(X_d, Z_f) \leqslant d_G(X_d, Y_e) + d_G(Y_e, Z_f)$,

which all follow immediately from the definition of Gromov distance, and hence the Gromov distance is a quasi-metric on the class of metric spaces.

THEOREM 2. *Let X_d be a metric space. If $A \subseteq B \subseteq X$, then*

$$B \subseteq \bar{A} \;\Rightarrow\; d_G(A_d, B_d) = 0.$$

PROOF. Take $\epsilon > 0$. It is enough to show that there is a 2ϵ-approximation between A_d and B_d. Let $R = \{\,(x,y) \mid x \in A,\ y \in B,\ d(x,y) < \epsilon\}$. Then $p_1 R = A$ since $A \subseteq B$, and $p_2 R = B$ since $B \subseteq \bar{A}$, and we have

$$xRy \text{ and } x'Ry' \;\Rightarrow\; |d(x,x') - d(y,y')| \leqslant 2\epsilon.$$

□

Note that the converse clearly does not hold: if X_d is a metric space with $A \subseteq B \subseteq X$ and $d_G(A_d, B_d) = 0$, it need not be the case that $B \subseteq \bar{A}$. For a counterexample, let X be $\mathbf{R}$ under the usual metric d and take $A = (0, \infty), B = (-1, \infty)$. Then $d_G(A_d, B_d) = 0$, since A and B are isometric, but $B \nsubseteq \bar{A} = [0, \infty)$.

COROLLARY 3. *For d the usual metric on $\mathbf{R}^n$, $d_G(\mathbf{Q}^n_d, \mathbf{R}^n_d) = 0$.*

PROOF. Follows directly from Theorem 2. □

THEOREM 4. *Let X_d be a compact metric space. For every $\epsilon \in \mathbf{R}^+$ there exists a finite set $S \subseteq X$ such that*

$$d_G(X_d, S_d) \leqslant \epsilon.$$

PROOF. By compactness, X can be covered by finitely many balls of radius ϵ, so we can find $\{x_1, x_2, \ldots, x_n\} = S$, say, with

$$X = B(x_1, \tfrac{1}{2}\epsilon) \cup B(x_2, \tfrac{1}{2}\epsilon) \cup \ldots \cup B(x_n, \tfrac{1}{2}\epsilon).$$

Now the expression

$$xRx_i \;\Leftrightarrow\; x \in B(x_i, \tfrac{1}{2}\epsilon)$$

defines an ϵ-approximation between X and S. Thus

$$d_G(X_d, S_d) \leqslant \epsilon$$

for the given $\epsilon \in \mathbf{R}^+$, as required. □

REMARK. It seemed to be evident that $d_G(X, X'') \leqslant d_G(X, X')$ for metric spaces with $X' \subset X'' \subseteq X$. However, we have the following finite counterexample:†

Set $X = \{x_0, x_1, x_2, x_3\}$,
$d(x_1, x_2) = d(x_2, x_3) = 1$,
$d(x_1, x_3) = 2$, and
$d(x_0, x_i) = 100$ for $i = 1, 2, 3$.

Take $X' = \{x_1, x_3\}$ and $X'' = \{x_1, x_2, x_3\}$. For a relation $R \subseteq X \times X'$ and $x' \in X'$, write $R\,x' = \{x \in X \mid x R\, x'\}$. Then by the relation shown in Figure 1 we know $d_G(X, X') \leqslant 98$ (in fact this is best possible), but, by considering the possible elements x of X which satisfy $x R\, x_2$ in the case for X'', it is clear that $d_G(X, X'') = 99$.

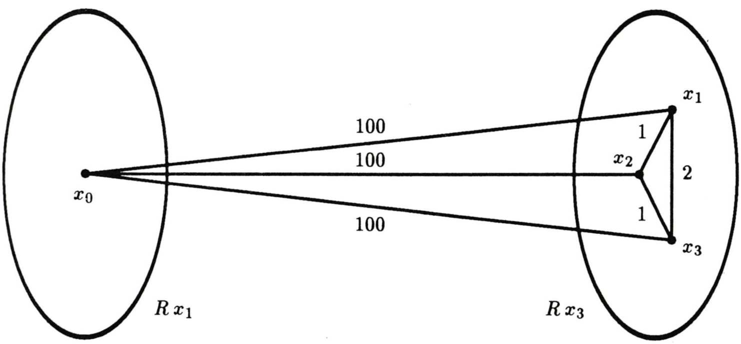


Figure 1.

DEFINITION. The ***diameter*** of a metric space X_d, written ϕX, is defined by

$$\phi X = \sup\{\, d(x, x') \mid x, x' \in X\}.$$

THEOREM 5. *Let X_d and Y_e be metric spaces with finite diameters. Then*

$$|\phi X - \phi Y| \leqslant d_G(X_d, Y_e) \leqslant \max\{\phi X, \phi Y\}.$$

PROOF. Without loss of generality we assume that $\phi X \geqslant \phi Y$. We begin with the first inequality.

Let $R \subseteq X \times Y$ be an ϵ-approximation and take $\delta \in \mathbf{R}^+$. Choose $x_1, x_2 \in X$ such that $d(x_1, x_2) \geqslant \phi X - \delta$, and choose $y_1, y_2 \in Y$ such that $x_1 R\, y_1$ and $x_2 R\, y_2$. Then

$$\epsilon \geqslant |d(x_1, x_2) - e(y_1, y_2)| \geqslant d(x_1, x_2) - e(y_1, y_2) \geqslant \phi X - \delta - \phi Y.$$

Since this is true for all $\delta > 0$, we obtain

$$d_G(X_d, Y_e) \geqslant \phi X - \phi Y = |\phi X - \phi Y|.$$

† F. Paulin, personal communication, 1990.

To show the second inequality, let $R \subseteq X \times Y$ be any relation for which $p_1 R = X$ and $p_2 R = Y$. If $x_1 R y_1$ and $x_2 R y_2$, we have

$$|d(x_1, x_2) - e(y_1, y_2)| \leqslant \max\{d(x_1, x_2), e(y_1, y_2)\} \leqslant \max\{\phi X, \phi Y\}.$$

Hence $d_G(X_d, Y_e) \leqslant \max\{\phi X, \phi Y\}$. □

The Gromov distance between two finite metric spaces

We look for an algorithm to compute the Gromov distance between finite metric spaces. Let X_d and Y_e be metric spaces, with $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. We compute the number of binary relations R between X and Y with $p_1 R = X$ and $p_2 R = Y$ as follows:

Compute the number of relations on $X \times Y$ with $p_1 R = X$. If R is such a relation, then R connects every $x \in X$ with the points of a subset of Y. All subsets of Y except $\emptyset$ have to be considered, leaving us with $(2^m - 1)$ possibilities. Since $|X| = n$, there are $(2^m - 1)^n$ binary relations on $X \times Y$ with $|X| = n$, $|Y| = m$, and $p_1 R = X$.

Now consider the second condition, $p_2 R = Y$. Let $N(n, m)$ be the number of binary relations R on $X \times Y$ with $p_1 R = X$ and $p_2 R = Y$. We want to find a recursive formula satisfied by $N(n, m)$.

If $p_2 R = A \subseteq Y$ with $|A| = r \leqslant m$, then the number of relations $R \subseteq X \times Y$ with $p_1 R = X$ and $p_2 R = A$ is equal to $N(n, r)$. The number of subsets A of Y with $|A| = r$ is $\binom{m}{r}$, so we obtain the formula

$$(2^m - 1)^n = \binom{m}{1} N(n,1) + \binom{m}{2} N(n,2) + \ldots$$
$$\ldots + \binom{m}{m-1} N(n, m-1) + \binom{m}{m} N(n,m)$$

which gives

$$\begin{aligned}
1^n &= N(n,1) \\
3^n &= 2N(n,1) + N(n,2) \\
7^n &= 3N(n,1) + 3N(n,2) + N(n,3) \\
&\ \vdots \\
(2^m - 1)^n &= \tbinom{m}{1} N(n,1) + \tbinom{m}{2} N(n,2) + \ldots \\
&\quad \ldots + \tbinom{m}{m-1} N(n, m-1) + \tbinom{m}{m} N(n,m)
\end{aligned}$$

We obtain a linear system of m equations in the m unknowns $N(n,1)$, $N(n,2), \ldots,$ $N(n, m-1)$, $N(n,m)$. Cramer's rule yields

$$N(n,m) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 1^n \\ 2 & 1 & 0 & & 3^n \\ 3 & \binom{3}{2} & 1 & & 7^n \\ \vdots & & & \ddots & \vdots \\ m & \binom{m}{2} & \binom{m}{3} & \cdots & (2^m - 1)^n \end{vmatrix}$$

We aim to develop an algorithm to compute the Gromov distance between two finite metric spaces. We know that there are $N(n,m)$ relations R which satisfy

$$p_1 R = X \text{ and } p_2 R = Y.$$

The following proposition reduces the number of relations to be investigated.

PROPOSITION 6. *Let $R \subseteq X \times Y$ be some ϵ-approximation between two metric spaces X_d and Y_e. If $S \subseteq R$ and $p_1 S = X$, $p_2 S = Y$, then S is also an ϵ-approximation between X_d and Y_e.*

PROOF. Take R as above. Then for all (x_1, y_1), $(x_2, y_2) \in R$,

$$|d(x_1, x_2) - e(y_1, y_2)| \leqslant \epsilon.$$

Hence, for all (x_1', x_2'), $(y_1', y_2') \in S$,

$$|d(x_1', y_1') - e(x_2', y_2')| \leqslant \epsilon. \qquad \square$$

DEFINITION. Let X_d and Y_e be two metric spaces with $|X| = n$ and $|Y| = m$ as above. For each $x \in X$ we choose one $x^* \in Y$, and for each $y \in Y$ we choose one $y^* \in X$. In this way we obtain a set

$$S = \{(x_1, x_1^*), (x_2, x_2^*), \ldots, (x_n, x_n^*), (y_1^*, y_1), (y_2^*, y_2), \ldots, (y_m^*, y_m)\}.$$

We call such a relation a *basic relation* between X_d and Y_e. It will be convenient to write $S(x)$ for x^*.

To compute Gromov distances it turns out to be sufficient to investigate these basic relations. How many such relations S are there?

Every S is associated with a map $f : X \to Y$ and a map $g : Y \to X$.
The number of possible maps $f : X \to Y$ is m^n.
The number of possible maps $g : Y \to X$ is n^m.
Hence the number of basic relations is at most $m^n.n^m$.

Suppose we have finite metric spaces X_d and Y_e. Assume that $|X| = n$ and $|Y| = m$, and further that $n \geq m$.

We write down all the basic relations

$$S_1, S_2, \ldots, S_{m^n.n^m},$$

possibly with some repetition, and for each $k \in \{1, 2, \ldots, m^n.n^m\}$ we determine

$$\epsilon_k = \sup_{i \neq j} \left\{ \left| d(x_i, x_j) - e(S_k(x_i), S_k(x_j)) \right| \right\}.$$

Then, by Proposition 6, we know

$$d_G(X_d, Y_e) = \inf \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{m^n.n^m}\}.$$

THEOREM 7. *Take* $A = \{a_0, a_1, a_2, a_3, \ldots\} \subseteq \mathbb{Z}$ *with* $0 = a_0 < a_1 < a_2 < a_3 < \ldots$, *and define* $K = \sup_n (a_{n+1} - a_n)$. *Then, for d the usual metric on* $\mathbb{Z}$,

$$d_G(\mathbb{Z}_d, A_d) = K - 1.$$

PROOF. Define R by $xRa_n \Leftrightarrow a_n \leqslant x < a_{n+1}$. Then R is a $(K-1)$-approximation between $\mathbb{Z}_d$ and A_d, so

$$d_G(\mathbb{Z}_d, A_d) \leqslant K - 1.$$

For the other inequality, let R be some ϵ-approximation; it is enough to show that for any $p \geqslant 0$,

$$\epsilon \geqslant a_{p+1} - a_p - 1.$$

Let $u = \max\{x \in \mathbb{Z} \mid \exists q \leqslant p \text{ with } xRa_q\}$, and choose $q \leqslant p$ with uRa_q. Then there is an r such that $(u+1)Ra_r$. For such an r, $r > p \geqslant q$, so $r \geqslant p+1$ and $a_r \geqslant a_{p+1}$.

