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Editorial

This is the 42nd edition of Eureka (any archivist from the 21st century unaware of the deep significance of this number should refer to "The Hitch-hiker's Guide to the Galaxy" by D. Adams). It is the last edition the current editors are producing and we wish our successors well. We would also like to thank the secretaries at DAMTP for doing all the typing. This year Eureka offered £5 for each published article written by an undergraduate, with distinct lack of success. The offer still stands, so get writing! Contributions should be sent to C J Budd, St John's, or N Boston, Trinity.

An Elementary Proof of $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

by T J Ransford

I am greatly indebted to Mr. J. Scholes for having communicated to me the proof which follows. According to him, it has been around for a long time, decades at least, but despite this I do not believe it to be all that widely known (see for example [1] §13.6). As well as being 'elementary', the proof is also easy - the two do not always go together!

It relies on the following formula:

$$\cot^2\left(\frac{\pi}{2m+1}\right) + \cot^2\left(\frac{2\pi}{2m+1}\right) + \ldots + \cot^2\left(\frac{m\pi}{2m+1}\right) = \frac{1}{3}m(2m-1) \qquad (1)$$

(m any integer ≥ 1)

To show this, let n be an integer ≥ 1, x any real number, and expand $\cos nx + i\sin nx = (\cos x + i\sin x)^n$ to obtain:

$$\sin nx = \binom{n}{1} \sin x \cos^{n-1} x - \binom{n}{3} \sin^3 x \cos^{n-3} x + \ldots \qquad (2)$$

(or for a truly elementary proof, which avoids complex numbers, induct on n). For m as above, set $n = 2m+1$ and $x = \frac{r\pi}{2m+1}$ $(r = 1,2,\ldots,m)$. In each case $\sin nx = 0$, $\sin x > 0$, so we may divide through (2) by $(\sin x)^n$ to give

$$\binom{2m+1}{1} \cot^{2m} x - \binom{2m+1}{3} \cot^{2m-2} x + \ldots = 0 \text{ for } x = \frac{r\pi}{2m+1} \ (r = 1,2,\ldots,m)$$

Otherwise put, the m roots of the polynomial equation

$$\binom{2m+1}{1} t^m - \binom{2m+1}{3} t^{m-1} + \ldots = 0$$

are precisely $t = \cot^2(\frac{r\pi}{2m+1})$ $(r = 1,2,\ldots,m)$. The sum of these roots is therefore $\binom{2m+1}{3} / \binom{2m+1}{1} = \frac{1}{3} m(2m-1)$, which gives (1)

To finish off the proof, add m to both sides of (1) and use the identity $\csc^2 y = 1 + \cot^2 y$ to obtain

$$\csc^2(\frac{\pi}{2m+1}) + \csc^2(\frac{2\pi}{2m+1}) + \ldots + \csc^2(\frac{m\pi}{2m+1}) = \frac{1}{3} m(2m+2) \qquad (3)$$

If $0 < y < \pi/2$, then $0 < \sin y < y < \tan y$, hence $\csc y > 1/y > \cot y$. Applying this to (1) and (3) in turn leads to the double inequality

$$\frac{1}{3} m(2m-1) < (\frac{2m+1}{\pi})^2 + (\frac{2m+1}{2\pi})^2 + \ldots + (\frac{2m+1}{m\pi})^2 < \frac{1}{3} m(2m+2)$$

that is:

$$\frac{\pi^2}{6} (1 - \frac{1}{2m+1})(1 - \frac{2}{2m+1}) < \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{m^2} < \frac{\pi^2}{6} (1 - \frac{1}{2m+1})(1 + \frac{1}{2m+1})$$

QED.

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2. U.S.S.R. Olympiad Problem Book (Q233) (Freeman & Co., 1962).

On the College Societies

by P Taylor

In Cambridge we are uniquely fortunate. Our sister society, the Invariants, provides a service in the university of Oxford, whilst the colleges of London boast many ancient mathematical societies. We are blessed with both, although I fear we place too little value on our assets. An annual programme of fifty speaker meetings is impressive, but in practice it serves as little more than a purveyor of mathematical general knowledge which could equally well be included in the Tripos. Mathematical participation by undergraduates is sparse, and few can remember which colleges belong to which societies.

History makes occasional reference to many undergraduate mathematical societies, but of the five college societies now surviving, the Trinity Mathematical Society is the oldest, having been founded by the late Professor E.A. Milne as an undergraduate on his return from war service in 1919. This was at the suggestion of Professor Hardy: apparently a similar society had existed before 1914. The tradition of the TMS has been unbroken ever since, and it has met during every term except Easter 1942 and 1943. The Adams Society, named after the Johnian mathematician who should have discovered Neptune, was founded by J.J. Hyslop in 1923, and the Quintics and New Pythagoreans began later that decade.

Briefly during the war the Quintics and New Pythagoreans met jointly with the Adams and TMS respectively, but then folded, to be re-established in 1947. The Adams Society also had a hiatus in the mid sixties. The historical agreement, of admitting each other's members to their meetings, began in October 1936, just before the foundation of the Archimedeans.

Until 1961, mathematicians in Clare, King's, Magdalene and Trinity Hall used to join the TMS as associate members. The declining self-respect at that time of the TMS, its failure to enfranchise its associate membership and the foundation of Churchill College led to the establishment of the Tensors, which last term (on the basis of my research) celebrated its 150th meeting.

It was once common in the TMS and the Adams Society (I'm not so familiar with the others) for meetings to be addressed by officers and other undergraduates - Milne gave the first paper to TMS - but this healthy custom has unfortunately died, and it is difficult to see how it might credibly be re-established. In my opinion all six societies have lost sight to a large measure of their original purpose - to encourage undergraduates to participate in living mathematics.

The other problem faced by the college societies is finance. Excluding now the TMS and the Adams Society, which have both been campaigning recently to increase their derisory grants from Trinity and St. John's, the other societies lose out by being neither University nor college societies (in the sense used outside our own circle). Thus colleges charge them University rates for hire of rooms, whilst they are ineligible for grants from either source. Combined with the custom of holding open meetings, the embarrassment of a double subscription (this year £5 for Archimedeans and £1 for most college societies) leads to severe financial hardship. The Quintics, the New Pythagoreans and the Cambridge branch of the Mathematical Association have all this year foregone the luxury of a printed programme card for this reason.

The illfated subcommittee which rewrote the Archimedeans' constitution last year tried to address itself to this problem. Several schemes were proposed, all of which involved finance passing through our own Society, and each of them in turn was dropped. In retrospect this was a good thing, not only because such heated discussion could never have resulted in a sound solution, but also because to admit the principle of our funding them would just make the problem worse, since it would then become impossible for even TMS and Adams to obtain grants from their own colleges. In my view the Society's policy should be that the College Societies are exactly that, and should be financed by capitation from their respective colleges.

The college societies are an irreplacable asset, and it is our duty to respect their value to Cambridge Mathematics. It is remarkable how the existence of the structure is able to support an otherwise (arguably) moribund sister, but it would be foolish and irresponsible to assume either that the societies are in any real sense independent of one another, or that their continued existence can be guaranteed for the future. It would be naive to suggest that precisely the same sisters will always form the family - history teaches us differently - and we should always be asking ourselves whether any particular college is in the right society, and whether a society should be dominated entirely by one college. However in my opinion we may soon face a more fundamental question, the toll which will be taken of our sorority by poverty and apathy unless some action is taken to prevent it.

Achieving the Skinny Animal[1]

Frank Harary[2]
Overseas Fellow, Churchill College
University of Cambridge, 1980-1981

Dedicated to Professor Sir William Hawthorne,
Master, Churchill College

Abstract

The summary of my animal achievement games (generalized Tictactoe) in the Mathematical Games column of the Scientific American for April 1979 contains some wrong information which was supplied by me to Martin Gardner. We present the correct results here, and show in particular that the Skinny animal has board number 7 and move number 6.

1. History

One day a biologist walked into my office at the University of Michigan and asked if I could help him with a combinatorial problem. He wanted to investigated a theory of evolution for very small animals, beginning with one cell and growing just one cell at a time. He decided that it would be necessary to oversimplify the shape of the cell deliberately and chose square cells. He then drew the animals shown in Figure 1, (which I later named as indicated, with a little bit of help from my friend and colleague Ronald C. Read who said, "I don't know what to call the others, but I should call this one Fatty "). After we quickly constructed the 5-cell animals he was content and departed without leaving his name. The book [3] by Golomb called these animals 'polyominoes'. Several open questions concerning their enumeration were presented in [4] and [8].

1. An invited address to The Archimedeans at Trinity College, Cambridge on 17 February 1981.

2. Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109, U.S.A.

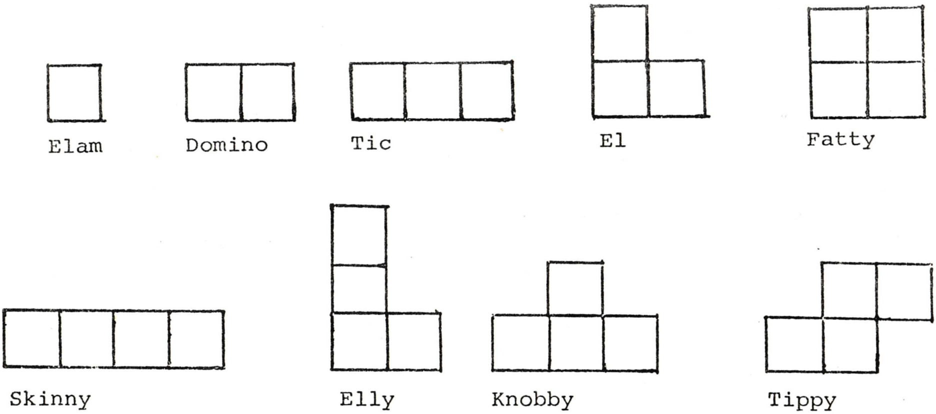

Figure 1. The smallest animals, with names.

When I first proposed achievement and avoidance games for graphs in [5], I did not realize that this would soon lead to generalizations of the traditional game of Naughts and Crosses (called Tictactoe in the United States)

2. Achieving Animals

Let A be a given animal and consider a b x b playing board. We adopt the convention that the first player to move is called Oh and that his moves are labelled $O_1, O_2, \ldots$ while the second player, Ex, makes moves $X_1, X_2, \ldots$ The winner, if any, is the player who first completes a copy of the objective animal A using only his own squares marked O or X. This is called the game of achieving animal A on a b^2 board; more briefly it is the achievement game (A,b). Note that any rotation or reflection of A is still regarded as animal A.