Now finally

$$\begin{aligned} \epsilon &\geqslant \big||(u+1) - u| - |a_r - a_q|\big| \\ &= |1 - (a_r - a_q)| \\ &= a_r - a_q - 1 \\ &\geqslant a_{p+1} - a_p - 1. \end{aligned}$$

□

THEOREM 8. *Let* $f : \mathbf{R} \to \mathbf{R}$ *with* $f(x) = f(-x)$ *for all* $x \in \mathbf{R}$, *and* $f(0) = 0$. *Let* $G = \{(x, f(x)) \mid x \in \mathbf{R}\}$. *If there exists an* ϵ*-approximation between* G *and* $\mathbf{R}$, *then for* $x > \frac{3}{2}\epsilon$,

$$|f(x)| \leqslant \sqrt{3\epsilon x + \frac{9}{4}\epsilon^2}.$$

PROOF. Assume R is an ϵ-approximation between G and $\mathbf{R}$. Take $x > 0$, and define $A = (0,0)$, $B = (x, f(x))$, and $C = (-x, f(x))$ (see Figure 2).

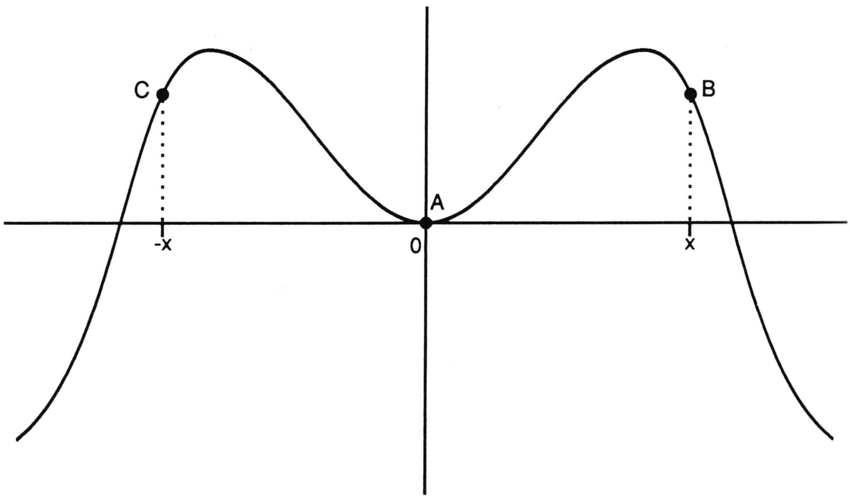
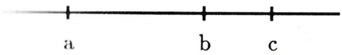
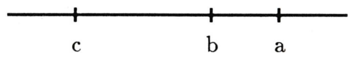


Figure 2.

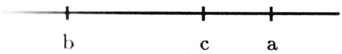
We know there exist $a, b, c \in \mathbf{R}$ with aRA, bRB, and cRC. According to the positions of $a, b, c \in \mathbf{R}$, the following cases have to be investigated:



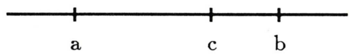
or



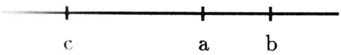
$$||C - A|| + \epsilon \geqslant |c - a| = |c - b| + |b - a| \geqslant ||C - B|| + ||B - A|| - 2\epsilon$$



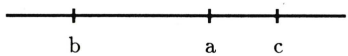
or



$$||A - B|| + \epsilon \geqslant |a - b| = |a - c| + |c - b| \geqslant ||A - C|| + ||C - B|| - 2\epsilon$$



or



$$||B - C|| + \epsilon \geqslant |b - c| = |b - a| + |a - c| \geqslant ||B - A|| + ||A - C|| - 2\epsilon.$$

We obtain the following inequalities:

$$||C - A|| + 3\epsilon \geqslant ||C - B|| + ||B - A||$$

or

$$||A - B|| + 3\epsilon \geqslant ||A - C|| + ||C - B||$$

or

$$||B - C|| + 3\epsilon \geqslant ||B - A|| + ||A - C||,$$

and further

$$\sqrt{x^2 + f(x)^2} + 3\epsilon \geqslant \sqrt{x^2 + f(x)^2} + 2x$$

or

$$\sqrt{x^2 + f(x)^2} + 3\epsilon \geqslant \sqrt{x^2 + f(x)^2} + 2x$$

or

$$2x + 3\epsilon \geqslant \sqrt{x^2 + f(x)^2} + \sqrt{x^2 + f(x)^2}.$$

Thus

$$3\epsilon \geqslant 2x$$

or

$$2x + 3\epsilon \geqslant 2\sqrt{x^2 + f(x)^2},$$

but we know $2x > 3\epsilon$ by assumption. Hence $x + \frac{3}{2}\epsilon \geqslant \sqrt{x^2 + f(x)^2}$, and so

$$|f(x)| \leqslant \sqrt{3\epsilon x + \frac{9}{4}\epsilon^2}$$

as required. $\square$

Note that $|f(x)| \leqslant \sqrt{3\epsilon x + \frac{9}{4}\epsilon^2}$ for all $x > \frac{3}{2}\epsilon$ does not imply that the canonical relation R_f, given by $xR_fA \Leftrightarrow A = (x, f(x))$, is an ϵ-approximation between G and $\mathbf{R}$.

As a counterexample, we show that R_f is not a 1-approximation in the case where $f(x) = \sqrt{|x|}.\cos x$, even though

$$\sqrt{|x|}.|\cos x| \leqslant \sqrt{|x|} \leqslant \sqrt{3\epsilon x + \frac{9}{4}\epsilon^2}$$

for $x > \frac{3}{2}$ and $\epsilon \geqslant \frac{1}{3}$. If it were, then

$$xR\,(x, \sqrt{|x|}.\cos x) \text{ and } yR\,(y, \sqrt{|y|}.\cos y),$$

so

$$\left| |x-y| - \sqrt{(x-y)^2 + (\sqrt{|x|}.\cos x - \sqrt{|y|}.\cos y)^2} \right| \leqslant 1$$

for all $x, y \in \mathbf{R}$. However, taking $x = 2\pi$ and $y = \pi$ we obtain a contradiction.

THEOREM 9. *Let $f : \mathbf{R} \to \mathbf{R}$. The following assertions are equivalent:*

i) *R_f is an ϵ-approximation.*

ii) *$|f(x) - f(y)| \leqslant \sqrt{\epsilon^2 + 2\epsilon|x-y|}$ for all $x, y \in \mathbf{R}$.*

PROOF. Assertion i) is equivalent to the statement that, for all $x, y \in \mathbf{R}$,

$$\begin{aligned}
& \left|\sqrt{(x-y)^2 + (f(x) - f(y))^2} - |x-y|\right| \leqslant \epsilon \\
\Leftrightarrow\quad & \sqrt{(x-y)^2 + (f(x) - f(y))^2} \leqslant \epsilon + |x-y| \\
\Leftrightarrow\quad & (x-y)^2 + (f(x) - f(y))^2 \leqslant \epsilon^2 + 2\epsilon|x-y| + |x-y|^2 \\
\Leftrightarrow\quad & |f(x) - f(y)| \leqslant \sqrt{\epsilon^2 + 2\epsilon|x-y|}.
\end{aligned}$$

□

THEOREM 10. *Let $f : \mathbf{R} \to \mathbf{R}$ be bounded, and set $G = \{(x, f(x)) \mid x \in \mathbf{R}\}$. Then $d_G(\mathbf{R}_d, G_{d'})$ is finite, where d and d' are the usual metrics on $\mathbf{R}$ and $\mathbf{R}^2$ respectively.*

PROOF. Assume $f : \mathbf{R} \to \mathbf{R}$ is bounded, and pick A such that $|f(x)| \leqslant A$ for all $x \in \mathbf{R}$. Then $|f(x) - f(y)| \leqslant 2A$ for all $x, y \in \mathbf{R}$. Taking $\epsilon = 2A$, we get

$$\sqrt{\epsilon^2 + 2\epsilon|x-y|} \geqslant \sqrt{\epsilon^2} = 2A \geqslant |f(x) - f(y)|,$$

and hence by Theorem 9 we know R_f is a $2A$-approximation. □

References

Paulin, F. (1988). "Topologie de Gromov équivariante, structures hyperboliques et arbres réels", *Inventiones Mathematicae* Vol. 94, pp. 53–80.

Gromov, M. (1981). *Structures Métriques pour les Variétés Riemanniennes.*

Problems Drive 1992

Vin de Silva and Oliver Riordan

THE RULES

Pairs played as teams and had to answer twelve questions in one and a quarter hours. Every five minutes they were given a new question, and the questions were removed from them after ten minutes; at the end, teams were given ten minutes to invent any more plausible answers. The scoring system was more incomprehensible than usual, irrational and infinitesimal components being obtainable. The winners traditionally receive a bottle of port and are expected to set the questions the following year.

The questions were considered harder than usual, although they appear here with a few hints removed. The quotations from *Doctor Who* episodes, of at best dubious relevance, have been left in.

CAPRICORN

Stubborn old goat!
The Web of Fear

Julius Silverstein has a circular plot of land, 20 yards in diameter. He has a goat tethered to the inside of the perimeter fence. How long is the tether if it allows the goat to graze on all but 100 square yards of the plot?

AQUARIUS

I may have had a knock on the head, but that's a dashed queer story!
The Talons of Weng-Chiang

Lady Gnomyall was playing with complex polynomials. She wrote down the following equation:

$$z^n + z + c = 0. \tag{1}$$

She then called her husband, Polly.

"I have two numbers in my head," she said; "let's call them a and c. a is an integer, and c is a complex number with $|c| = 1$."

Lady Gnomyall added that, with her chosen value for the constant c, equation (1) has one or more solutions $z = z_0$ with $|z_0| = 1$ precisely when $n \equiv a \pmod{1992}$, that is, when 1992 divides $(n - a)$.

How many possible values of c could she have chosen?

NOTE. For each positive integer m, there are $m \prod(1 - 1/p)$ integers (up to congruence modulo m) coprime to it, with the product taken over distinct prime factors p of m.

PISCES

Nothing in the world can stop me now!
The Underwater Menace

Professor Zaroff has come up with a new threat to the sanity of humanity. It is a 3D Xerox machine, or a prototype, at least.

The contraption is immersed in a special liquid. It has four emitters, which emit beams of light of ingeniously chosen wavelengths. When the machine is activated, any region of liquid which is hit by all four beams instantly and permanently solidifies. The resulting object may then be retrieved at leisure.

The four emitters are positioned at the vertices of an imaginary regular tetrahedron. They can be pointed independently in any direction (one takes care to avoid having infinite regions of intersection). All four emitters emit beams of the same shape, although there is a parameter that can be modified (for all four beams together).

The beams are shaped like regular (single) cones, *except* that they have cross-sections shaped like equilateral triangles, rather than discs. The apex of each is identical to the apex of (say) a triangular pyramid $OABC$, where $OA = OB = OC$ and $AB = BC = CA$. The parameter that can be altered is the angle $\theta = AOB = BOC = COA$; this may vary from 0° to 120°. All four beams must have the same angle.

Zaroff selects a value for the parameter θ. He points the emitters cunningly. Then he activates the machine. Moments later, he retrieves a regular polyhedron.

What possible pairs $(\theta, \text{polyhedron})$ are there?

NOTE. There are five convex regular polyhedra:
Tetrahedron: 4 equilateral triangles, meeting 3 to a vertex.
Hexahedron: refer to *Regular Complex Polytopes* by Coxeter.
Octahedron: 8 equilateral triangles, meeting 4 to a vertex.
Dodecahedron: 12 regular pentagons, meeting 3 to a vertex.
Icosahedron: 20 equilateral triangles, meeting 5 to a vertex.

ARIES

Your master will be angry if you kill me ... I'm a genius!
The Seeds of Death

Recall that, in logic, $(A \Rightarrow B)$ is true unless A is true and B is false. Recall also that $\perp$ is always false.

A Frog and a Toad play a game. They start with a row of N symbols $\perp$, separated by $(N-1)$ symbols $=$, where N is a positive integer:

$$\perp \; = \; \perp \; = \; \dots \; = \; \perp$$

Frog and Toad then take turns to convert an $=$ of their choice to a one-way implication sign ($\Leftarrow$ or $\Rightarrow$), and then bracket off the terms on each side of the new implication. Of course, brackets already placed must be respected. For example,

$$(\perp = \perp = \perp) \Rightarrow (\perp = \perp = \perp = \perp)$$

might become, among other things,

$$(\perp = \perp = \perp) \Rightarrow ((\perp = \perp) \Leftarrow (\perp = \perp))$$

or

$$((\perp = \perp) \Rightarrow \perp) \Rightarrow (\perp = \perp = \perp = \perp).$$

When all $=$ symbols have been converted, the truth value of the resulting logical sentence is computed. Toad wins if it is true, Frog wins if it is false. For each positive integer N, determine who wins on best play if

a) Toad starts;
b) Frog starts.