We illustrate an achievement game with the animal Elly of Figure 1. It is obvious that neither player can achieve Elly on a 3 x 3 board as it is very well known that the subanimal Tic cannot be achieved on a 3^2 board. Hence we now consider the achievement game(Elly, 4). In the 4^2 board of Figure 2, Oh makes a rational first move O_1 in any of the four central squares. Player Ex replies with X_1 in a central square adjacent to O_1.

Then Oh puts O_2 in the other central square next to O_1 and Ex realizes that he/she is in trouble. For no matter which end of the O_1-O_2 domino Ex picks, Oh will complete a Tic, as in Figure 2, by moving at the other end of the domino, and thus create a double threat in the two squares marked Θ. So Ex resigns.

Θ	O_3	Θ	
	O_2		
	O_1	X_1	
	X_2		

Figure 2. Achieving Elly.

We say that the <u>board number</u> of Elly, written b(Elly), is 4 since Oh can complete an Elly on a 4^2 board but not on a 3^2 board. The <u>move number</u> of Elly, written m(Elly), is the smallest possible number of moves with which Oh can make Elly on this 4^2 board. Thus b(A) = m(A) = 4 when animal A is Elly.

3. Achieving Skinny

Based on the information which I supplied hastily to Martin Gardner for his annual April Fool's column in 1979, he reported in [2] that when animal A is Skinny, b(A) = 6 and m(A) = 8. However I was wrong about both the board number and the move number of Skinny! Finally it became patently clear that only an exhaustive check of all the possibilities would constitute a genuine proof. The correct values will now be derived.

A conventional labeling of the squares of a b^2 board Figure 3 uses letters a,b,... for the rows and numbers 1,2,...for the columns.

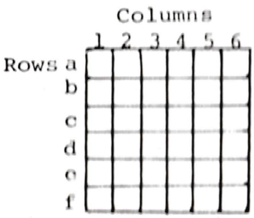

Figure 3. Notation for the squares of a board.

Theorem When A is the Skinny Animal, $b(A) = 7$ and $m(A) = 6$.
Proof It takes only a moment to verify that Oh cannot achieve Skinny on a 4^2 board or a 5^2 board. Figure 4 shows a 6^2 board in which the four central squares are marked C, the twelve middle squares M, and the outer squares are unmarked.

	M	M	M	M	
	M	C	C	M	
	M	C	C	M	
	M	M	M	M	

Figure 4. The central and middle squares of a 6^2 board.

Clearly the best opening move for Oh is to put O_1 into a C - square. Now there are three possibilites. If Ex decides to move X_1 in another C - square, Oh can win in only six moves as illustrated by the following game.

move	1	2	3	4	5
Oh	c3	d3	e2	d2	c2
Ex	c4	b3	e3	d4	resigns

If X_1 is placed in any of the 20 outer squares, the move is essentially wasted and Oh again proceeds to win in just six moves as in the game above.

But if Ex puts X_1 in any one of the twelve M - squares then Oh will not succeed in achieving Skinny and the result will be a draw! For after a few moves, Oh will make a threat which Ex can defend with a counterthreat, thus taking the attacking momentum from Oh. (Actually if Oh continues attacking now, he may enable Ex to win.) Thus with rational play by Ex, Oh cannot achieve Skinny on a 6^2 board.

Turning to a 7^2 - board, Oh places O_1 in the unique central square. No matter where X_1 is now located, O_2 is marked next to O_1 in a general direction"away" from X_1. The 7^2 - board now provides enough space for Oh to force Skinny in six moves as in the illustrated game above.

Thus when A = Skinny, we have $b(A) = 7$ and $m(A) = 6$.

4. Appendix

We conclude with nine brief observations concerning animal achievement games.

I. Fatty is not a winner.
It was shown in [2] how to tile an infinite plane with dominoes to form a blocking pattern for Fatty by having Ex reply to each Oh - move by completing the domino begun by Oh. As any Fatty in the plane must contain a whole domino, Oh cannot win.

II. An animal A is a <u>minimal nonwinner</u> if Oh cannot achieve A, but any proper subanimal obtained from A is a winner. Including Fatty, 12 minimal nonwinners were displayed in [2] together with blocking patterns for each(including three patterns which need to be permuted in order to block the animals indicated). Of course any superanimal of a minimal nonwinner cannot be a winner either.

III. There are now 11 animals which have been proved to be winners. Using the letter - names (following Golomb [3]) of the three animals shown in Figure 5, their board and nove members are listed in Table 1.

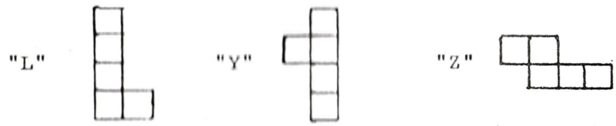

Figure 3. Three more winners.

Table 1. The board and more numbers of the proved winners.

A	b	m
Elam	1	1
Domino	2	2
Tic	4	3
El	3	3
Skinny	7	6
Elly	4	4
Knobby	5	4
Tippy	3	5
"L"	7	7
"Y"	7	6
"Z"	6	6

The results for Skinny and the three letter - animals correct those listed in [2].

IV. There is exactly one animal which has not yet been proved to be a winner or nonwinner: Snaky (so named by Martin Gardner),shown in Figure 6. When I proposed the phrase "True Conjecture" for a conjecture about which one feels extremely confident, I did not know that Euler himself had already used it a few centuries earlier.

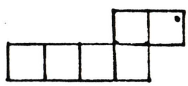

Figure 6. Snaky

True Conjecture Snaky is a winner.

My plausibility consideration for this assertion is that my former student, Dr. Geoffrey Exoo, won his last twenty games of Snaky achievement played on a sheet of "graph paper". However, it still remains to prove this and to determine the exact values of b and then m for Snaky.

After this conjecture has been proved,perhaps by an exhaustive computer programme, we will know that:

(a) There are exactly 12 winning animals;
(b) There are exactly 12 minimal nonwinners, each of which is blocked by a domino tiling pattern. (The symmetric appearance of this result would be quite pleasing.)

V. Let c(A) be the number of cells in animal A. Then A is called economical if A is a winner and m(A) = c(A).

Theorem There are exactly six economical animals: Elam, Domino, Tic, El, Elly and Tippy.

VI. It was shown in [6] that on a toroidal 5^2- board, the move number of Skinny is 8. The toroidal board and move numbers of the other animals are being invesigated.

VII. For the achievement game of "4 in a row" including diagonals as well as Skinny, the board and move numbers are both 5.

VIII. A fascinating book by Berlekamp, Conway and Guy [1] on mathematical games, presenting many new results, is about to appear.

IX. Many new achievement and avoidance games will be presented in the book [7] which is now being actively written.

ACKNOWLEDGEMENT

The corrected results for Skinny were achieved with the help of incisive comments by chess experts Adam Feinstein, David Levy and Kevin O'Connell.

Primes in Arithmetic Progressions

by N Boston

In 1837 Dirichlet published a memoir in which he proved that there are infinitely many primes in each arithmetic progression $kn + h$, $n = 1,2,\ldots$, k,h coprime. The methods he used were of an analytic nature (see, e.g., [1] I-IV) and indeed began the development of analytic number theory. The search for an elementary proof, however, has been successful so far only for special values of k and h (see [2], p.124) and below we concentrate primarily on cases $h = \pm 1$ and see how formulae the reader can easily verify by induction can lead to important results.

The idea is to choose a sequence of non-negative integers, $P(n) = aP(n-1) + bP(n-2)$, a,b coprime integers, $P(0) = 0$, $P(1) = 1$, and from this a subsequence in which the terms are divisible by distinct primes of the required form. Examples of such sequences include the natural nos. ($a = 2$, $b = -1$), Mersenne nos. ($a = 3$, $b = -2$), and the Fibonacci nos. ($a = b = 1$, see below), and not surprisingly there are many useful properties shared by them, that must first be investigated to ensure the correct choice of a, b, etc. later on.

It will be helpful to bear in mind the following table of Fibonacci nos:

n	P	Q	R	(mod n)	n	P	Q	R	(mod n)
1	1	1	1	-	7	13	13	13	-1
2	1	1	1	-	8	21	7	7	-1
3	2	2	2	-1	9	34	17	17	-1
4	3	3	3	-1	10	55	11	11	1
5	5	5	5	0	11	89	89	89	1
6	8	4	1	1	12	144	6	1	1

The first observation to be made is that the n for which P(n) is divisible by a given m occur in A.P., 1st. term 0, and this is proved via

(1) $P(n) = P(n-r)P(r+1) + bP(n-r-1)P(r), \quad r = 0,1,\ldots,n-1,$

(2) $(P(n), P(n+1)) = 1, \quad n = 1,2,\ldots$

so

(A) $\exists$ a function of m, c(m), such that $m|P(n)$ iff $c(m)|n$, where we take $c(m) = 0$ if m divides only P(0), e.g. the Mersenne nos. with m = 2.

The behaviour of this function is our new concern (clearly it suffices to consider $c(p^r)$, p prime) and is constrained firstly by a theorem analogous toFermat's ([3], p.63), viz.

(B) p prime. (i) $c(p) = 2$ iff $p|a$, $= 0$ iff $p|b$,

(ii) if $p\nmid a,b$, then $p = 2$ implies $c(p) = 3$, else put $d = a^2 + 4b$ then $c(p)|(p-(\frac{d}{p}))$ (Legendre's symbol, [3], p.68; $(\frac{d}{p}) = d^{\frac{p-1}{2}} (\mathrm{mod}\, p)$).

(ii) is proved by solving the auxiliary equation $x^2 - ax - b = 0$ to obtain, in terms of the discriminant d,

(3) $P(n) = (\binom{n}{1}a^{n-1} + \binom{n}{3}a^{n-3}d + \binom{n}{5}a^{n-5}d^2 + \ldots)/2^{n-1}, \quad n = 1,2,\ldots,$

so, e.g., in the table $(\frac{5}{p}) = (\frac{p}{5}) = p^2 \pmod 5$ by quadratic reciprocity ([3], p.76) implies ;$c(5) = 5$; if $p = \pm 1 \pmod 5$, $c(p)|p-1$ (i.e. $p|P(p-1)$); if $p = \pm 2 \pmod 5$, $c(p)|p+1$ (i.e. $p|P(p+1)$).

The remaining phenomenon we shall want controlled is the possibility of repeated divisors and for this we use

(C) (i) If p is an odd prime, then $c(p) = c(p^2) = \ldots = c(p^k) \neq c(p^{k+1})$ implies $c(p^r) = p^{r-k}c(p^k)$, $r = k, k+1, \ldots$

(ii) We may have $c(2) \neq c(2^2)$ but $c(2^2) = c(2^3)$ (see table). If however, $c(2^2) = c(2^3) = \ldots = c(2^k) \neq c(2^{k+1})$, then $c(2^r) = c(2^k).2^{r-k}$ $r = k, k+1, \ldots$

This is proved via

(4) $P(rm) = \binom{r}{1}P(1)u(vb)^{r-1} + \binom{r}{2}P(2)u^2(vb)^{r-2} + \ldots + \binom{r}{r}P(r)u^r$,

where $u = P(m)$, $v = P(m-1)$,

taking $u = hp^k$, $p\nmid h$, and (4) (mod p^{k+2}).