TAURUS

Oh, my word!
The Ice Warriors

The regular octahedron has 8 equilateral triangular faces, meeting 4 to a vertex. The regular icosahedron has 20 equilateral triangles, meeting 5 to a vertex.

a) Let ABC be a face of a regular octahedron. A small insect starts at vertex A and crawls in a straight line towards edge BC. Whenever it reaches an edge, it continues crawling in a 'straight line', in the obvious sense that the incident angle with the edge is the same as the departing angle. We assume that it never crawls onto a vertex of the octahedron.

By choosing the initial direction of its journey, the insect can exercise some control over the faces it visits and the order in which it visits them. In fact, every face can be visited by choosing a suitable direction and prolonging such a 'straight line' journey sufficiently. (The insect must always begin its journey on ABC in the manner described.)

Consider the following statement:

'If you choose a face, the insect can arrange to visit it crossing at most N edges.'

What is the least N for which the statement is true?

b) Answer the same question for the regular icosahedron.

REMARK. The answer to the corresponding question for the regular tetrahedron is $N = 2$.

GEMINI

Three of 'em! I didn't know when I was well off!
The Three Doctors

I have a cube. Suppose I wish to mark each face with one of the letters Z, O, G, such that there are precisely N ways of spelling out the word 'ZOG', tracking from square to adjacent square. For which N in $1, 2, \ldots, 10$ is it possible to do this?

CANCER

I used my special technique ... keeping my eyes open and my mouth shut.
The Tomb of the Cybermen

For the purposes of this question, it may be assumed that the surface of the planet Mondas is, as its natives believe, a copy of the Euclidean space $\mathbf{R} \times \mathbf{R}$. It is not a very good copy, though, because, owing to the presence of a large crab at the origin $(0, 0)$, the natives use a different metric (distance function) on the planet. The distance between two points $\mathbf{x}$ and $\mathbf{y}$ is given by

$$d(\mathbf{x}, \mathbf{y}) = 25\frac{|\mathbf{x} - \mathbf{y}|}{|\mathbf{x}|.|\mathbf{y}|},$$

where $|\mathbf{x}|$ is the positive square root of $(x_1^2 + x_2^2)$ as usual.

Lady C. lives at $(3, 4)$ and her lover lives at $(2\frac{1}{2}, 5)$. The lover wishes to visit Lady C., but he wants to go via the main road, which consists of all points of the form $(t, 0)$. How long is the shortest route he can take?

LEO

That's the trouble with good ideas; they only come a bit at a time.
Revenge of the Cybermen

With a_0 and n fixed, maximise

$$\frac{1}{[a_0, a_1]} + \frac{1}{[a_1, a_2]} + \ldots + \frac{1}{[a_{n-1}, a_n]},$$

where $a_1, a_2, \ldots, a_n$ is any set of positive integers satisfying $a_0 < a_1 < a_2 < \ldots < a_n$, and where $[p, q]$ denotes the least common multiple of the integers p and q,

a) when $a_0 = 1$ and $n = N$, for each positive integer N, and
b) when $a_0 = 5$ and $n = N$, for each positive integer N.

VIRGO

Leonardo! ... You remember the Mona Lisa:
that dreadful woman with no eyebrows who wouldn't sit still!
City of Death

$ABCDE$ is a convex pentagon.

$AB^2 = 16$; $BC^2 = 5$; $CD^2 = 4$; $DE^2 = 2$; $EA^2 = 41$.

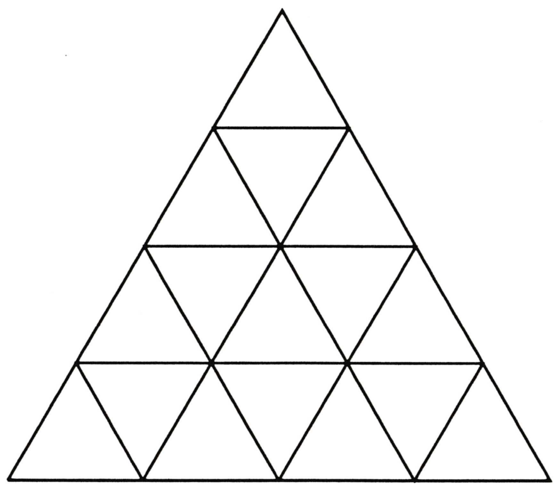
$AC^2 = 29$; $CE^2 = 10$.

Calculate the area of $ABCDE$.

LIBRA

You're just a pathetic bunch of tin soldiers
skulking about the galaxy in an ancient space-ship!
Revenge of the Cybermen

Little Tom, the baker's son, is waging a military campaign. The battlefield is a triangular grid of size N (a positive integer). An example grid, with $N = 4$, is shown below.



Soldiers (of the stannic variety) stand at the vertices of the grid. They are so arranged that each grid parallelogram is 'balanced'; that is to say, each pair of opposite vertices of the parallelogram has the same total number of soldiers on them as the other.

Of course, little Tom has fractional soldiers where required—the inevitable consequence of long battles past.

Suppose the three main vertices hold A, B, and C soldiers in some order. Little Tom would like to know how many soldiers there are in total. What is the answer (in terms of A, B, C, and N)?

SCORPIO

Have you heard of the Tong of the Black Scorpion?
The Talons of Weng-Chiang

What is the area of the largest square that can be covered by three unit squares (which are allowed to overlap but may not be cut or deformed in any way)?

SAGITTARIUS

Doctor—be careful!
The Evil of the Daleks

Fill in the gaps in the following sequences:

a) 10, 20, 120, 440, ..., ...
b) 1, 3, 5, 7, 9, 15, 17, 21, 27, ..., ...
c) 3, 4, 5, 6, 8, ..., 12, 15, 16, ...
d) 0, 1, 2, 4, 5, 8, 9, ..., ...

A *small* bonus will be awarded for a correct non-obvious solution to the following:

e) 4, ..., 4, ..., 4, 4, 4, 4, 4, 4.

The bottle of port was won this year by Timothy Luffingham (Trinity, now Warwick) and Tom Leinster (New College, Oxford), who scored

$$\frac{57}{10}+\frac{1}{16}(2\sqrt{2}-1)(\sqrt{5}+1)$$

out of a maximum possible score of $12+\epsilon$.

Solutions to the problems are on pp. 55–56.

Let's Draw the Integers

The Faculty

Here is a taste of the lecturing style found in the Cambridge University undergraduate Mathematical Tripos, mostly from the year 1991–92. Punctuation may mislead.

Fluids

"If you take a bag of water and squeeze it, the bag bursts."
"Another way of getting rid of terms you don't like is setting them to zero."

Algebra

"I don't think that this is the time to encourage Australian sympathies by having tildes waltzing around our mathematics."
"Every interesting corollary leads to a good definition."
"Now, another Lemma 10."

Probability

"I'm going to make the convention that whatever is obviously true *is* true."
"Arrange n married couples around a table; place the wives in odd positions."
"In general, this is true. It's only in certain cases that it isn't."
"This is the case $n = 2$. It also works when $n = n$."

Galois Theory

"I was thinking about this in my bath the other day and it was obvious."
"I've just added another element to F_{16}."
"I think this probably proves something, but I can't think quite what."

Number Theory

"This work was actually paid for by somebody—not very much, but then I don't think it's very valuable."
"I'll just move the origin, which is neither here nor there."
"This course is intended primarily for those about to give up Mathematics."

Numerical Analysis

"One year is very short compared with 25 000 years."
"Do unto U as you do unto L."
"Knowing that an iteration will converge 55 years after the sun becomes a white dwarf will be of very little consolation to someone trying to predict tomorrow's weather."

Complex Variable

"This is morally equivalent and probably more equivalent than that."

Principles of Dynamics

"If we lived in a world where circular orbits were not stable, spiralling into the sun would be a lot more disastrous than the Greenhouse Effect."
"Here are the wheels, which we'll ignore in the analysis."

Markov Chains

"The geese are eaten by the foxes—or rabbits."
"Here, $16 = (30 \times 48)/90$."

Algebraic Geometry

"Theoretical physicists need to know some algebraic geometry—or at least some algebraic geometers."

"Zorn's first name is Max, which is just what it ought to be."

Logic and Set Theory

"The language has an infinite supply of variables, which will be written as x and y."

"You probably all know about Russell's paradox, and if you don't I'll be reminding you of it in a few moments' time."

Mathematical Methods

"If I sit on that piece of chalk, like a spaceman, the components will be fixed."

"We know this is a yellow closed curve ... "

"Anybody with sufficient symmetry will be spherical."

"This equation is number '23' in what I hope is an obvious notation."

Rings

"I *have* done it correctly—I've done it twice."

"So the result follows. Well, the result follows by the following lemma."

Analysis

"... because Cauchy doesn't know where you are."

"We've got so many conditions on the board, it seems we could end up with almost anything."

"You've got to show that if you take a y in there, say z, ... "

Graph Theory

"There are millions of ways of constructing spanning trees, but one is easier than the other."

Statistics

"So let's draw the integers."

Riemann Surfaces

"Right, we now come to the second misprint."

"This is the statement we're trying to prove ... and here's one I made earlier."

Differential Analysis and Geometry

"Do you all know what c_n is? Ah—I'm not sure *I* do. It doesn't really matter what c_n is, so long as it's something."

"Quite a mess, this, isn't it?"

"There's really no substitute for preparing lectures."

Stochastic Processes

"This relies on a property of the exponential distribution called the 'lack of memory', which you've probably all forgotten."

Electromagnetism

"Imagine you're an electron lying on your back ... "

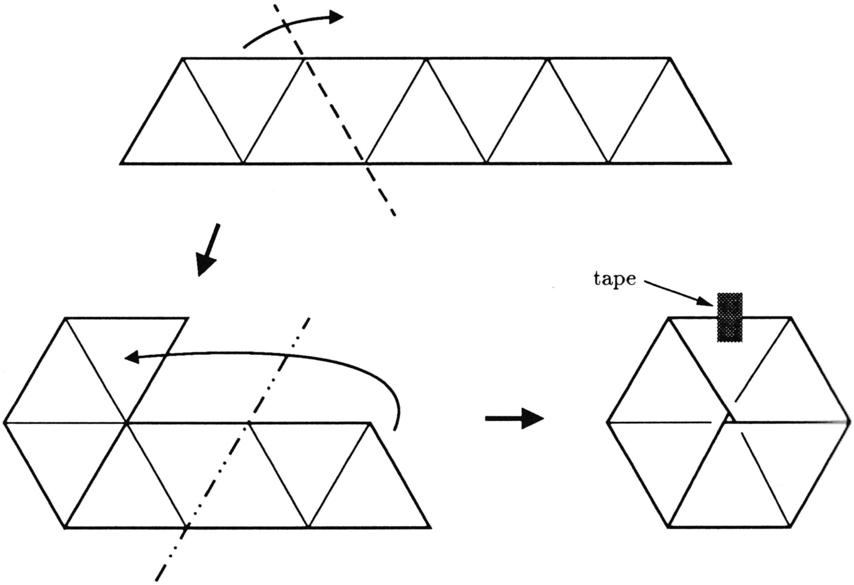
"The only reason we stop this is that we don't know what to call the next one in the sequence."

Flexagons

Vin de Silva

Hexaflexagons

What can you do with a strip of paper? Arthur Stone, a postgraduate at Princeton in 1939, folded it into a *flexagon.*

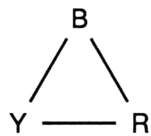


We use the standard origami notation:

fold in front: — — — — — — — — — — — —
fold behind: — · · — · · — · · — · · — · · — · ·

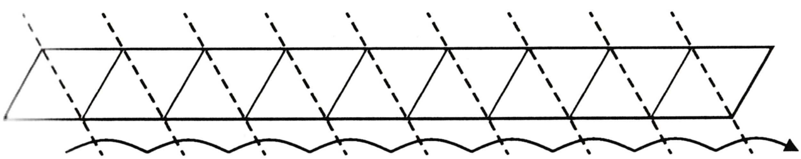
This is flat and hexagonal and has two sides, which we paint red and blue. By flexing it along its creases, an uncoloured face can be found, and we promptly colour it yellow.

Clearly this is a mathematical artefact of considerable interest, and so we invent a notation for it.

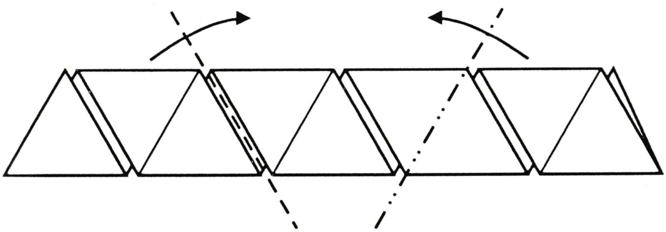


Each edge represents a *state* of the flexagon, and each vertex represents a colour. There is an edge for the initial Red-Blue state, and two for the other states, which are obtained cyclically. Stone thought about it overnight and realized that a more complicated flexagon could be made. The hexagon has six-fold rotational symmetry, whereas a single flex has only three-fold symmetry. There is room for another flex.