Lastly, we need to know that given a divisor n of $p\pm 1$ we can ensure $c(p) = n$ by appropriate choice of a,b (mod p). This is verified by a method similar to that used to prove the existence of primitive roots (mod p) ([3], p.85), showing that Q(n), a function constructed like the cyclotomic polynomials ([4], p.156) by $\prod_{d|n} Q(d) = P(n)$, $n = 1,2,\ldots$, considered as a polynomial, $a^{\phi(n)} + \ldots$, in a and b, must have its full quota of zeros in Z_p^*. The first few Q(n) are $1, a, a^2+b, a^2+2b, a^4+3a^2b+b^2, a^2+3b, \ldots$ and for $a = b = 1$ these are included in the table up to $n = 12$.

Our aim is to isolate primes p dividing P(n) with $c(p) = n$ so that by (B) $p \equiv 0,\pm 1 \pmod n$, but whereas Q(n) does get rid of obvious cases of $p|P(n)$ with $c(p) < n$, there is still the possibility of repeated divisors, e.g. the 4 dividing Q(6) in the table where 2 has already appeared in P(3). Stripping Q(n) of prime factors p with $c(p) < n$, we obtain R(n) (see table), a product of primes $0,\pm 1 \pmod n$ so itself $0,\pm 1 \pmod n$.

The magnitude of R(n) is kept in check relative to Q(n) by (C) and so a detailed argument by Birkhoff and Vandiver, Annals of Maths. 1904, showing that for the Mersenne nos. $Q(n) > n$ if $n > 6$, ensures $R(n) > 1$ if $n > 6$. (This solves the QARCH problem, $(2^n-1)\nmid\prod_{0<r<n}(2^r-1)$, $n = 7,8,\ldots$ since R(n) divides the LHS but not the RHS). In this case $\left(\frac{d}{p}\right) = 1$, so that choosing primes p_r dividing R(rn), $r = 7,\ldots$ these are distinct and $c(p_r) = rn$ implies by (B) that $p_r \equiv 1 \pmod n$. It is however easier to prove that there are infinitely many such primes by taking $a = x+1$, $b = -x$, for which consult [4], p.157.

To illustrate the power of the above methods we shall now show:

Thm: There are infinitely many primes $-1 \pmod{p^r}$, p an odd prime, r a positive integer.

Pf: If $p \nmid d$, $R(p^k) = Q(p^k) = P(p^k)/P(p^{k-1}) = P(p) \pmod p$ by (4)

$$= \left(\frac{d}{p}\right) \pmod p \text{ by (3).}$$

$R(n) = 0, \pm 1 \pmod n$ so $R(p^k) = \left(\frac{d}{p}\right) \pmod{p^k}$.

Now choose $a,b > 0$ s.t. $\left(\frac{d}{p}\right) = -1$. Then since $R(p^k) = -1 \pmod{p^k}$ and any of its prime factors are $0, \pm 1 \pmod{p^k}$, we may choose distinct primes q_k s.t. $q_k = -1 \pmod{p^k}$ and $c(q_k) = p^k$. $q_r, q_{r+1}, \ldots$ are then the desired primes.

An example (*) of the use of this is if $p = 3$, $r = 1$, when, since $\left(\frac{5}{3}\right) = -1$, we get from the table $2, 17, \ldots$. Note how an upper bound on $P(n)$ leads to one on the 1st. prime $-1 \pmod{p^r}$. For a stronger bound due to Linnik, viz. $\exists\, C$ s.t. if $(a,q) = 1$, $\exists$ prime $p = a \pmod q : p < q^C$, consult Prachar, "Primzahlverteilung", Springer (1957), Chapter 10 (ref. from [1]).

It remains to see for which other n we can prove there are infinitely many primes $-1 \pmod n$. The Fibonacci nos. suffice for $n = 2^r$, any $r = 2,3,\ldots$, since $P(2) = P(8) = P(32) = \ldots = 1 \pmod 4$ whereas $P(4) = P(16) = \ldots = -1 \pmod 4$, whence $Q(2^r) = -1 \pmod 4$ so $R(2^r) = -1 \pmod{2^r}$, $r = 2,3,\ldots$, and continue as above.

Unfortunately, without repeated divisors, n not a prime power, $R(n) = Q(n) = 1 \pmod n$, since putting $n = kq^r$, q an odd prime, $q \nmid k$, $k > 1$, $r \geq 1$, $Q(n) = P(n)/(P(k)\prod_{q|d|n} Q(d))$, and by (4) $P(n)/P(k)$ is $P(q^r) \pmod q$ as is the product expression, the only terms not necessarily $1 \pmod q$ being by induction $Q(q), Q(q^2), \ldots Q(q^r)$.

If, however, $n = mp^r$, $m > 2$, $p \equiv -1 \pmod m$ prime, then choosing $a,b > 0$ with $c(p) = m$, $Q(mp^k) \equiv 1 \pmod m$, $R(mp^k) \equiv -1 \pmod m \equiv -1 \pmod{mp^k}$ for sufficiently large k. The reason for this last comment is the possibility of (finitely many) fluke repetitions, e.g. $m = 3$, $p = 2$ with the Fibonacci nos. where $Q(12) \equiv 0 \pmod 3$ due to $c(3) = 4$. In particular, taking $r = 0$, if we know of one prime $-1 \pmod m$, then there are infinitely many.

An example of the more general use is if $n = 15$, when we take $m = 3$, $p = 5$, $a = 2$, $b = 1$. Finally, if $b = 1$, then putting $n = 2r+1$ in (1) gives by (2) a sum of coprime squares, so that any odd factor of $P(2r+1)$ is $1 \pmod 4$, so that, e.g., (*) above shows there are infinitely many primes $5 \pmod{12}$.

I am grateful to Professor Bumby for his comments on an earlier draft of this work.

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Pi and Pentagon

by A Tan

In an excellent article, Te Selle (1) discussed how one can compute values for pi from regular polygons inscribed in and circumscribed on a circle of diameter 1. By successively choosing polygons with number of sides double that of the previous polygon, the perimeter of the polygon is shown to approach the value of pi. As the initial polygons, the author chose the square and the hexagon, both of whose perimeters are easily calculated. Another polygon whose perimeter can be calculated is the regular pentagon. In this note, the results are reported for the case when the starting polygon is a regular pentagon.

The perimeter of a regular pentagon inscribed in a circle of diameter 1 is known to be $5\sqrt{5-\sqrt{5}}\,/\,2\sqrt{2}$. Table 1 shows the length of one side of the inscribed polygon s_n, the length of the apothem a_n and the perimeter of the polygon I_n, when the number of sides of the polygon n = 5, 10, 20 .... This table is supplementary to the first two tables of Te Selle. Table 2 is an extension of Te Selle's third table, which shows the numerical values of the perimeter of the circumscribed polygon C_n in addition to those of I_n. The table includes polygons with n = 5, 10, 20 ... in addition to those of Te Selle. Also included in the table is the triangle with n = 3.

Reference:

(1) David W. Te Selle, Pi, Polygons and a Coputer, Math. Teacher, 63, 128-132, 1970.

<u>TABLE 1</u>

n	s_n	a_n	I_n
5	$\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$	$\frac{1}{4}\sqrt{\frac{3+\sqrt{5}}{2}}$	$\frac{5}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$
10	$\frac{1}{2}\sqrt{2-\sqrt{\frac{3+\sqrt{5}}{2}}}$	$\frac{1}{4}\sqrt{2+\sqrt{\frac{3+\sqrt{5}}{2}}}$	$5\sqrt{2-\sqrt{\frac{3+\sqrt{5}}{2}}}$
20	$\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{\frac{3+\sqrt{5}}{2}}}}$	$\frac{1}{4}\sqrt{2+\sqrt{2+\sqrt{\frac{3+\sqrt{5}}{2}}}}$	$10\sqrt{2-\sqrt{2+\sqrt{\frac{3+\sqrt{5}}{2}}}}$
40	$\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{\frac{3+\sqrt{5}}{2}}}}}$	$\frac{1}{4}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\frac{3+\sqrt{5}}{2}}}}}$	$20\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{\frac{3+\sqrt{5}}{2}}}}}$

TABLE 2

n	I_n	C_n
3	2.59808	5.19615
4	2.82843	4.00000
5	2.93893	3.63271
6	3.00000	3.46410
8	3.06147	3.31371
10	3.09017	3.24920
12	3.10583	3.21539
16	3.12145	3.18260
20	3.12869	3.16769
24	3.13263	3.15966
32	3.13655	3.15172
40	3.13836	3.14807
48	3.13935	3.14609
64	3.14033	3.14412
80	3.14079	3.14321
96	3.14103	3.14271
128	3.14128	3.14222
160	3.14139	3.14200
192	3.14145	3.14187
256	3.14151	3.14175
320	3.14154	3.14171
384	3.14156	3.14166
512	3.14157	3.14163
640	3.14158	3.14162
768	3.14158	3.14161
1024	3.14159	3.14160
1280	3.14159	3.14160
1536	3.14159	3.14160
2048	3.14159	3.14159

Lagrangians in Action

by C N Corfield

Part Ib and II mathematicians have met Lagrangians in mechanics and perhaps doubt their utility. This article shows how they can be exploited to gain some fundamental insights into wave motion. Let us start with a simple system:

$$\ddot{u} + f(\varepsilon t)u = 0 \quad \text{where} \quad \varepsilon \ll 1 \;, \quad f > 0$$

ε has been put into f to emphasize that it varies <u>slowly</u>.

We expect u to oscillate with frequency $\omega = \sqrt{f}$ and to change on a time scale given by $T = \varepsilon t$. What we don't know is the effect changes of f have on the oscillation. Let us write

$$u = a(T) \cos \int^t \omega(\varepsilon t')dt' \qquad \omega(\varepsilon t') = \sqrt{f(\varepsilon t')}$$

Note the integral in the phase. This is because $u \propto \cos\theta \Rightarrow \frac{d\theta}{dt} = \omega$ and ω is not constant.

To get information about $a(T)$ we can use the Lagrangian:

$$L = \tfrac{1}{2}\dot{u}^2 - \tfrac{1}{2}fu^2$$

The motion of the system extremises (Hamiltons principle):

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} L \, dx \, dt$$

To find out about slow changes we can replace L by its average over one period in the above integral

$$\mathcal{L} = L_{\text{average}} = \tfrac{1}{4}a^2\omega^2 - \tfrac{1}{4}fa^2$$

ω, a, f are all functions of slow time T .