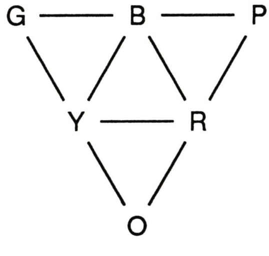
This is Stone's construction for the ***hexahexaflexagon***, or hexahexa (the one shown above, with three faces, is called the ***trihexaflexagon***). We start with a strip twice as long as before:



'Rolling' the strip as shown, we obtain a smaller strip which we can then fold as a normal trihexaflexagon.

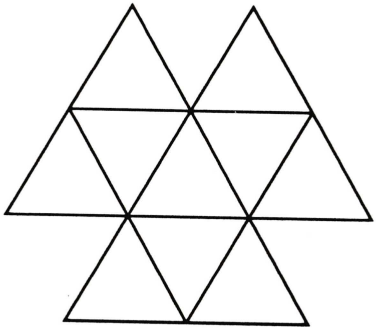


The result is another flexagon, a little fatter than the first. It has three principal faces, Red, Blue, and Yellow, which can be permuted cyclically by flexing. There are also three secondary faces. When in the Red-Blue state, it is possible to produce a Purple face with a single well-chosen flex. There are, similarly, Green and Orange faces, which can be obtained from the other principal states. We symbolise the relationships as follows:



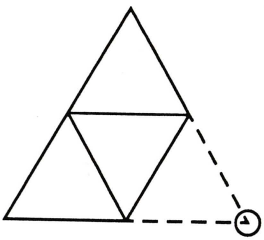
In any of the outer positions, there is no choice as to which face is produced next; for instance, if we are in the Purple-Red state, Blue will appear after a flex (though either Purple or Red can be chosen to disappear to make way for it).

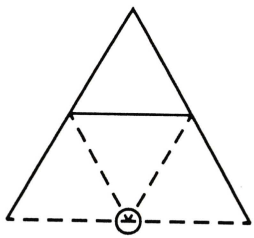
We can make an even bigger flexagon from a strip twice as long again. Using the 'rolling' procedure, reduce it in length to a hexahexa strip, and finally fold it as a hexahexa. This has twelve faces and the following diagram:



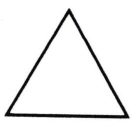
The obvious generalisation of this procedure will produce flexagons with 3, 6, 12, 24, 48, ... faces. We may also wish to consider the 'free' flexagon with $3 \times 2^\infty$ faces. For the larger flexagons, the state diagrams overlap themselves, and hence are undrawable. We could get round this problem by distorting the triangles to make them fit, but there is no great need. Besides, a generalisation described later fits more naturally if the triangles are left undistorted.

How can we make a flexagon with five faces? We could start with a hexahexa and remove one face. Suppose we play with a hexahexa. When our chosen disgraced face (circled below) shows up, we sabotage it by applying a drop of glue to one of its facelets. The face will never be seen again once it flexes away, and the result is the *pentahexaflexagon* on the left.





It's important to sabotage an outer face, otherwise you end up with the right-hand state diagram. Here, the surviving states split into three groups that don't communicate with each other. Depending on the state you reach when you flex the errant away, the result will be a flexagon that behaves like one of these (the last two are degenerate and don't flex):



or

or

DEFINITION. A *proof* is a plausible argument of some kind.

DEFINITION. A *triangle diagram*, or TD, is defined recursively as follows:

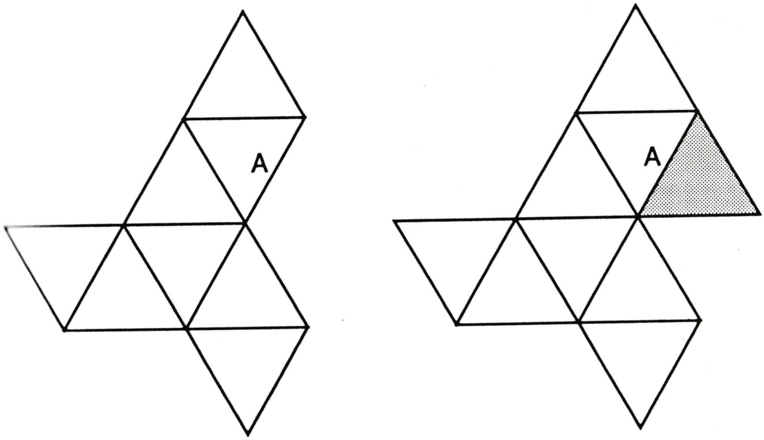
i) A single triangle is a TD.

(ii) If you attach a new triangle to a TD at one of its boundary edges, then the result is also a TD.

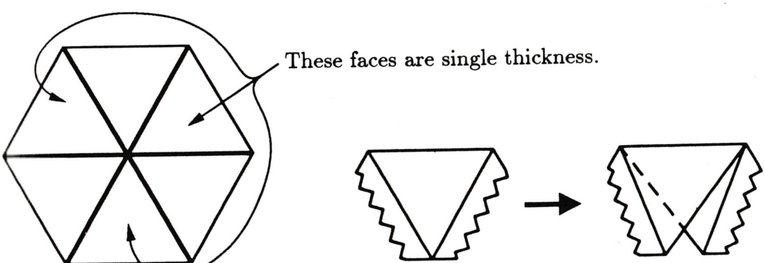
We now prove that triangle diagrams correspond precisely to non-degenerate hexa-flexagon forms.

The easy bit is to show that every TD is the state diagram for some flexagon. The proof of this involves sabotage. Take a 3×2^n-type flexagon whose state diagram includes our chosen TD as a subdiagram. Sabotage appropriate faces to reduce the diagram to the chosen TD. That's all.

There is also a more constructive proof. We know we can produce a flexagon whose state diagram is a single triangle (the trihexa). I shall explain how to modify a flexagon so that its state diagram acquires an extra triangle at any given edge. The result then follows by induction.



Suppose we would like to add the shaded triangle to the state diagram above. Flex the flexagon to the edge marked A. In this state, the six sectors of the flexagon come in two sets of identical triplets, by symmetry. It turns out that, in an 'outer' state, one of the sets of triplets consists of sectors that are a single thickness of paper. (Proof by induction, working alongside this construction.) We doctor each of these, replacing the single triangle by a pair of hinged triangles as shown. It's as if we were reversing the gluing process of the sabotage method, splitting single triangles as if they were really two stuck together.



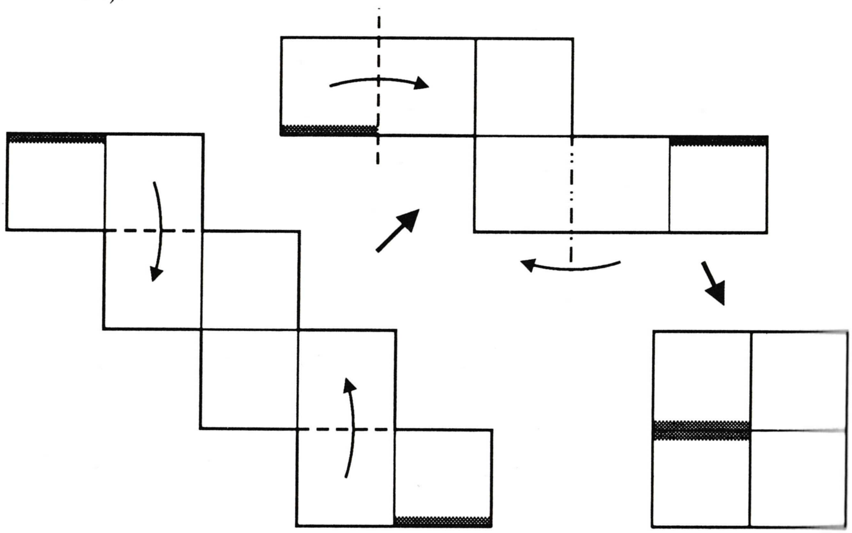
You have to be careful about which of the added triangles connects with the left and which connects with the right, otherwise it all goes horribly wrong. In fact, it's enough simply to be consistent and always have the top triangle of each new pair attached to the left.

Conversely, every hexaflexagon has a state diagram that is a triangle diagram. This follows after a bit of hand-waving and practical experimentation. There are two main points. Locally, the structure is made up of little three-cycles. The existence of a flex actually guarantees a three-cycle. Secondly, there are no large-scale interactions of states. There is essentially only one way to reach a given state from any other. This is because when we exit from a three-cycle its mechanism becomes folded away in the recesses of an adjacent three-cycle, and can only be re-accessed by returning to that and flexing to the correct state.

This completes our classification of hexaflexagons. □

Tetraflexagons

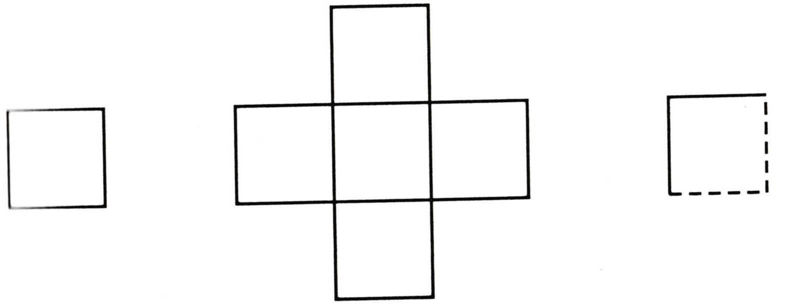
The basic *tetraflexagon* is folded as shown (the two shaded edges are connected together at the end):



Flexing is achieved by folding in half and unfolding in the 'wrong' direction. There are four faces which are cyclically permuted by flexing. Accordingly, we give the flexagon the state diagram in (a).

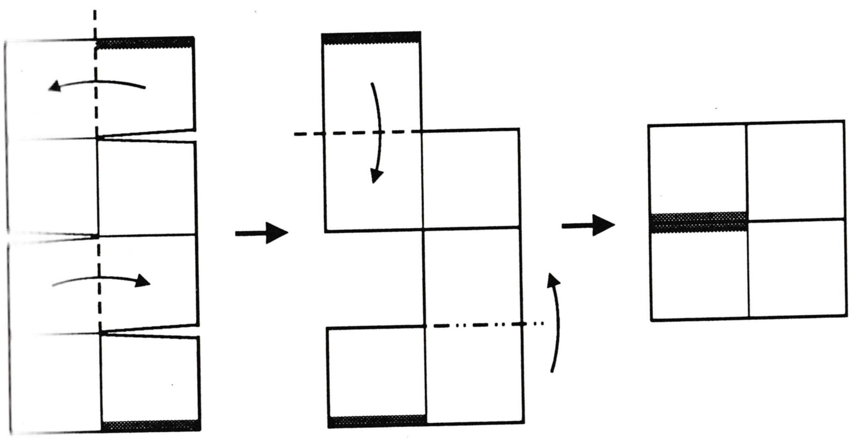
Analogously to the triangle diagrams for hexaflexagons, we can generate a large family of tetraflexagons based on *square diagrams*. Either existence proof may be reused here: for the 'sabotage' proof, we may note that a zigzag chain of $8 \times 3^{n+1}$ squares may be folded into a zigzag chain of 8×3^n squares by a process akin to the 'rolling' used to shorten the strips of triangles. If we start with a 24-square chain, roll it, and then fold as above, the resulting 12-face tetraflexagon will have state diagram (b). The state diagram for the next case, the 72-square strip, is all but undrawable, because of overlaps.

Squares in the state diagram may be partially sabotaged. The simplest example of this, obtained from the basic tetraflexagon, is the *tritetraflexagon*. This has three faces and two states, and is represented by (c).

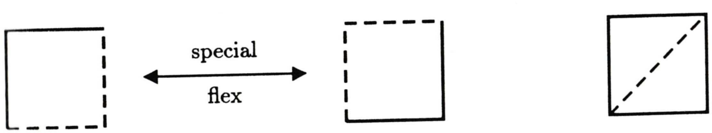


(a) (b) (c)

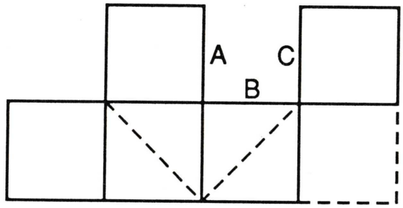
There is a second mode of flexing that may be exploited. The simplest examples of tetraflexagons which allow this have four faces; one possibility is:



This flexagon is effectively a combination of two flexagons which communicate through a 'special flex'—we introduce the notation on the right to represent its state diagram. The exact nature of the special flex is left as an exercise.



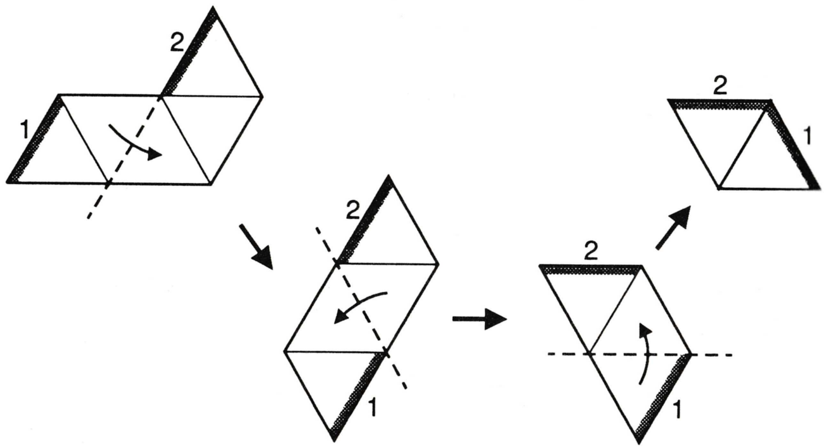
It is easy to incorporate this new flex into our scheme. Overleaf is an example of a possible state diagram for a tetraflexagon—note that, for instance, the states A, B, and C do not communicate directly with each other.



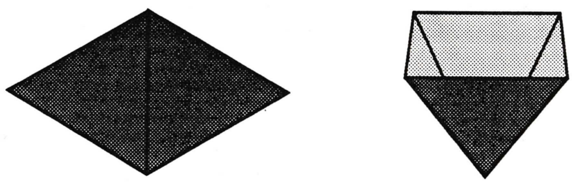
After a bit of case-checking, it turns out that tetraflexagons correspond exactly to such state diagrams (modulo a small fudge factor or two). Thus we have a classification for tetraflexagons. □

Flexaflops

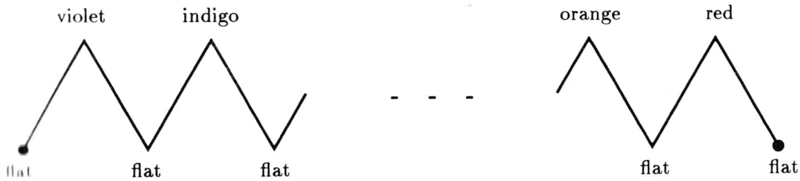
We now simplify things and consider ***flexaflops***, which are the least (interesting flexagons) or the (least interesting) flexagons, according to taste. Here is an example—fold two copies of the net shown and join them together, edge 1 of the first copy to edge 2 of the second, and vice-versa:



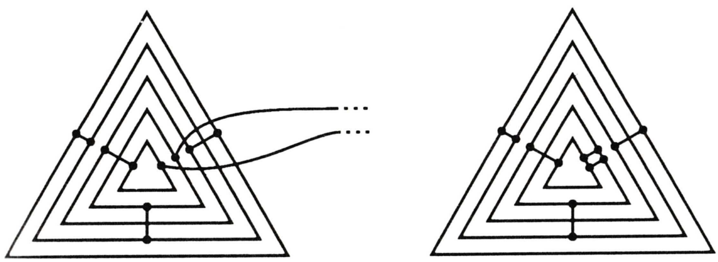
The outside of the flexaflop never changes colour. Everything happens on the inside, which may be viewed by opening the flexaflop out into a hollow pyramidal shape.



The transition from flattened form to pyramid potentially occurs in two ways; likewise, there are two ways of flattening the pyramid shape. This degree of choice means that we may be able to ***flop*** the flexaflop along a chain of distinct states, with different sets of faces exposed in the pyramid states. In this particular version, there are four such states, so we may call it the ***tetratetraflop***. We can construct flexaflops with arbitrarily long state diagrams, but these are always chains with two ends.



We now consider the mechanical structure of the tetratetraflop.



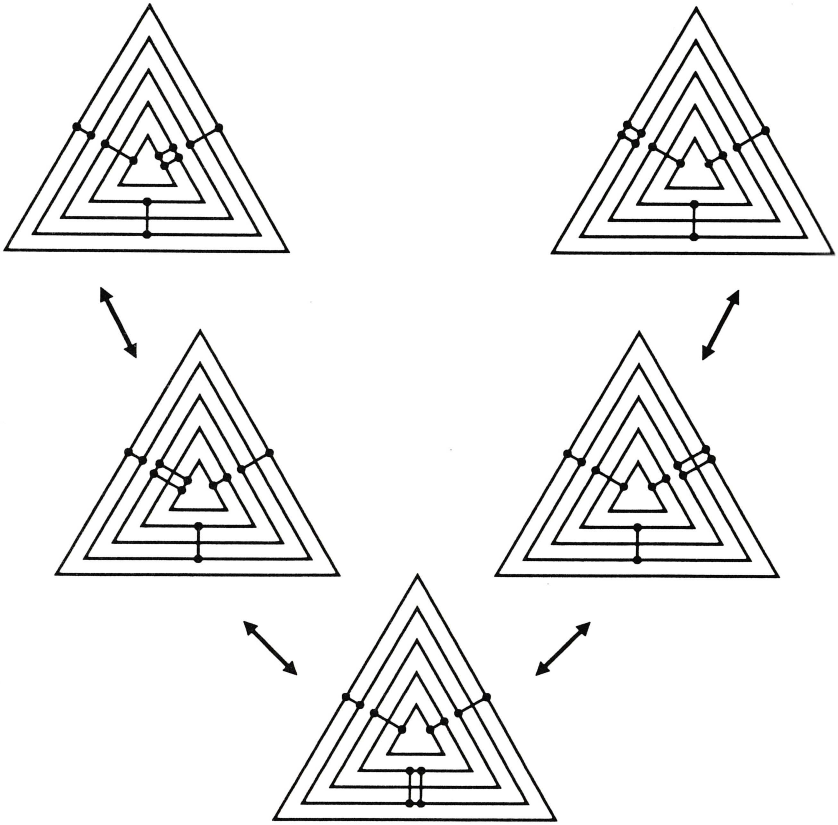
(d) (e)

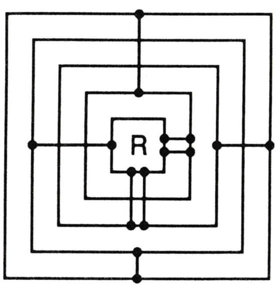
One half of the tetratetraflop is depicted schematically in (d). Triangles nearer to us are drawn smaller, and a line connecting two edges indicates that those two edges are joined. There are two connections to the other half of the flexaflop (edge 1 and edge 2 in the construction diagram opposite). Since edge 1 of one half meets edge 2 of the other, there is a sense in which the two free ends of (d) may be connected together; this is done in (e), the line being doubled to indicate that the faces which are joined by it are on different halves.

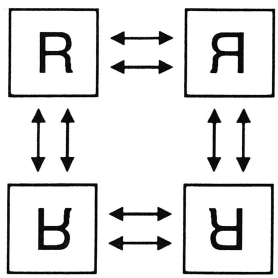
The payoff comes when we try flopping the flexaflop to see how its structure diagram changes (see over).

The pattern of lines on the diagram remains unchanged, except that the doubled line is a different one with each flop. Experimenting with a model will make the reasons for this clear. During a flop, each half splits into two sections which then regroup as the flop is completed. The regrouped halves are identical to the halves at the beginning of the procedure, except that the two pairs of edge connections directly involved in the flop exchange status, one becoming doubled and the other undoubled.

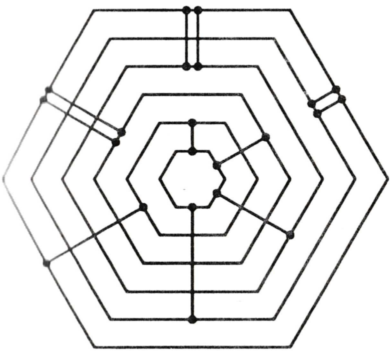
There are flexaflops whose structure diagrams have more than one multiple line. Shown overleaf is the structure for the ***tetraoctaflop***, from which the reader may infer how to construct a working model. This time four identical copies of the basic unit need to be made, and they should be oriented in the manner suggested. Viewing the internal colours involves opening the octaflop out into a square tube. Again, only a finite chain is possible—loops do not occur.

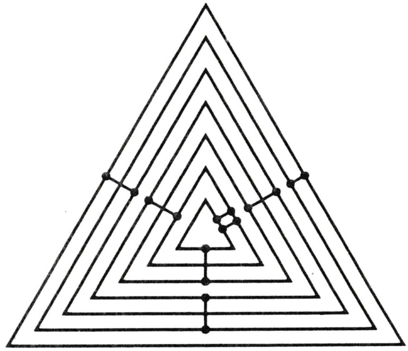






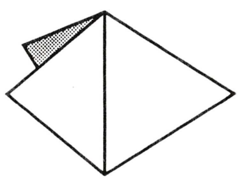
There is a generalisation to n dimensions. Working with n-dimensional hypercubes as the basic cell, we can build a flexaflop which operates in $(n+1)$ dimensions, with a structure diagram that involves n doubled connections. A chain of faces (of arbitrary length) can be made to appear on the interior. Here is an example structure diagram in the case $n = 3$; the six sides of the hexagon correspond to the six faces of the cubes that are the basic units. Finally, another tetraflop is shown on the right, hopefully to make it clear how chains of arbitrary length are obtained.

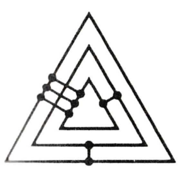




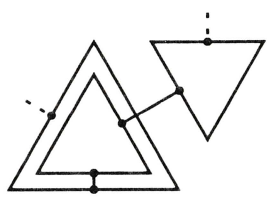
Generalisations

The structure diagrams of the previous section apply equally well to flexagons in general. Here is a possible structure diagram (f) for the trihexaflexagon:

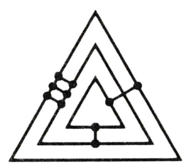




(f)



(g)



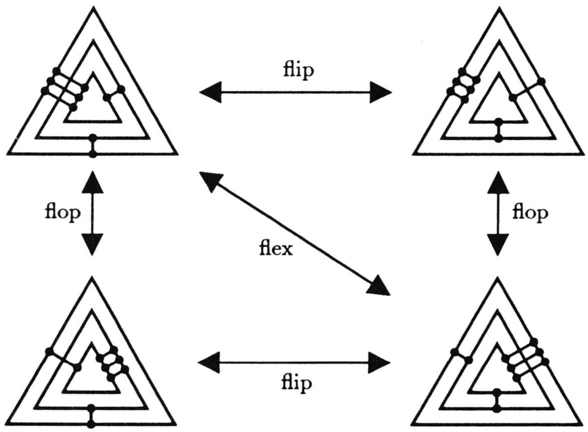
(h)

Three copies of this unit are needed to construct the whole flexagon. The tripled connection is made three times, in cyclic fashion, between the three parts. The flexagon state represented by this diagram is a 'half-way' stage in the middle of a flex; only one colour is visible (compare the top diagram). The significant difference between this and the flexaflop which has the same structure diagram (except of course for the multiplicity of the distinguished edge) is that there are new ways of transforming the diagram. It is the introduction of the third section which gives it the extra freedom. When a regular tetraflexaflop is opened, the total angle at the centre is $2 \times 2 \times 60° = 240°$, which is not enough for it to be possible to flatten it. With three sections, the angle is $3 \times 2 \times 60° = 360°$, which is just right. In terms of these 'half-way' structure diagrams, the extra move allowed—the ***flip***—consists of reversing one of the edge connections and moving the front triangle (or triangles) to the back. This mechanism is illustrated in the sequence (f) → (g) → (h). The multiple connection remains on the same side after the flip.

The precise conditions for when a flip or a flop can be made are easily stated. Define k to be the number of sections connected together at the multiple edge (so $k = 3$ for hexaflexagons, $k = 2$ for tetraflops), and θ to be the angle between the edges on which the two connections involved, one single and one multiple, are found. (Since hexaflexagons are made out of equilateral triangles, $\theta = 60°$ in all cases.) Then:

i) There must be a pair of adjacent triangles such that the gap between them is spanned by the multiple line and one other line, and no others.
ii) If $k\theta \leqslant 180°$ then a flop is possible, by making the multiple line single, and vice versa.
iii) If $k\theta = 180°$ then a flip is possible—take all the triangles in front of the gap and move them to the back of the pile.
iv) If $k\theta > 180°$ then nothing is possible—'the mechanism jams'. (This is not always strictly true, but nothing much is lost in assuming it.)