We now use the Euler Lagrange equations twice:

(1) The E - L equation for a is:

$$\frac{d}{dT}\frac{\partial\mathcal{L}}{\partial a_T} - \frac{\partial\mathcal{L}}{\partial a} = 0 \quad \text{where} \quad a_T = \frac{da}{dT}, \text{ yielding}$$

$$\tfrac{1}{2}a(\omega^2 - f) = 0 \Rightarrow \omega = \sqrt{f} \quad \text{as expected.}$$

(2) The second use requires subtlety. For a roughly periodic variable $u \propto \cos \alpha(x,t)$ we have

$$\alpha(x_o + x,\ t_o + t) = \alpha(x_o, t_o) + x\frac{\partial\alpha}{\partial x} + t\frac{\partial\alpha}{\partial t} + \ldots\ldots = \alpha_o + (\omega t - kx) + \ldots$$

where $\omega = \frac{\partial\alpha}{\partial t}$, $k = -\frac{\partial\alpha}{\partial x}$. Hence the averaged Lagrangian is a function of a, $\frac{\partial\alpha}{\partial t}$, $\frac{\partial\alpha}{\partial x}$. The E - L equation for the phase α is $\frac{\partial}{\partial T}\frac{\partial\mathcal{L}}{\partial\alpha_t} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\alpha_x} - \frac{\partial\mathcal{L}}{\partial\alpha} = 0$ where $\alpha_t = \frac{\partial\alpha}{\partial t}$, $\alpha_x = \frac{\partial\alpha}{\partial x}$

In our example this reduces to $\frac{\partial}{\partial T}\frac{\partial\mathcal{L}}{\partial\alpha_t} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial\alpha_t} = \text{const.}$

But $\frac{\partial\mathcal{L}}{\partial\alpha_t} = \frac{\partial\mathcal{L}}{\partial\omega} = \tfrac{1}{2}a^2\omega$, this quantity is called the <u>wave action</u>.

Since we already know that $\omega(T) = \sqrt{f(T)}$ we see that

$$a(T) = a(o)\left(\frac{f(o)}{f(T)}\right)^{\frac{1}{4}}$$

$$\Rightarrow u = a(o)\left(\frac{f(o)}{f(T)}\right)^{\frac{1}{4}} \cos \int_o^t \sqrt{f(\varepsilon t')}\, dt'$$

Part II students will recognise this as the WKBJ approximation.

The wave action is an important quantity in wave dynamics, so we will look at it more closely. The E - L equation for the phase was

$$\frac{\partial}{\partial T}\frac{\partial\mathcal{L}}{\partial\alpha_t} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\alpha_x} = 0 \quad \text{but} \quad \frac{\partial}{\partial\alpha_t} = \frac{\partial}{\partial\omega} \quad \text{and} \quad \frac{\partial}{\partial\alpha_x} = -\frac{\partial}{\partial\kappa} \Rightarrow \frac{\partial}{\partial T}\frac{\partial\mathcal{L}}{\partial\omega} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\kappa} = 0$$

Let $A = \frac{\partial\mathcal{L}}{\partial\omega}$ and $F = -\frac{\partial\mathcal{L}}{\partial\kappa}$. F is the <u>wave action flux.</u>

(In 3 dimensions we have $\frac{\partial A}{\partial T} + \nabla.\underline{F} = 0$).

For linear waves, i.e. waves that satisfy a linear differential equation, the averaged Lagrangian is $\mathcal{L} = a^2 G(\omega, \kappa)$.

The E - L equation for a gives $G(\omega,\kappa) = 0$ which is the <u>dispersion relation</u> for ω and κ . We also have

$$A = \frac{\partial \mathcal{L}}{\partial \omega} = a^2 \frac{\partial G}{\partial \omega}$$

$$F = -\frac{\partial \mathcal{L}}{\partial \kappa} = -a^2 \frac{\partial G}{\partial \kappa} = a^2 \frac{\partial G}{\partial \omega}\frac{d\omega}{d\kappa} \quad \left(\text{since} \quad 0 = dG = \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \kappa} d\kappa\right)$$

$$= A \frac{d\omega}{d\kappa} = AC_g$$

$C_g = \frac{d\omega}{d\kappa}$ is the group velocity and is the speed at which energy travels and is in general different in magnitude (and direction for 3 dimensional waves) from the phase speed $\frac{\omega}{\kappa}$.

The conservation relation for wave action is thus

$$\frac{\partial}{\partial T} A + \frac{\partial}{\partial X}(AC_g) = 0 \quad \left(\text{or} \quad \frac{\partial}{\partial T} A + \nabla . (A\underline{C}_g) = 0 \quad \text{in 3 dimensions}\right).$$

The action has a simple expression in physical variables. Let

$$u = f(\alpha(x,t)) \quad \text{where} \quad f \text{ has period } 2\pi \ .$$

$$u_t = \frac{\partial \alpha}{\partial t} f' = \omega f' \ , \qquad u_x = \frac{\partial \alpha}{\partial x} f' = -kf'$$

$$\Rightarrow L = L(u, u_t \ u_x) = L(f, \omega f', -kf')$$

$$\text{Hence } \mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L \, d\alpha = \frac{1}{2\pi}\int_0^{2\pi} L(f, \omega f', -kf') d\alpha$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \omega} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial L}{\partial \omega} d\alpha = \frac{1}{2\pi}\int_0^{2\pi} f' \frac{\partial L}{\partial u_t} d\alpha$$

$$= \frac{1}{\omega}\frac{1}{2\pi}\int_0^{2\pi} \omega f' \frac{\partial L}{\partial u_t} d\alpha = \frac{1}{\omega}\frac{1}{2\pi}\int_0^{2\pi} ut \frac{\partial L}{\partial u_t} d\alpha$$

$$= \frac{2}{\omega} E_k \quad \text{where } E_k \text{ is the averaged kinetic energy}$$

(recall L = (kinetic) - (potential) and "$\dot{q} \frac{\partial L}{\partial \dot{q}} = 2(\text{kinetic})$").

This is a fundamental result that it is $\frac{E_k}{\omega}$, rather than the total energy E , that is conserved as the medium varies. For linear waves $E_k = E_{pot}$ so $\frac{E}{\omega}$

is also conserved. Note that this agrees with the well known result for radiation

$$E = \hbar\omega \Rightarrow E/\omega = \hbar = \text{constant}.$$

Another approach to the wave action is to think of it as an adiabatic invariant. For slowly changing periodic systems the quantity

$$A = \frac{1}{2\pi} \oint p dq \ , \ p = \frac{\partial L}{\partial \dot{q}} = \text{generalised momentum}$$

is a constant throughout the slow changes. In our system

$$\frac{1}{2\pi}\oint p dq = \frac{1}{2\pi} \oint \frac{\partial L}{\partial u_t} du = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial L}{\partial u_t} \frac{du}{d\alpha} d\alpha = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial L}{\partial u_t} f' \ d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial L}{\partial \omega} d\alpha = \frac{\partial \mathcal{L}}{\partial \omega}$$

If you have found this too intricate, read the article again. The maths is simple even if the use is subtle. The Lagrangian has allowed us to see two different processes happening simultaneously:

(1) the rapid oscillations

(2) the slow drift of phase and change in amplitude and how they are related.

We have also obtained a very general conserved quantity, the wave action A , which is not always obvious from a given differential equation describing some system.

The HO Scale Structure of Space-Time

by C J Budd

The progress in both mathematics and physics has allowed us to predict with astonishing accuracy events on the large scale such as planetary motion and on the small scale concerning subatomic particles. On the H-0 scale progress has been very slow and such fundamental problems as the arrival of the next bus and the prediction of who the next Miss World will be have defied 20th century mathematics. However, some remarkable breakthroughs have been made in other areas of industrial and social topology. The purpose of this paper is to summarise some of these.

The British Rail Metric:

The first discovery was that British Rail actually had a topology (which BR conjectures is connected, but no-one has proved this) and is even metrisable by the 'BR metric' (Readers are reminded that a metric is an expression of the distance between A and B satisfying

i) $d(A,B) = 0$ if $A = B$

ii) $d(A,B) = d(B,A)$

iii) $d(A,C) \leq d(A,B) + d(B,C)$ "The Triangle Inequality")

The BR metric is distinguished by the property that if A,B are any stations and L is London then

$$d(A,B) = d(A,L) + d(L,B) \tag{1}$$

The triangle inequality is especially significant in this context and is known as 'The Scenic Route'. The metrisation of BR leads to an interesting result: Since it has a metric it is Hausdorff, hence the single point sets (the stations) are always closed.

Attempts to topologise Cambridge have yielded at least three non-homeomorphic topologies. These depend on whether you are walking, cycling, driving or going by taxi. Denote these by Cw, Cc, Cd, Ct respectively. There are evidently maps from Cc, Cd, Ct to Cw known as the 'Flat Tyre Operators' (FTOs). Inverses may be constructed, but are messy and cover your hands in oil. The Ct topology is metrisable and $d_t(A,B)$ is the greatest upper bound of $\{d(A,B)\}$ for all other metrics. Cw approximated to the Euclidean metric, but a more accurate representation of Cambridge at 3am would be $d(A,B) = C(d_e(A,B))$ where d_e is the Euclidean metric and C is a convex function which depends on your ability to climb walls and bribe porters. The topology Cc can change if you are in a hurry. Illegal cycling makes Cc similar to Cw . If one wishes to remain legal a Euclidean metric is no longer possible, since the legal Euclidean distance from Trinity to John's is more than ten times that from John's to Trinity. At high speeds all the topologies have the BR metric where L = Addenbrooke's.

The Land Sharing Lemma:

Any neighbourhood in Cambridge contains a local. Hence Cambridge comes packed with locals; we deduce that Cambridge is locally compact. Adding in Girton - a point at infinity - produces a one point compactification. Any point in Cambridge lies on a piece of open land owned by one of the colleges and this generates an open cover. By compactness this cover has a finite subcover. Draw a map of this cover and apply the 'Four Color Theorem' to show that Cambridge is completely covered by land owned by the four colleges Trinity, John's, King's and Clare.

We come now to the main result of this paper.

The Loss of Large Numbers Theorem:

A compact set is sequentially compact. Consider a sequence of events taking place in the unit Cambridge ball. All events in a ball occur in June but converge in May Week. As May and June are disjoint we conclude that there is no convergent subsequence. Hence the unit ball is not sequentially compact and therefore not compact. Thus Cambridge is infinite dimensional. However, as Cambridge is spanned by thirty one colleges we see that:

$$\text{Infinity} \leq 31$$

Corollary: From the land sharing lemma it follows that infinity = 4.

k th Powers of the Roots of Polynomials

by P Taylor

One frequently needs to express the sum of the k^{th} powers of the roots of a polynomial equation in terms of its coefficients. The problem is in principle straightforward (it was solved by Newton) but in practice the iterative solution is tedious and error-prone. The following mnemonic, whose proof is left to the reader, allows the solution to be written down without intermediate calculation.