A flip and a flop combine to give a flex, three of which return the flexagon to its initial state. The two operations may be performed in either order:

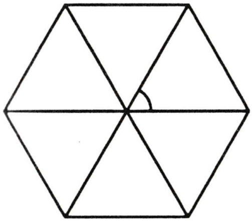


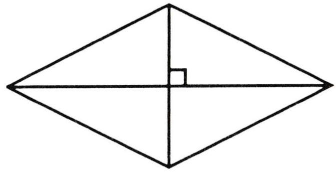
The generalised hexaflexagon

An elegant classification can be obtained for those flexagons whose facelets are triangular. Apart from the detailed structure, the parameters we can alter are k, the number of identical sections used, and θ, ϕ, and χ, the angles of the triangle.

In the regular hexaflexagon, $k\theta = k\phi = k\chi = 180°$, which allows the maximum freedom of flexing; the other cases we will consider will be subject to certain restrictions in manipulation. We have seen how these restrictions apply to the 'half-way' structure diagrams. However, what we really want are generalised rules for the state diagrams discussed in the first two sections.

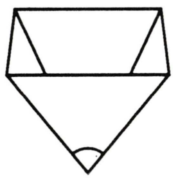
Recall that a *state* is a flattened configuration of the flexagon, with one face on each side. The angle at the centre, θ, occurs $2k$ times, so $k\theta = 180°$. Here are two possible examples of states:

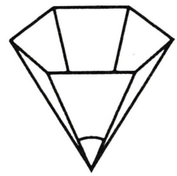




States are represented by solid edges in the state diagram.

We now introduce the notion of a ***semi-state***. This occurs when the flexagon is partially opened out but cannot be flattened further. All flexaflop states are semi-states. $k\theta < 180°$, and we denote it by a broken edge, or, rather, one half of a broken edge. Here are two examples:

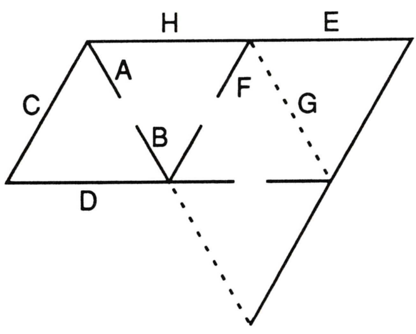




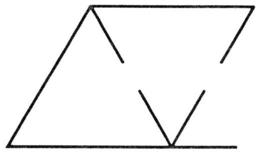
The colour on the outside is the colour of the vertex belonging to the half-edge. Crossing from one half of a broken edge to the other corresponds to turning the semi-state inside out. This is impossible to do directly if the flexagon is made of rigid material, though it might be possible to flex from one semi-state to the other.

Finally, there are ***null states*** which can't be reached because $k\theta > 180°$, but which would be there otherwise. These are drawn dotted.

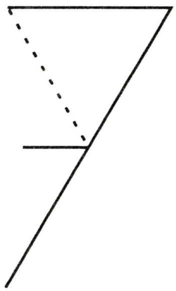
Here is an example of something that might be a state diagram:



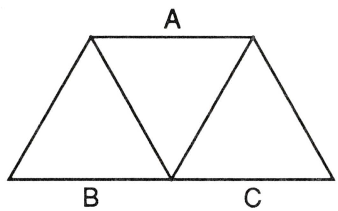
The rule for flexing is simple. From any state (or semi-state) we are permitted to go to an immediately adjacent state (or semi-state) sharing a common vertex. For example, C and H can be reached from A, A and F can be reached from H, and only H can be reached from F. G is a null state. In this case it turns out that B can be reached from A indirectly, going via C and D. We find that the flexagon breaks into two non-communicating bits:



and



Actually, there is no flexagon with a state diagram like the one shown above. Inspection of working flexagons gives us a rule for the occurence of states, semi-states and null states. Recall that state diagrams can always be projected onto a grid of equilateral triangles (though structurally it is somewhat misleading to do so), with overlaps if necessary. The rule is that parallel states belong to the same category. The following diagram, together with a corresponding hexaflexagon, explains why this is so.



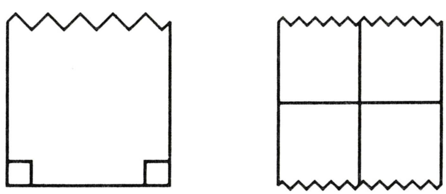
Suppose this flexagon is in state A. Notionally label 'θ' the angles of the component triangles that lie at the centre of the flexagon. Now, on flexing from A to B (or A to C), it turns out that the θs are all at the centre once again.

Thus, once we've decided what category of state each of the three sets of parallel lines will have, we will know what structure diagrams are possible. Specifically, any triangle diagram can occur, with its edges solid, broken, or dotted, provided all parallel edges are the same.

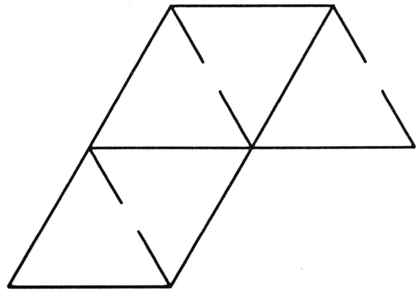
If we write S for solid, B for broken, D for dotted, then there are ten possibilities: (S,S,S), (S,S,B), (S,B,B), (S,S,D), (S,B,D), (S,D,D), (B,B,B), (B,B,D), (B,D,D), and (D,D,D). We will now obtain geometric realisations by calculating the corresponding possible values for k, θ, ϕ, and χ.

(S,S,S): $k\theta = k\phi = k\chi$, so $\theta = \phi = \chi = 60°$ and $k = 3$, and we have the regular hexaflexagon.

(S,S,B): $k\theta = k\phi = 180°$; $k\chi < 180°$. The only solution is $k = 2$, $\theta = \phi = 90°$, $\chi = 0°$. Surprisingly, this can be realised, in the form of truncated infinite triangles.

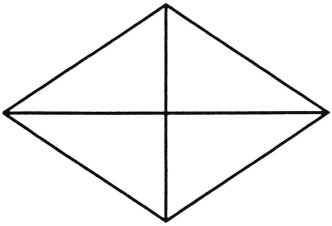


An example state diagram is

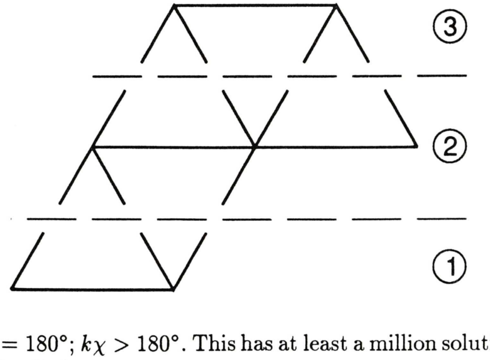


which looks suspiciously similar to certain tetraflexagon diagrams ... These are in fact precisely tetraflexagons where one set of edges is disallowed from having any hinges (e.g., by making all those edges jagged).

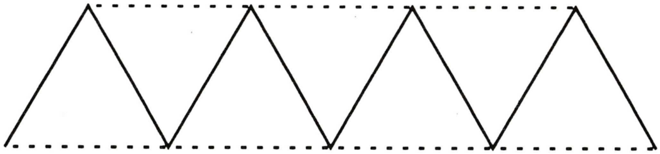
(S,B,B): $k\theta = 180°$; $k\phi, k\chi < 180°$. Solution $\theta = 90°$, with $\phi, \chi < 90°$ and $k = 2$. This is a (really quite interesting) class of flexagons with right-angled facelets.

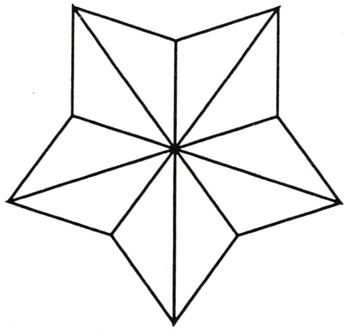


The following might be a state diagram, but it breaks up into three bits.



(S,S,D): $k\theta = k\phi = 180°$; $k\chi > 180°$. This has at least a million solutions; the flexagons they generate aren't all that exciting.

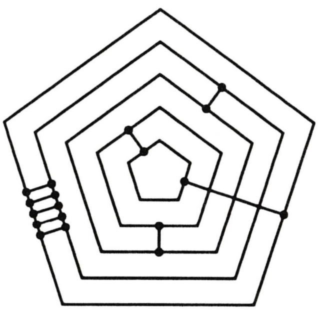


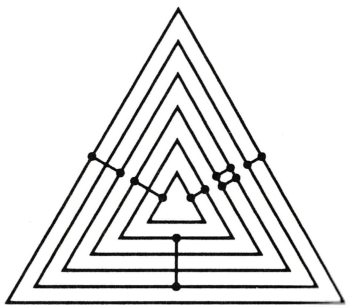


(B,B,B): $k\theta, k\phi, k\chi < 180°$, so $\theta, \phi, \chi < 90°$ and $k = 2$. This is the humble tetraflop.

Those were the five interesting cases; the other five are left as exercises.

The way forward?





Perhaps not.

An Infinite Grammatical Solecism

John Beasley and Mark Wainwright

In this paper we prove

THE TERMINATING PREPOSITION THEOREM. *For all $n \in \mathbb{Z}^+$, there exists an English sentence ending with exactly n successive prepositions.*

Consider the old grammarians' rule, "You should not use prepositions to end sentences with." From this, we can deduce that "You should not use prepositions to end sentences about using prepositions to end sentences with with", and in fact it is clear that this process can be repeated indefinitely. Specifically, consider the plural noun clause

(C_1) sentences

and define

(C_{n+1}) sentences about using prepositions to end (C_n) with .

Then (C_n) is a plural noun clause for all n, by induction, and

(S_n) You should not use prepositions to end (C_n) with.

is a valid English sentence, ending with n successive occurrences of the word "with". We have thus proved the theorem. $\square$

Crux Numerorum

Ampenditus

Satisfy the equations below by filling each blank with a number no greater than C.

I	II	III	IV	V
	VI			
VII	VIII		IX	
	X		XI	
XII		XIII		

I a = VI a + XII a

I d = X a × V d

III d = VII a + VIII d

IV a = IX a × XI d

Complex Bases

Adam Chalcraft

Suppose you have one of those old-fashioned scales with two pans. By putting objects into the pans, you can tell which one is heavier, or whether they are the same weight.

Now suppose that you want to be able to use these scales to weigh out any whole number of ounces up to, say, 40. What is the smallest number of weights you can get away with, and what values should they have? If you have not seen this question before, you might like to take some time to figure out the answer before reading on.

The trick is that you can put some of the weights into the same pan as the object you are trying to weigh, so each weight can act as positive (the other pan), negative (the same pan), or zero (not used). Because there are three choices, we are in 'base three', so we can get away with just four weights, of values 1, 3, 9, and 27. You might like to check this.

The difference between this and the normal base three is that the set of 'digits' is $\{-1, 0, 1\}$ instead of $\{0, 1, 2\}$. This is a perfectly good base, and you can do all the usual things with it, such as long multiplication. Indeed, it is better in some ways, because you can express all the integers with it, not just all the non-negative ones. I will refer to this as Example 1.

EXAMPLE 2. Instead of changing the digit-set, we could see what happens if we change the power. A useful example is power -2, with digit-set $\{0, 1\}$. The 'places' are now worth 1, -2, 4, -8, and so on. The integers are written

...	-3	-2	-1	0	1	2	3	4	5	6	...
	1101	10	11	0	1	110	111	100	101	11010	

The representations are slightly longer, but also include negative integers. Arithmetic is only slightly harder than in normal base 2. In addition, for example, $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, and $1 + 1 = 110$, so you put down 0, and carry 1 into both the next column and the one after that.

EXAMPLE 3. As one last example before the theory, take the power to be $(-1 + i)$. The digit-set is still $\{0, 1\}$. This can represent all the Gaussian integers (complex numbers where both the real and imaginary parts are integers). You might like to try writing -1 in this base before reading on.

Theory

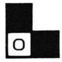
From now on, we will only be considering complex bases, like Example 3.

DEFINITIONS. A *base* $b = (p, D)$ has two parts: a *power* p, and a *digit-set* D. The power p is a Gaussian integer. The digit-set D is a finite set of Gaussian integers. D must contain 0, because all finite representations have an implicit infinite string of zeros off to the left.