Example: to find $\sum_{i=1}^{n} x_i^4$ where $\{x_i\}$ are the roots of $f(x) = x^n + a_1 x^{n-1} + \dots + a_n = 0$.

		(1)	(2)	(3)	(4)
$2^{4-1} =$	8	1000	-	4	a_4
	9	1001	+	1	a_3a_1
	10	1010	+	2	a_2a_2
	11	1011	-	1	$a_2a_1a_1$
	12	1100	+	3	a_1a_3
	13	1101	-	1	$a_1a_2a_1$
	14	1110	-	2	$a_1a_1a_2$
2^4-1	15	1111	+	1	$a_1a_1a_1a_1$

Solution: $-4a_4 + a_3a_1 + 2a_2a_2 - a_2a_1a_1 + 3a_1a_3 - a_1a_2a_1 - 2a_1a_1a_2 + a_1a_1a_1a_1$

Method: (1) List the k-digit binary numbers (2) mark + or - according as the number of '1''s is even or odd (3) mark the place of the rightmost '1', (4) decode the number, replacing the sequence '$10^{(r-1)}$' ('1' followed by r-1 '0''s) by a_r . The sum of the k^{th} powers is then the sum of the resulting terms.

The Liaison Group

by P Tompkins and P Taylor

When the Archimedeans was founded, just before the war, it saw its role as to provide undergraduate mathematicians with a voice to express their opinions on the way they were being taught, and to extend their horizons beyond this University to industry and to other parts of the academic world. In its early days the Society was active in pursuit of students' interests both locally (setting up the Interfaculty Coordinating Committee) and nationally (in the Mathematical Association and the National Union of Students).

The Society seems to have lost that initial enthusiasm after the war and become rather parochial, and the business of representing undergraduate opinion to the Faculty Board has been undertaken at least since the 1960s by independent bodies. Reading through Eureka, there is remarkable regularity in the topics which seem to have been of concern - lecture notes, quality of lecturing and supervision, blackboards, questionnaires and common rooms. Maybe these are things undergraduates instinctively think ought to be problems or perhaps they are genuine and beyond human capacity to solve.

The old Liaison Committee was formed in the ebullient days of the late sixties by Colin Myerscough, a former editor of this journal and no mean mathematician. There were elections to this for senior and junior representatives but full student representation on faculty boards did not start until the mid-seventies. Some time in the early seventies it earned itself some "pocket money" by selling (you guessed it) lecture notes, but by 1981 it was all but defunct and it was proposed to make over its remaining finances to "QARCH" when, following the lack of any candidates for formal election to the Committee, it was formally wound up.

The Archimedeans, taking more to heart the Society's original objects, began to involve itself actively in promoting members' interests within the Faculty and the three Faculty Board student members were Officers or ex-Officers of the Society, Mark Muffet (Trinity), Richard Taylor (Clare) and Peter Tompkins (Trinity). An open meeting was held by the Society and it was decided to replace the Committee with an open Liaison Group. It mooted suggestions of holding coffee meetings early in the Michaelmas Term in each college to advise freshmen on whose lectures to attend and offer such informal help as their second and third year colleagues could provide. This operated in a few colleges and had been tried by Ian White in Trinity the previous year with some success.

The main success, however, of the Liaison Group in its first year of existence was the Course Summaries, descriptions of lecture courses in 3 or 4 pages, giving the theorems, formulae and motivation of each. Full lecture notes, it was argued, had been tried before and were expensive and required considerable work. It was aimed to provide these for the first term's courses and, following an extremely popular reception when they were distributed free of charge in Michaelmas, a small charge was made to cover production costs in Lent and we now have an almost complete set of summaries for all of the IA and IB courses. It is aimed to replace the reading lists sent out in March before matriculation with our IA summaries. Particular thanks must go to Ian White and Shane Voss (Trinity) for the coordination of production.

The qualification for lecturing in the University, despite the complaints of decades of undergraduates, is still academic rather than teaching ability. Consequently, although much lecturing is of exceptional standard, a few lecturers still show little awareness of their audiences. The problems of feedback from students to lecturers were given much thought and the time and organisation involved and ill-feeling generated by the last Liaison Committee questionnnaire on lecturers when the results were published decided against another student-run one. Instead, a circular was distributed encouraging lecturers to use questionnaires for themselves. Unfortunately, the response to this was poor and the matter has to be reconsidered. A problem is the text of the one available for the use of lecturers, which is bland and (unintentionally) discourages any meaningful response; indeed it tends rather to invite sarcasm. The response is poor and by no means all lecturers use it (notably those who most need it don't) and confidence in the system is low on both sides. The subject comes up repeatedly at Liaison Group meetings but anyone who has tried to use a questionnaire will know how difficult it is to gain anything from them.

Currently we are trying, without apparent success, to obtain use of a room to be used as a general common room between lectures. This would be of practical value to those from distant colleges but it is hoped that it could provide a focus for mathematical discussion between students from different colleges and also perhaps as "neutral ground" for members of the two departments.

A major questionnaire was distributed to ascertain the feeling of students on the suggestion raised on the Faculty Board that calculators be permitted in Tripos examinations. Apart from the interesting social trends this produced, showing much greater ownership and agreement to their use amongst first-years, this demonstrated that whilst a majority were prepared to allow their introduction, almost nobody wanted questions to require their use and the Faculty Board subsequently agreed to the introduction for the first time in 1983 without changing the style of questions. The well-worn topic of formula sheets for examinations has been resurrected and work continues on trying to see if there is a case for them.

Our meetings, usually in the Committee Room in DPMMS, are open to all and are advertised through the Newsletter and the Lecture-Card system. We should very much like to hear comments on any of the course summaries and should also be pleased to hear from readers of Eureka who have been involved in such activities in the past and might be able to advise us on their successes and failures.

The work done this year should provide a strong basis for the future.

Problems Drive 1982

Questions by R Pennington and N Inglis

1. A k-digit number n has property (*) if and only if the last k digits of n^2 are equal to n, i.e. if and only if $n^2 - n$ is divisible by 10^k. Find all 4-digit numbers with property (*). How many 5-digit numbers are there with property (*)? How many 6-digit numbers have property (*)?

2. All mathematicians may be divided into 'pure' and 'applied' mathematicians; similarly they may be divided into 'sane' and 'insane', where

   pure mathematicians tell the truth about their beliefs,

   applied mathematicians lie about their beliefs,

   sane mathematicians' beliefs are correct, and

   insane mathematicians' beliefs are incorrect.

   A conversation between four mathematicians runs as follows:

   A: I am insane.

   B: I am pure.

   C: I am applied.

   D: I am sane.

   A: C is pure.

   B: D is insane.

   C: B is applied.

   D: C is sane. Describe the four mathematicians.

3. Find the 8th-degree equation whose roots are the fourth powers of the roots of the equation:

$$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 = 0 .$$

4. Mr. Clogger and his sons and grandsons form a cricket team of eleven players. Each of Mr. Clogger's sons has at least one son. The no. 1 is the father of K.W.R., and M.J.K. is no. 6's uncle. F.S. and no. 9 are brothers. No. 2 is the father of G. No. 8 is L.'s son. A.P.E. and no. 11 are brothers. I.T. and no. 10 are cousins. No. 4 is L.E.G.'s son. I.V.A. is no. 5's nephew. No. 7 is the father of J.W.H.T., and no. 3 is W.G.'s son. Given that nobody bats at a higher number than anybody with more initials, what is K.W.R. Clogger's batting number?

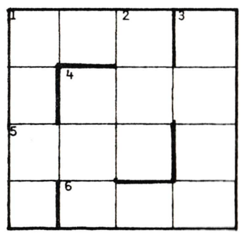
5. The race track in the capital of Archimedea was built as follows: a regular heptagon of side 100 Archimedean cubits was constructed, and the inscribed and circumscribed circles formed the inner and outer edges of the track.

The track was then surfaced at a cost of $40/\pi$ eurekas per square cubit. How much did the surfacing cost?

$(\text{cosec}(\pi/7) = 2.305+)$

6. Complete the cross-number problem.

No answer has a zero initial digit.



ACROSS	DOWN
1. Twice 4 down.	1. 3 down plus 4 across.
4. Two less than twice 2 down.	2. See 4 across.
5. One ninth of 3 down.	3. See 5 across.
6. See 4 down.	4. Three times 6 across.

7. Show, by means of proof or counterexample, whether all plane quadrilaterals can be tessellated with themselves so as to cover the entire plane.

8. Find as many sets as possible of three consecutive positive integers less than 10000, each divisible by the cube of an integer greater than 1.

9. This question was circulated in glorious full colour, not reproduced here.

C				F	I				L	O				R
	3	3	4			2	5	1			4	6	4	
	2	6	1			6	5	6			3	1	3	
	3	4	6			1	2	5			4	4	2	
B				E	H				K	N				Q
	5	2	2			2	1	1			5	5	6	
	2	2	1			5	3	5			1	4	4	
	1	3	6			3	4	5			6	6	3	
A				D	G				J	M				P

1 = blue
2 = green
3 = orange
4 = red
5 = white
6 = yellow

The diagrams show three pairs of faces from a dismantled Rubik cube. Reassemble it. (It suffices to list the corners to be joined, e.g. X=Y, etc.)

10. A magic product square is a square array of distinct positive integers which has the property that the products of all the elements in

(i) any one row,

(ii) any one column, and

(iii) either main diagonal,

are separately equal to a constant, the row product N .

Find the 3×3 magic product square with the smallest row product.

11. A battered newspaper cutting contains the following league table for the 'Home Internationals' football league, 19 ..

Team	Played	Won	Drawn	Lost	Goals for	Goals against	Points
.	.	.	1	.	4	.	.
.	.	.	2	.	2	1	.
.	.	.	.	.	3	4	.
.	.	.	0	.	1	5	.

Dots represent missing information. The accompanying article gives some extra facts:

(i) Two points were awarded for a win and one for a draw,

(ii) England and Northern Ireland each conceded exactly one goal in the first half of every match they played,

(iii) No team scored three or more goals in a match, and only Scotland won a match or matches by a margin of more than one goal,

(iv) England's equaliser against Wales was rather fortuitous,

and (v) In the completed competition, each team would play (or have played) each other team exactly once.

Fill in as much of the missing information as possible.

12. An Archimedean with less than E100 to his name withdraws all his money from his bank account one day in order to buy an expensive gift. Unfortunately, the cashier accidentally reverses the number before giving him the money (so if he had had E25.36 in his account, he'd have been given E63.52). Sadly, the relevant shop is closed that day, so he deposits the remainder of the money (after spending 97q on transport) in his account.