A base is said to be *injective* iff no Gaussian integer can be written in more than one way in the base, *surjective* iff every Gaussian integer can be written in it, and *bijective* iff it is both injective and surjective.

Although it was not mentioned at the time, all of the examples above are injective, but only Example 3 is surjective. An interesting problem is that of characterising injective bases.

A useful way of imagining a base is to picture the digit-set as a shape composed of squares in $\mathbf{C}$, one of which is marked as 0. For example, the digit-set $\{0, 1, i\}$ could be written as follows:



For a given base (p, D), let S_n be the set of Gaussian integers which can be written in n (or fewer) digits—so with $D = \{0, 1, i\}$ the set above is just S_1. There are two ways of getting from S_n to S_{n+1}: you can either add the new digit on the left end or on the right end.

Adding the new digit on the left end corresponds graphically to taking $|D|$ copies of S_n and sticking them together. We can write this as

$$S_{n+1} = S_n + p^n D, \tag{1}$$

where the notation $S_n + p^n D$ means $\{s + p^n d \,|\, s \in S_n \text{ and } d \in D\}$. This form is, incidentally, very useful for drawing the set on a computer, which can 'copy' the entire screen and 'paste' it back offset by $p^n d$.

Adding the new digit on the right end corresponds to 'exploding' the set S_n by multiplying each of its members by p, and then putting a copy of D on each point (with 0 on the point itself). More precisely,

$$S_{n+1} = pS_n + D. \tag{2}$$

These visual forms are helpful in understanding injection and surjection. The base is injective when the sets being unioned in (1) and (2) are always disjoint, and the base is surjective when any given disc centred at 0 is eventually covered by S_n for some n.

Equation (2) also shows that if $b = (p, D)$ is a bijective base, then the graphical form of D tiles the plane. Not only that, but it tiles the plane using only translations by multiples of p.

More precisely, let the set of Gaussian integers be $\mathbf{Z}[i]$, and let $\langle p \rangle = p\,\mathbf{Z}[i]$. If the base is surjective, then

$$\mathbf{Z}[i] = \langle p \rangle + D. \tag{3}$$

If the base is bijective, then these copies of D are disjoint. If these copies of D are disjoint, then the base is injective. If the copies of D are disjoint, and D and p satisfy (3), then we say that *D tiles at size p*.

An interesting corollary of this is that if $b = (p, D)$ is bijective, then $|D| = |p|^2$. By Pythagoras, we know that $|D|$ must be the sum of two squares, so $|D|$ is one of $\{2, 4, 5, 8, 9, 10, \ldots\}$. (By the way, a number is the sum of two squares if and only if every prime factor which is congruent to 3 (mod 4) appears an even number of times.)

A useful way of saying that D tiles at size p is to say that D contains precisely one number from each equivalence class of $\mathbf{Z}[i]$ (mod p). In other words, $|D| = |p|^2$, and no two members of D differ by a member of $\langle p \rangle$.

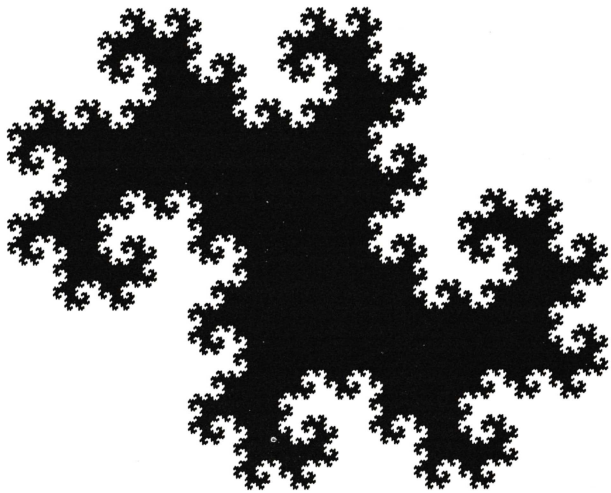


Figure 1.

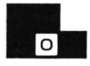
The Pretty Bit

Returning to the graphical forms, there is a modification we can make to the picture which is conceptually better for some purposes. Instead of looking at S_n, we look at $E_n = p^{-n} S_n$. In other words, after increasing S_n to S_{n+1}, scale the entire picture down by a factor of p. The recursive formulae for this are

$$\begin{aligned} E_{n+1} &= E_n + p^{-(n+1)} D \\ \text{and } E_{n+1} &= p^{-1}(E_n + D) . \end{aligned} \tag{4}$$

If we watch this as n tends to infinity, the picture remains bounded, and eventually tends to a fractal E_∞, with the boundary crinkling more as n increases.

In Example 3, the set E_∞ is the fractal sometimes called the 'double dragon'. This is shown in Figure 1. A rather uglier example is given in Figure 2, which is generated from the power $(2i - 1)$ and the following digit-set:



Since S_n tiles at size p^n, E_n tiles at size 1 for all n, so E_∞ tiles at size 1. The notation is meant to suggest that these fractals could be named *Escher sets.*

The second formula in (4) above is what is sometimes called an 'Iterated Function System'. It defines the set E_∞ by the property that $E_\infty = p^{-1}(E_\infty + D)$, and implies

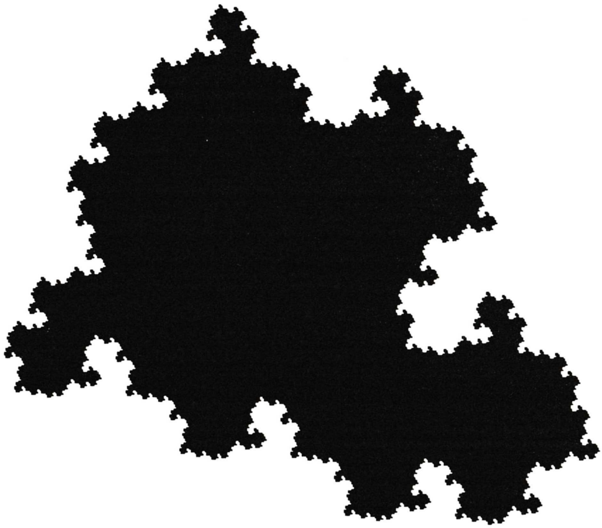


Figure 2.

that the 'Hausdorff' dimension of E_∞ is 2. It also gives a rather elegant algorithm for plotting E_∞ by computer:

```
z:=0
Repeat
    Plot z
    Let d be a random member of D
    z:=(z+d)/p
Forever.
```

The Tricky Bit

Suppose, then, we have $b = (p, D)$, and D tiles at size p. We know that b is injective, but it might not be surjective. One quick check is that b cannot be surjective if all the members of D are divisible by some z with $|z| > 1$. What else can we do?

Take any Gaussian integer z and write $z \in pw + D$. There is exactly one value of w for which this is true. Write $w = f(z)$. Suppose we keep iterating the function f. Since $f(z) = p^{-1}(z - d)$, the iterates cannot stay forever very far from 0 (in fact, they cannot stay further away than $(|p| - 1)^{-1}\Delta$, where Δ is the maximum value of $|d|$ for $d \in D$) and so they must eventually cycle.

Now $z \in S_n$ if and only if $f(z) \in S_{n-1}$, so z is captured in some S_n if and only if the values of $f^n(z)$ eventually hit 0 (where they will then stay).

All we have to do, therefore, to see if b is surjective is to check every point in a circle of radius $(|p| - 1)^{-1}\Delta$ around 0, and see if the iterates of f hit 0 or not. Although this procedure is not very pretty, it is finite.

As a short-cut, consider the values of z (other than 0) which are on cycles of length 1, i.e., those z with $f(z) = z$. Then $z \in pz + D$, i.e., $z(1-p) \in D$. So there is a non-zero z on a cycle of length 1 if and only if $\langle 1-p \rangle$ intersects D in some point other than 0. In other words, the following condition is necessary for b to be surjective:

$$\langle p-1 \rangle \cap D = \{0\}. \tag{5}$$

Similarly, z is on a cycle of length 2 if and only if $z \in p\,(pz+D)+D$, that is, $z(1-p^2) \in pD + D$. So the following condition is also necessary for b to be surjective:

$$\langle p^2 - 1 \rangle \cap S_2 = \{0\}. \tag{6}$$

This generalises to the following necessary condition:

$$\forall n \geqslant 1,\ \langle p^n - 1 \rangle \cap S_n = \{0\}. \tag{7}$$

Now we have included cycles of all lengths, so this condition is also sufficient.

More Examples

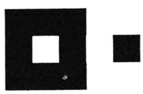
We now have enough ammunition to attack a few examples.

Suppose we wish to consider small powers p. We know $|p|^2 \geqslant 2$. In Example 3, we saw a bijective base with $p = -1+i$. Are there any other bijective bases with $|p|^2 = 2$?

First, is there a bijective base with $p = 1+i$? If $p = 1+i$, then $p-1 = i$, so $\langle p \rangle = \mathbf{Z}[i]$, so by condition (5) we know that the digit-set can only contain 0, so the answer is no.

Secondly, is there a bijective base with power $(-1+i)$ but a different digit-set? The digit-set must contain 0, and the other digit must be one of $\{1, i, -1, -i\}$ to avoid a common factor of D. The answer is therefore essentially no again.

EXAMPLE 4. By way of a fairly random example, consider the following digit-set. I have deliberately not marked the 0, because I want to discuss where it could go. The shape tiles at size 3.



The power could be 3, -3, or $3i$ (or $-3i$, but the set has reflectional symmetry in a horizontal axis, so this is essentially the same as $3i$).

If $p = 3$, then $p-1 = 2$, and a quick look at the figure shows that every square in the figure is a multiple of 2 away from some other square—so, wherever you put the 0, some other point would be on a multiple of 2. The power 3 cannot therefore give a bijective base.

If $p = -3$, then $p - 1 = -4$, which rules out two points for where 0 could be. At the next stage, $p^2 - 1 = 8$. This in fact rules out one more square (the one nearest the isolated square), and then the process stops, leaving six valid squares for the 0.

If $p = 3i$, then $p - 1 = 3i - 1$, which rules out two points. It turns out that this is all; the 0 can go in any of the other seven squares. The result (wherever the 0 goes) is shown in Figure 3.

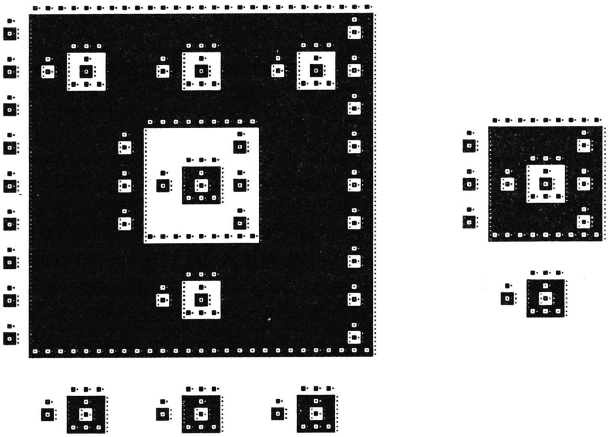


Figure 3.

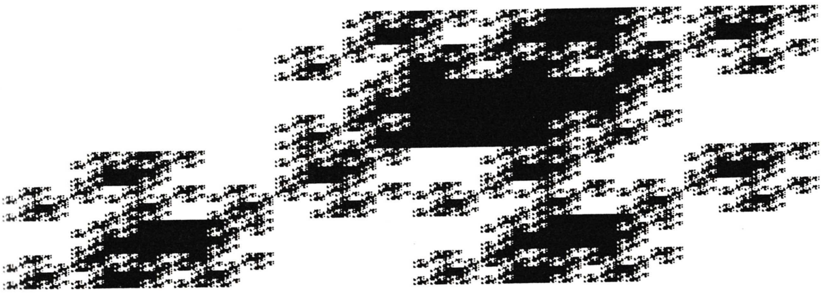


Figure 4.

An Infinite Family

One fairly natural question is whether there can be infinitely many bijective bases with the same power. To answer that question, here is an explicit example. The bases $b_n = (p, D_n)$ are defined by $p = -2$ and $D_n = \{0,\ 1,\ i,\ 2n+1+i\}$ for all $n \in \mathbb{Z}$. For example, D_1 is as follows:

By the discussion following (3), b_n is injective. Suppose that b_n is not surjective. Let $z = x + iy$ be a Gaussian integer which cannot be written in base b_n, and consider the iterates $f^n(z)$.

First, consider the imaginary part. If $f(z) = z' = x' + iy'$ then either $y' = -y/2$ or $y' = -(y+1)/2$ (whichever is an integer). If you iterate this, everything goes to 0, so $f^n(z)$ is eventually real. Without loss of generality, therefore, z is real.