The following day, he withdraws it again. The cashier makes the same mistake as before, and after paying 97q for transport again, the Archimedean spends half of his remaining cash on the gift.

He then deposits the remainder in his account, which now contains as much as it did the previous morning.

How much was the gift?

(The Archimedean currency is the eureka, divided into 100 qarches; E1.00 = 100q)

"Fox and Hounds" or "Kiddies' Körner"

by R Thetford

Many of you will no doubt be familiar with a board game known as "fox and hounds" or "fox and geese". For those who aren't, I shall review the rules briefly. (Throughout what follows, "anywhere", "wherever", etc refer to black squares only).

The Simple Game

Two players, white (W) and black (B) face one another across a chess board. W has four pieces (hounds), initially placed on his back rank (on the four black squares); B has one piece (a fox), placed wherever he likes on his back rank.

The hounds can move like ordinary draughtsmen (i.e. one square diagonally forwards). The fox can move like a king in draughts, i.e. one square diagonally in any direction. B has first move. No player may miss a turn; no two pieces may occupy the same square; there is no capturing.

W's object is to trap the fox by surrounding it, so that it cannot move. (Using the edge of the board, if necessary). B's object is to break through the cordon of hounds, and get his fox to the far end of the board (i.e. to W's back rank).

With a little study, this can be shown to be a certain win for W , provided that he plays sensibly.

The Harder Game

A rather more interesting game is to have two foxes and six hounds. (Conveniently, white pawns and black bishops from a chess set). Play is once again confined to the black squares, but we need a few extra rules:

(a) B may distribute W's pieces anywhere on W's back two ranks, and his own two foxes anywhere on his own back rank. B moves first, again.

(b) B need only get one fox to W's back rank to win.

(c) W , on the other hand, must trap both foxes.

(d) Any trapped fox is immediately removed from the board.

There are some variations on rule (d):

(i) The fox must be surrounded by hounds (or the edge of the board)

(ii) The fox may be surrounded by pieces of any type

(iii) A fox is deemed to be trapped only if it cannot move, and it is B's turn.

Version (ii) seems to provide the most interesting games. It is possible to reach positions like:
(H = hound, F = fox)
where it is B's move:
One fox is forced to deliver the coup de grace to the other! (Which is immediately removed from the board, and then W moves).

H				H
	F		F	
H		H		H

This game is rather more tricky than the "simple game", and it is very easy to lose concentration even when you think you have a winning strategy.

It is of course possible to put even more pieces on the board (7 vs. 3 is the next interesting case), and with four foxes, a stalemate is possible (W to move and unable to). (This can arise in the other games only if either the fox is trapped, or it has broken through the line of hounds but has not yet reached the back rank - B wins in this case). Some other variants: try including varying numbers of super-foxes or super-hounds, with powers such as being able to move two squares on every n^{th} move, for some suitable n , or else being allowed to capture. One final, lighthearted, suggestion: try playing two simultaneous games, one with the roles reversed, taking place on the white squares!

Enhancing Series

by C N Corfield

For many problems the answer is a series, say $\Sigma c_k z^k$. If it is a typical tripos problem you stop there, but if you need some numbers your work has just begun. Suppose you end up with the series

$$f(z) = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 \;....$$

What is $f(1)$?

$$f(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \;....$$

All you have to do is to sit down with your calculator for a few days and start adding and adding and adding .... In case you have forgotten you wanted 6 figure accuracy which requires several hundred thousand terms (even the legendary Pennington would probably (?) baulk at this). But don't give up! Here are two easy techniques to enhance the convergence of the series.

(1) Shanks method

Let $S_n = \overset{n}{\Sigma} c_k z^k$. We now suppose the S_n are approximately in a geometric progression, so that if $S_n = A + BC^n$ and $|C| < 1$ then $A = \overset{\infty}{\Sigma} c_k z^k$. In our example $z = 1$ $c_{2k+1} = \frac{(-1)^k}{2k+1}$. We can estimate A by writing

$$A = \frac{S_n^2 - S_{n-1}S_{n+1}}{2S_n - (S_{n-1} + S_{n+1})}$$

(To see this just plug in $S_n = A + BC^n$ into the formula). Now let $S_n = 1 - 1/3 + 1/5 \;...\; + \frac{(-1)^n}{2n+1}$ and refine A as follows:

Sn	F(Sn)	FF(Sn)	FFF(Sn)
1			
0.6666667	0.7916667		
0.8666667	0.7833333	0.7855263	
0.7238095	0.7863095	0.7853625	0.7853998
0.8349206	0.7849206	0.7854108	
0.7440115	0.7856782		
0.8209346			

Now for the surprise. The correct answer is $f(1) = \pi/4$ since $f(z) = \tan^{-1}z$. Multiply the last estimate by 4 to get $4 \times 0.7853998 = 3.1415992$ which is π to six figures! Not bad from a mere seven terms of the original series

(2) <u>Padé approximants</u>

We try and write $c_k z^k = \dfrac{\sum^M a_k z^k}{\sum^N b_k z^k}$ i.e. the ratio of two polynomials of arbitrary (but fixed) degrees M and N. To get a_k, b_k just multiply up and put

$$(\sum^N b_k z^k)(\ c_k z^k) = \sum^M a_k z^k$$

and truncate the LHS at z^{n+M}. By equating coefficients and putting $b_o = 1$ we have $M+N+1$ equations for $M+N+1$ unknowns. For example, with $M = 5$, $N = 4$, we put

$$z - \frac{1}{3}z^3 + \frac{1}{5}z^5 + \ldots = z\ \left(\frac{1 + a_2 z^2 + a_4 z^4}{1 + b_2 z^2 + b_4 z^4} \right)$$

Here we have cut corners by taking out a factor of z and noting that $a_1 = a_3 = b_1 = b_3 = 0$. Now multiply up:

$$(1 + b_2 z^2 + b_4 z^4)(1 - \frac{1}{3}z^2 + \frac{1}{5}z^4 - \frac{1}{7}z^6 + \frac{1}{9}z^8 \ldots) = 1 + a_2 z^2 + a_4 z^4$$

$z^0 : 1 = 1$

$z^2 : b_2 - 1/3 = a_2$

$z^4 : b_4 - \frac{1}{3}b_2 + \frac{1}{5} = a_4$

$z^6 : -1/3b_4 + 1/5b_2 - 1/7 = 0$

$z^8 : 1/5b_4 - 1/7b_2 + 1/9 = 0$

Solve these to get $a_2 = 7/9$, $a_4 = \frac{64}{945}$, $b_2 = \frac{10}{9}$, $b_4 = \frac{5}{21}$ and so

$$z - \frac{1}{3}z^3 + \ldots \simeq z \left(\frac{1 + 7/9z^2 + \frac{64}{945} z^4}{1 + 10/9z^2 + 5/21z^4} \right)$$

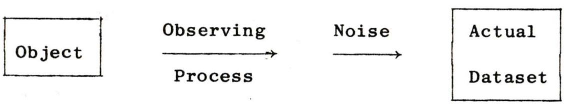
Now put $z = 1$ in the approximant to get 0.7855856 compared with $\pi/4 =$ 0.7853982 i.e. an error of $< 2.10^{-4}$, which is a good result. Straight adding of the series would need several thousand terms to do as well as we have using the first five!

Image Processing by Maximum Entropy

by J Skilling

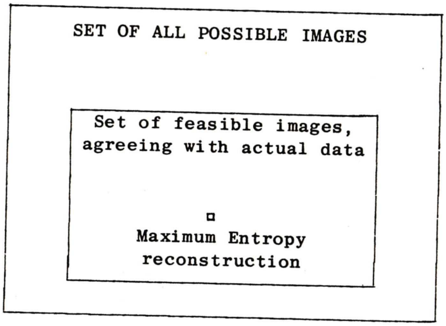
The Maximum Entropy technique extracts exactly that amount of information about an image that is reliably obtainable from given experimental or observational data. It gives, in this sense, an optimal reconstructed image.

Usually the data one has of an object are incomplete and noisy. For example, a blurred photograph fails to reproduce rapidly varying structure within an object, and also suffers from noise in the photographic recording process. Again, in interferometry one measures only some of the Fourier components of the image, to some finite accuracy.



Many conventional image reconstruction techniques attempt to use an inverse filter, to undo directly the blurring or other degradation imposed by the observing process. Invariably, inverse filters are noise amplifiers, often to an alarming degree, and this behaviour has to be controlled explicitly.

Maximum Entropy adopts a deeper approach. First, the set of all possible images is considered: this is a multi-dimensional set with one degree of freedom for each pixel of the image. (A pixel is a "picture element". A photograph may be divided into, say, 256 x 256 elements; the one degree of freedom is the intensity of the image in that pixel.) To each of these images, there corresponds a (simulated) dataset which the observing system would have produced (without noise) of that image. Normally most of these simulated datasets are wildly incompatible with the actual dataset, but some will agree with it within the experimental errors. These latter datasets define the feasible images. Usually, this set of feasible images is still inadmissibly large.

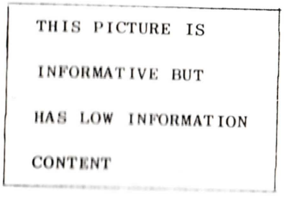


Maximum Entropy affords a general criterion for selecting one particular feasible image. To use it, one defines the entropy of the image as:

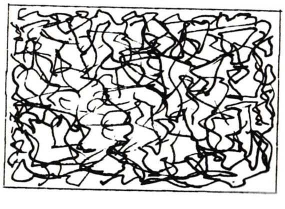
$$S = - \Sigma p_i \log p_i$$

$$(p_i = f_i / \Sigma f = \text{proportion of total flux in pixel } i)$$

which is just (minus) the information content. One then selects that particular feasible image which has maximum entropy (i.e. minimum information content). Note that a high information content does not imply that an image is highly informative:



Low information content



High information content

Maximum Entropy gives the following advantages:

1) There must be evidence in the data for any structures which appear in the reconstructed image, since no less-structured image can fit the data. This makes the reconstruction uniquely safe.

2) The intensities f in the pixels of the image are automatically positive.

3) Noise is smoothed from both bright and faint regions of the image.

4) With good data, the optimal reliable amount of super-resolution is attained.

5) Reconstruction artefacts such as sidelobes and diffraction patterns are suppressed.

6) Outliers in the data (perhaps due to experimental error) can easily be picked out and dealt with appropriately.

7) Having minimum information content, the maximum entropy reconstruction is uniquely easy to comprehend, i.e. informative.

It has also proved possible to use the positivity constraint inherent in maximum entropy to recalibrate observational equipment after the event, and thus to "bootstrap" the entire reconstruction process.