Now z is real, and so $f(z) = z'$ is also real, so either $z' = -z/2$ or $z' = -(z+1)/2$ (whichever is an integer). This goes to 0 as before, and therefore $f^n(z)$ eventually goes to 0.

This shows that b_n is bijective. E_∞ in the case b_1 is shown in Figure 4.

Questions

Although this article gives some necessary and sufficient conditions for a base to be bijective, it falls far short of categorising bijective bases.

We could start by considering bases for $\mathbb{Z}$, rather than $\mathbb{Z}[i]$. I do not know of a categorisation of bijective bases for $\mathbb{Z}$, but I think it can be solved. If a base is bijective for $\mathbb{Z}$, then $|D| = |p|$. Examples 1 and 2 are bijective for $\mathbb{Z}$.

At the other end of the scale, we could consider bases for the quaternions, or for any other ring which can be thought of as $\mathbb{Z}^n$. (For the quaternions, $n = 4$.) If the base is bijective, then $|D| = |p|^n$.

Note that there is no problem with evaluating $\sum_{i=0}^{m} d_i\, p^i$ in the quaternions, even though they are not commutative, so long as we are careful to stick to this definition. Non-associative algebras, such as the Cayley numbers, need slightly more care.

Aside from the mathematics, there is the question of whether this has any practical use. Not normally an accusation you would want levelled at an otherwise insular piece of mathematics, but in this case there is a chance of an application in computer complex arithmetic or 2-dimensional graphics.

Acknowledgements

Example 3 is not new; I believe that Donald Knuth has presented it before. Thanks are also due to Richard Tucker, who proved condition (5), and to David Moore and Charles Brown, who helped me by drawing fractals on Archimedes computers.

Book Reviews

The Crest of the Peacock
Non-European Roots of Mathematics

by **George Gheverghese Joseph** Reviewed by **Anton Cox**

There is a widely-held belief that Mathematics started with the Greeks and eventually passed into Europe, via the custody of the Arabs during the Dark Ages. This view (which is still to be found in histories of the subject) is conclusively refuted in this fascinating book by Professor Joseph. The history that is presented here is a far richer one, stretching from India and China in the East to the Mayan and Inca civilisations of South America. The most surprising feature of this account is the level of sophistication attained by many of these cultures, often hundreds of years before similar achievements in the West. We learn of quasi-matrix methods in China in the first century AD, and of studies of Pascal's triangle in about 1000 AD, while Indian texts over 1500 years old suggest that not all infinities are equal and attempt to produce a system similar to transfinite numbers. There are also surprising examples of, for example, infinite power series; even hints of a 12th century calculus!

This is a consistently fascinating and surprising book, and a well written one. Professor Joseph illuminates his case with a judicious amount of historical background and a clear, direct style; this is not a dry academic tome, but a genuinely interesting and occasionally amusing read. ("A *quipu* resembles a mop which has seen better days.") The book examines each civilisation separately, with plenty of worked examples to illustrate techniques used at the time, and suggests how different cultures may have influenced one another. It also discusses carefully the claims made by researchers, particularly for earlier periods where the historical evidence is sketchy and incomplete, and distinguishes clearly between fact and conjecture.

I would warmly recommend this book to anyone, not merely those interested in the history of Mathematics, as an enjoyable and enlightening read. It is safe to say that all will be surprised by what they learn about the remarkable historical achievements of non-Western mathematicians.

The Crest of the Peacock, by George G. Joseph, Penguin paperback, £8·99.

Index to Mathematical Problems
Volume I: 1980–1984

edited by **Stanley Rabinowitz** Reviewed by **Vin de Silva**

It's rather difficult to describe something as astonishing as this first volume of Stanley Rabinowitz's *Index to Mathematical Problems*, a monumental labour of love that will certainly become a standard reference book and an essential part of the problem-solving enthusiast's library.

It is, simply enough, a catalogue listing the problems that have appeared in any of 28 journals and 12 mathematical contests during the years 1980–1984. Rabinowitz undertook the project in response to the growing frustration he had felt at the almost complete lack of indexing and cross-reference between these sources, and the consequent difficulty of trying to track down results that had appeared or of finding out whether a given problem had been solved before.

The difficulty lies in conveying just how remarkably successful the *Index* actually is. Sometimes this sort of encyclopaedia can be harder to use than intended, because one is at the mercy of an alphabetical index (or some other particular scheme preferred by the author) which so often fails to locate whatever is required. Rabinowitz triumphs over this problem by having more than half a dozen indexes, including a chronology index and a keyword index listing what he calls "memorable phrases". The cross-referencing is comprehensive, and the book itself is, as a bonus, beautifully well-designed and laid out. There is only one thing missing from this treasure-house; there are no answers printed. But then, that would only spoil the fun.

Index to Mathematical Problems Volume I: 1980–1984, edited by Stanley Rabinowitz, MathPro Press hardback, \$49·95.

Beyond Numeracy
An Uncommon Dictionary of Mathematics

by **John Allen Paulos** Reviewed by **Daniel Denman**

Beyond Numeracy seeks to demonstrate that Mathematics is not a dry, dusty subject; that there is more to it than computation and rote learning; that it is, in fact, one of the 'liberal arts'. The book consists of a series of entries alphabetically arranged, none longer than five pages and most about half that, on subjects as diverse as *Matrices and Vectors*, *Coincidences*, and *Human Consciousness: Its Fractal Nature*. Most of the entries are concisely and entertainingly written; Paulos is fond of illustrating mathematical principles with folksy examples, and there is much talk of compound interest and street intersections. The message is clear: Maths is Real Life. Paulos is a man with a mission, but for the most part he writes without missionary zeal and with an admirable ability to pin down what really matters in a subject. He also appears to have a weakness for puns, and throws in some of the most agonising I've read in a very long while.

I have reservations, though, about the design of the book. Its strict alphabetical arrangement leads to some strange juxtapositions; a more thematic ordering would have prevented, for instance, the use in the chapter on quantifiers of the $\wedge$ symbol, which remains unexplained until *Tautologies and Truth Tables*. Paulos remarks in his introduction that "the entries are largely independent and lightly cross-referenced"; perhaps heavier cross-referencing would have eliminated such anomalies. Qualms, too, about a notation which results in, on the same page, "the probability of obtaining two heads is $\frac{1}{4} — \frac{1}{2} \times \frac{1}{2}$" and "the probability ... is 1−·28 or ·72". Perhaps, as well, though most points are dealt with in admirable detail, some are brushed over more quickly than they deserve; the chapter on *Infinite Sets* is a model of clarity, compared to which that on *Quantifiers in Logic* is uncomfortably dense. But these are minor drawbacks in a book which manages, by a combination of charm, humour, and lucidity, to make the most difficult concepts approachable. In *Beyond Numeracy*, as in its celebrated predecessor *Innumeracy*, Paulos intends not just to present and explain mathematical problems, but, more importantly, to redeem the popular conception of mathematics; that he succeeds in doing so is a testament to the clarity of his opinions and the clarity with which he sets them out.

Beyond Numeracy, by John Allen Paulos, Penguin paperback, £6·99.

Solutions to the Problems Drive

Vin de Silva and Oliver Riordan

CAPRICORN

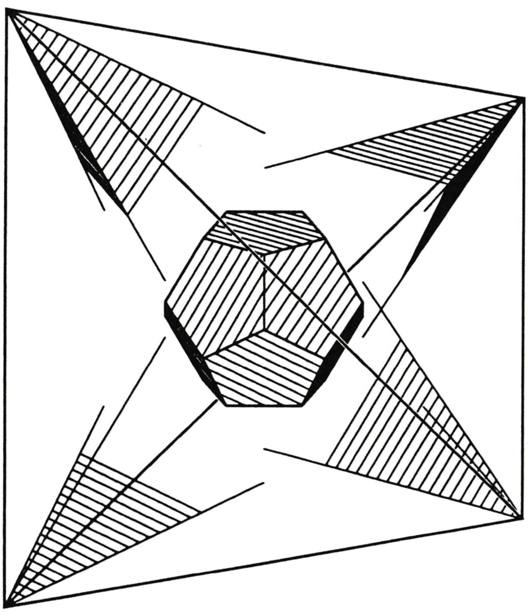
$10\sqrt{2}$ yards.
The problem with 100 replaced by x has no elementary exact solutions; this special case happens to have one.

AQUARIUS

328.

PISCES

There were five expected solutions: $(60°, \text{tetrahedron})$, $(120°, \text{tetrahedron})$, $(90°, \text{cube})$, $(108°, \text{dodecahedron})$, and $(36°, \text{dodecahedron})$. In fact ($\theta$, tetrahedron) also works for any $60° < \theta < 120°$. Here is the case $(36°, \text{dodecahedron})$:



ARIES

a) $N = 1$: Frog wins. Degenerate case, spotted by very few.
$N > 1$: Toad wins. An obvious winning strategy is to convert the line on the first move to

$$\perp \Rightarrow (\perp = \ldots = \perp).$$

b) For N odd, Frog wins.
For N even, Toad wins. Frog cannot play better than

$$(\perp\ =\ \ldots\ =\ \perp) \Rightarrow \perp$$

on the first move. However, this has the effect of reducing the N game to the $(N-1)$ game. The result follows by induction.

TAURUS

a) $N = 5$.
b) $N = 9$.

GEMINI

Only 2, 3, 4, 5, 6, and 8 are possible.

CANCER

$\sqrt{65}$. The metric $d(\mathbf{x}, \mathbf{y})$ reduces to the familiar Euclidean metric if we transform the plane by inversion in the circle $x_1^2 + x_2^2 = 25$.

LEO

a) Maximum is $(1 - 2^{-N})$.
b) Maximum is $\frac{1}{5}(1 - 2^{-N})$.

VIRGO

Area is $13\frac{1}{2}$.

LIBRA

$\frac{1}{6}(N+1)(N+2)(A+B+C)$.
Observe that, given A, B, and C, the distribution of soldiers is uniquely determined by the balance condition. Take 3 copies of the solution and superimpose them on each other, rotating one through 120° and another through 240° before doing so. The resulting grid is still balanced, and each main vertex holds $(A+B+C)$ soldiers. The unique balanced grid corresponding to this has $(A+B+C)$ soldiers at each of the $\frac{1}{2}(N+1)(N+2)$ vertices, and hence $\frac{1}{2}(N+1)(N+2)(A+B+C)$ soldiers in total. Divide by 3 to get the required answer.

SCORPIO

$\frac{1}{2}(1+\sqrt{5})$. This is undoubtedly the correct answer, although we have no satisfactory formal proof.

SAGITTARIUS

a) 3200, 20460 ($n!$ in base n);
b) 31, 33 (palindromic binary representations);
c) 10, 17 (constructible polygons);
d) 10, 13 (sums of two squares);
e) 1, 12 (numbers of episodes in *Mission to the Unknown* and *The Daleks' Master Plan*, the second and fourth stories in Season 3 of *Doctor Who*).

The Cover

Michael Greene and Richard Tucker

The eye-catching and unusual design on the cover is a successful (if parochial) example of a single image random-dot stereogram, or SIRDS. It was produced as a bitmap by a two-stage algorithm on an Archimedes computer, and printed on a Canon Bubblejet Printer. To those readers who have not encountered these infuriating objects before, I recommend its careful study.

The image is nearly periodic, repeating itself across any horizontal row. The period varies over the image; around the edge, it is about 15mm (the distance between the triangles at the bottom). To view the SIRDS, you must persuade your eyes to diverge to just the right amount that the right eye is focused on one part on the image and the left on a part one period to the left—this ensures that the eyes are looking at regions of the picture which agree almost exactly, and after some minutes your brain will decide (if you're lucky) that it is looking at a normal 3D object covered with random pixel shading. At first, you may find that the easiest way to do this is to stare at the two triangles at the bottom and relax your eyes; eventually, the *four* visible triangles (two for each eye) will merge into three. Try to focus on these—this is hard, but not impossible—and then shift your eyes up until you are looking at the main image.

The SIRDS should be held horizontal in good light at nearly arm's length, although bringing it closer may make it easier to view. It may help to focus directly on the centre of the image—pick an obvious blob which repeats across the SIRDS and merge together two adjacent copies. You'll certainly know when you've got it right.

So what should you see? If the SIRDS were exactly periodic, viewing it correctly would make it look completely flat. However, since some sections of the picture have periods slightly shorter than average, your eyes will have to move closer together on these sections to find regions which agree well, and the sections will appear to be closer to you. In this way, an amazing 3D effect is produced. If you have no luck at first, do persevere; it's well worth it.