An efficient and robust algorithm for maximum entropy has been developed, and is commercially available. It consists of about 2000 lines of standard Fortran, divided into about 70 subroutines to facilitate transfer to small computers, plus whatever convolution or other routines are needed to simulate a particular observing system. Since entropy is a non-linear function, the algorithm is necessarily iterative, and 10-20 iterations are normally needed. This means that the maximum entropy calculation runs 50-100 times slower than simply linear filters. The algorithm has been installed on various machines, with timings for optical deconvolutions shown below.

Maximum Entropy Computing Times
(Elapsed Times for 20 iterates)

	Programmed Size	Actual Time	Time Normalised to 512 x 512 deconvolution
IBM 3033	256 x 256	8 min*	32 min*
PDP 11/60	512 x 512	60 hr	60 hr
PDP 11/34	512 x 512	50 hr+	80 hr
PDP 11/34 + AP120B	512 x 512	2 hr	2 hr
NORD 50/10 combination	1024 x 1024	20 hr+	8 hr
VAX 11/70	512 x 512	4 hr+	5 hr

* CPU time only

+ Fourier transform data, not deconvolution.

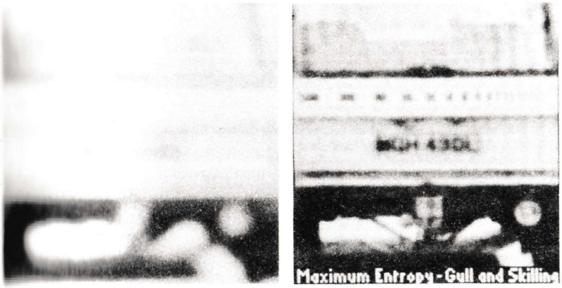
The algorithm has been applied to a wide variety of academic and commercial problems. Figures 1 and 2 below show examples of optical deconvolutions from blurred data.

Figure 2 shows how Maximum Entropy automatically gives an optimal trade-off between resolution and noise suppression: extra resolution will only be achieved if the signal/noise ratio is adequately large. Software is available for optical deconvolutions at resolutions of up to 512 x 512 pixels. Such deconvolutions have also been applied to gamma-ray, X-ray and optical astronomy.

Similar reconstructions have been applied in radionuclide emission tomography. Again, Maximum Entropy gives optimal results, even in the difficult cases of limited or sparse angles of view.

Other successful applications have been in interferometry (both very-long-baseline and conventional aperture synthesis) at resolutions up to 1024 x 1024 pixels.

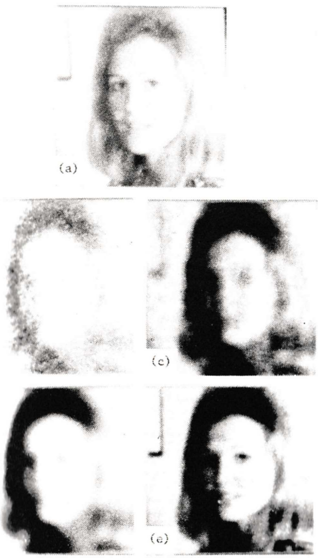
EXAMPLES OF MAXIMUM ENTROPY RECONSTRUCTION



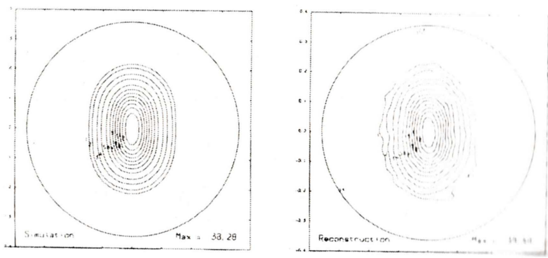
Reconstruction of photograph of
car number-plate

(left) Blurred original
(right) Maximum entropy reconstruction

Maximum entropy reconstruction from X-ray tomography data taken with an EMI brain-scanner.



(a) Test photograph of Susie.
(b) Degraded out-of-focus image;
low signal to noise.
(c) Reconstruction from (b); it is SMOOTHER than the original.
(d) Degraded image;
high signal to noise.
(e) Reconstruction from (d); it is SHARPER than the original.



Maximum entropy in plasma emission tomography. Simulated reconstruction of 5 Megawatt neutral injection beam for Joint European Torus, observed in $H\alpha$ emission by 12 transverse scanners.

Ordered Rings

by P Taylor

An <u>ordered ring</u> $(R + * >)$ is a nontrivial ring $(R + *)$, with a total ordering satisfying the conditions:

(i) $a > b \Rightarrow a+c > b+c$

(ii) $a > b \quad c > 0 \Rightarrow a*c > b*c \qquad c*a > c*b$

$a > b \quad c < 0 \Rightarrow a*c < b*c \qquad c*a < c*b$

for all $a,b,c \in R$. We observe immediately that an ordered ring is infinite, has characteristic zero and has no non-trivial zero divisors. Even if we drop condition (i) to leave a <u>semiordered ring</u> there are only three finite cases, namely $\{0\}$, $\{0,1\}$ and $\{-1,0,1\}$ with obvious relations.

It is not necessary to assume a priori that R has a 1, since we may adjoin one by defining

$$(a,n) + (b,m) = (a+b,n+m)$$

$$(a,n)*(b,m) = (a*b + nb + ma, nm)$$

on $R \times \mathbb{Z}$ and quotienting by the only nontrivial kernel, I, of multiplication by R (exercise for reader). Any nonzero monotonic ring homomorphism $\phi : R \to S$ $(1 \in S)$ may be extended to $\bar{\phi} : \bar{R} \to S$, where $\bar{R} = (R \times \mathbb{Z})/I$.

We may also assume $\mathbb{Q} \subseteq R$. For let $Z \supseteq \mathbb{Z}$ be in the centre of R, so Z is an integral domain with field of fractions K. We construct $\bar{R} = K \otimes_Z R \otimes_Z K$ and verify that $\bar{R}$ is the required ordered ring, in which we may write $k_1 k_2 r$ for $k_1 \otimes r \otimes k_2$. Moreover for $K = \mathbb{Q}$ we may extend any monotonic ring homomorphism to $\bar{R}$.

A natural topology is defined on R , as on $\mathbb{R}$, using open intervals $(a,b) = \{r \in R : a < r < b\}$. We also define a convex set , A , to be on such that, if $a, b \in A$ and $a < r < b$, then $r \in A$. Convex ideals are exactly the kernels of monotonic ring homomorphisms; moreover a nontrivial convex ideal is both open and closed, so the quotient topology on a quotient ring is discrete, which is not true of the natural topology induced by the ordering. An illustration of this is provided by $K((\chi))$ later.

It is usual [1,2] at this point to assume that R is commutative, i.e. an integral domain and (wlog) a field. Artin [1,3] showed that an integral domain can be ordered iff 0 cannot be expressed as a nontrivial sum of squares (in which case it is said to be formally real). Moreover the algebraic closure of a formally real field is of the form $K(i)$, where K is real closed (every odd degree polynomial has a root, and for all $c \in K \setminus \{0\}$, either c or $-c$ has a square root in K) and uniquely orderable. Conway [4] has given a hierarchical construction of the Field No (not a set) in which any commutative ordered ring may be embedded. We then, on the other hand, shall consider the commutative case known and shall investigate noncommutative ordered rings.

A ring is complete if every nonempty bounded set has at least upper bound, and Archimedean if, given $a,b > 0$, $\exists\, n \in \mathbb{Z}$ such that $a < nb$. By considering the set $\{1,2,...\}$, we see that every complete ring is Archimedean. From the following results we deduce that $\mathbb{R}$ and $\mathbb{Z}$ are the only complete rings: hence their standing in Analysis.

Relaxing these conditions, we say R is confined in $K \subseteq R$ (a field) if , given $r > 0$, $\exists\, m, M \in K$ such that $0 < m < r < M$. A confined ring is semicomplete if every densely bounded set ($A \subseteq R$ such that, given $K \ni \varepsilon > 0$, $\exists\, a \in A$, $b > A$ with $b-a < \varepsilon$) has a least upper bound (alternatively, every Cauchy sequence converges). Archimedean, then, means confined in $\mathbb{Q}$. We also note that a confined ring has no nontrivial convex ideal.

Any confined ring may be embedded in a semicomplete ring. A <u>section</u> is a pair of disjoint convex open sets (A,B), with A densely bounded by B and $A \cup B = R$ or $R \setminus \{c\}$. Addition, multiplication and ordering may be defined on the collection, $\bar{R}$, of sections, and R embedded in it. Moreover if $\phi : R \to S$ is a monotonic ring homomorphism with R, S confined in K and S semicomplete, ϕ may be extended to $\bar{\phi} : \bar{R} \to S$. This is essentially the Eudoxus-Dedekind construction for $\mathbb{R}$.

Given $0 < a \in R \supseteq K$, with R not necessarily confined, we introduce the notations:

$$o_K(a) = \{r \in R : \forall \varepsilon \in K, \varepsilon > 0 : -\varepsilon a < r < \varepsilon a\}$$

$$O_K(a) = \{r \in R : \forall\, k, \ell \in K : ka < r < \ell a\}$$

$$O^d_K(a) = \{r \in R : \forall \varepsilon \in K, \varepsilon > 0 \quad \exists\, k, \ell \in K : ka < r < \ell a \;\; |\ell - K| < \varepsilon\}$$

so that $o_K(a)$ is a convex ideal of both ${}_K(a)$ and ${}^d_K(a)$.

$$[a]_K = O_K(a)/o_K(a) \qquad [a]^d_K = O^d_K(a)/o_K(a)$$

Moreover $[a]^d_K$ is naturally embedded in $\bar{K}$, the semicompletion of K. In other words, there is a unique K-module homomorphism $\theta_a : O^d_K(a) \to \bar{K}$ with kernel $o_K(a)$ such that $a\theta_a = 1$. We write $\frac{x}{a}$ for $x\theta_a$ and note that $\frac{xy}{ab} = \frac{x}{a}\frac{y}{b}$, so θ_1 is a monotonic ring homomorphism.

In particular if R is Archimedean, so $K = \mathbb{Q}$, $o_K(a) = 0$ and $O^d_K(a) = O_K(a) = R$ for all $a > 0$, θ_1 embeds R in $\mathbb{R}$. Hence an Archimedean ring is commutative, and the uniqueness of $\mathbb{R}$ as the complete ordered field follows.

If K is an ordered field, we construct the <u>field of rational functions</u> over K as the field of fractions of the ring of polynomials, $K[x]$. Rational functions over K may be expressed as formal power series, and ordered in the obvious typographical fashion. The semicompletion of this field is $K((x))$, the set of all such power series, where the coefficients of powers of x may be any elements of K without restriction. The reason why K is good enough as a field of scalars and its semicompletion does not arise is that the topology of K as a subfield of $K((x))$ is discrete, and $K((x))$ is totally disconnected.

Finally we turn to the algebra of ordered rings, for which we shall need complex numbers. Of course $\mathbb{C}$ cannot be ordered, but nevertheless has many properties arising from the ordering on $\mathbb{R}$. In general, given an ordered ring $(R, +, *, >)$, we construct a complex ordered ring $(C, R, +, *, >)$, where $(C + * >)$ is a ring isomorphic to $R \times R$ with operations

$$(a,b) + (c,d) = (a+c,\ b+d)$$

$$(a,b)*(c,d) = (ac - bd,\ bc + ad)\ .$$

All of what has been said about ordered rings carries over mutatis mutandis to complex ordered rings: they have, in particular, characteristic zero and no non-trivial zero divisors, so all field extensions will be separable (see, e.g., [5]). Taking up the usual terminology from $\mathbb{C}$, if we consider the signs of the real and imaginary parts of $x,y,z \in R$ we find that $xy = yz$ implies $y = 0$ if x,z are in opposite quadrants. Indeed if the 'angle' between x,z is finite the same holds by considering powers.

Let R be an ordered or complex ordered ring containing a field K centrally, and let L be a normal algebraic field extension of K, with $R \supseteq L \supseteq K$. Then L is central. For the proof, wlog $L:K$ is finite.

If R is any entire K-algebra and $L:K$ is a finite normal separable extension with $R \supseteq L \supseteq K$ then we have a decomposition

$$R = \bigoplus_{\phi \in G} U_\phi$$

where $G = \Gamma(L:K)$ is the Galois group and $u \in U_\phi$ has the property, that $xu = u\phi(x)$ for $x \in L$. [In fact $E = \{\phi \in G : U_\phi \neq 0\}$ is a subgroup of G, corresponding by Galois' theorem to $L \cap Z(R)$, so if $Z(R) = K$, $E = G$]. We show that if R is ordered or complex-ordered then $E = 1$, so L is central.

$L:K$ is a splitting field extension for some polynomial, on whose roots G acts as a permutation group. Let $\phi \in E \setminus \{1\}$ act cyclically on $\{x_1, x_2, \ldots, x_r\}$, some set of roots and put $s = (x_1 + \ldots + x_n)/n$. Let $u \in U_\phi \setminus \{0\}$ and $y_i = s - x_i$, so $\Sigma y_i = 0$. Thus ϕ also acts cyclically on the y_i $(su = us)$, some two of which are in opposite quadrants, so $y_i u^{j-i} = u^{j-i} y_j$. But then $u^{j-i} = 0$.

We conclude with an alternative proof that algebraic numbers over $\mathbb{Q}$ are central, which does not require normal closures, but instead uses the analytical machinery developed earlier.

Let $u, x \in R$ satisfying $f(x) = 0$ over $\mathbb{Q}$ with f minimal and suppose $\varepsilon = xu - ux > 0$. Then $x^k u - ux^k = x^{k-1}\varepsilon + x^{k-2}\varepsilon x + \ldots + \varepsilon x^{k-1}$ and so $(x^k u - ux^k)\theta_\varepsilon = kx^{k-1}$. Hence $0 = [f(x)u - uf(x)]\theta_\varepsilon = f'(x)$, the formal derivative. But then $f'(x) = 0$ is a polynomial equation of lower degree than f satisfied by x, which is a contradiction. We conclude that θ_ε does not exist, so $\varepsilon = 0$.

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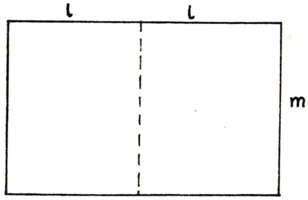
The Kit-Kat

by D McLean

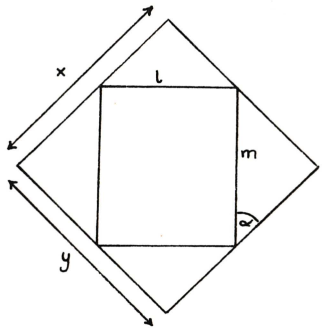
The Kit-Kat is an exceedingly beautiful mathematical model, admired in the better college bars for the way in which it is wrapped. Consider the application of a planar surface to (initially) a rectangular box.

For those who aren't already afficionados of this particular form of nourishment (perhaps because it is a weaker economic indicator than the Mars Bar) it must be explained that the Kit-Kat is wrapped by placing it almost diagonally across a piece of silver foil and folding in all four corners. The beauty of this is in the economic application in that it uses vastly less silver foil than were the bar initially placed parallel to the edges of the foil.

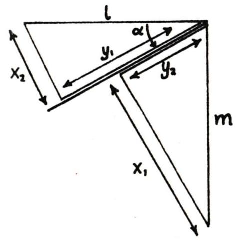
This offers a number of mathematical puzzles at different levels. The simplest case is to take a rectangle and to attempt to wrap it in foil.



A) A moment's thought offers an obvious solution: take a piece of foil twice one of the dimensions and fold in half.



B) An alternative is the simple diagonal method. This can be treated with varying degrees of complexity but for simplicity's sake, consider diagonal symmetry.

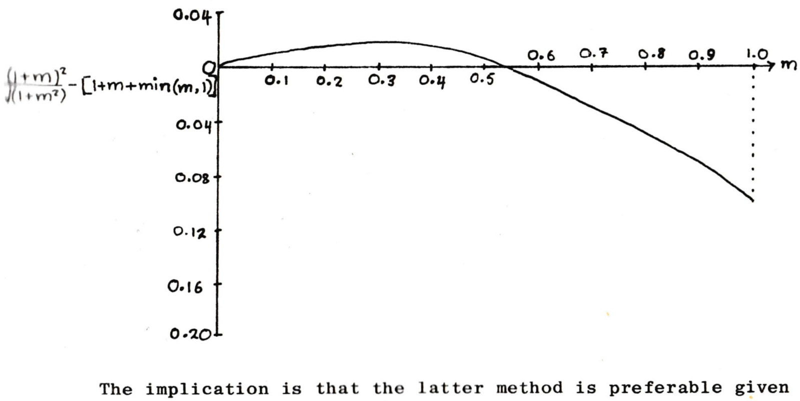


Here, $x_1 + x_2 = x \quad y_1 + y_2 = y$, of course. As α is free, it seems reasonable to set it such that there is no overlapping. i.e. either $y_1 = y_2$ or $x_1 = x_2$. Let $y_1 = y_2$ wlog. Now $(x_1 + x_2)$ is a diagonal. $x = \sqrt{\ell^2 + m^2}$.

Clearly $y = 2\ell m/\sqrt{\ell^2 + m^2}$ since $xy = 2\ell m$.

This has, however, cheapened the problem because to prevent a ruckled edge causing the inner rectangle to be visible we need add an extra element around the edge.

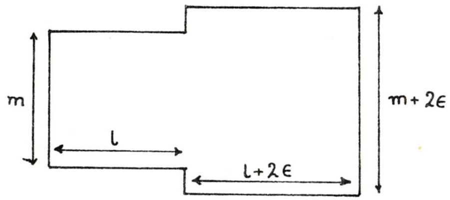
Neglecting 2nd order extra terms it can be seen that the extra required element for the first method varies as $[\ell + m + \min\{\ell, m\}]$ by folding the appropriate way. The second method requires an extra element varying as $x + y = \ell\left[(1 + \frac{m}{\ell})^2 / \sqrt{1 + (\frac{m}{\ell})^2}\right]$. Set $\ell = 1$ and $m \in (0,1]$ as the value for $m > 1$ is the same as that for $\frac{1}{m}$.



The implication is that the latter method is preferable given that the bar is a square or within sufficiently good approximation of one.

But, considering ε^2's (ε being minimal acceptable overlap) it is apparent that the first method requires wastage of $2\varepsilon^2$ whereas the latter requires wastage of only ε^2. A sufficiently generous choice of ε would make an increasing proportion of possible ratios of sides prefer the latter method.

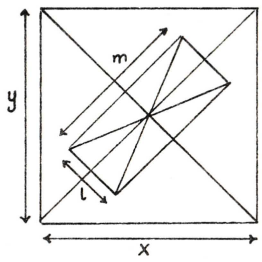
Also (A) requires more intricate cutting which is uneconomical.



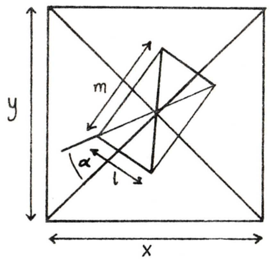
So far the only mathematics required is elementary trigonometry and the ability to draw a simple graph. Now we need (not strictly, but it's a lot easier) a little co-ordinate geometry.

Extend the problem to folding a planar sheet around a rectangular box. Intuitively the face covered first ought to be one of the largest two, but even given this assumption there is a clear analog of (A) above but for our alternative equivalent to (B) these are several possible starting points.

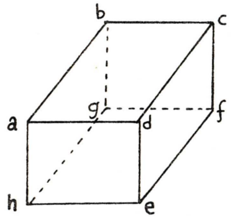
(i) Perhaps the most appealing (and certainly the easiest) is to assume that the diagonals of the wrapper bisect the angles made by the intersection of the diagonals of the largest face of the box.



(ii) Considering how easily our end overlaps can mess up this sort of calculation, it may be easier to assume that the intersections of the diagonals coincide but are off-set by some angle α, say.



(iii) The truly general solution. Map point a to (x_1, y_1), b to $(x_1 + m\cos\alpha,\ y_1 + m\sin\alpha)$ etc. and evaluate the product xy as a function of $x_1, y_1, \alpha, \ell, m, n,$ and ε.



Then for each set ℓ, m, n, ε we can minimise xy to find optimal x_1, y_1, α. Even here, applying the box to the cover, a particular side first, loses generality; as does folding the sides in a particular order. However permuting ℓ,m,n overcomes this problem.

Having completed the exercises implied above, don't relax too quickly. A Kit-Kat is not the shape of a rectangular box.

Solutions to Problems Drive

1. 9376; 1 five-digit number (90625), and 2 six-digit numbers (890625 and 109376).

2. A applied, insane

   B pure, sane

   C pure, insane

   D pure, insane

3. $x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 = 0$ or a multiple thereof. Yes, that is the original equation!

4. 6.

5. E100000 (10^5 eurekas).

6.

8	8	2	8
8	4	0	4
9	4	3	8
1	1	4	7

7. Tessellability may be proved by considering rotations by π about the midpoints of sides and translations by the diagonals of the quadrilateral (etc.).

8.

1375	1376	1377
4374	4375	4376
4912	4913	4914
5750	5751	5752
6858	6859	6860

9. A=Q, B=R, C=I=O, D=J=P, E=G, F=H, K=M, L=N.

10.

2	36	3
9	6	4
12	1	18

or a rotation or reflection thereof.

11.

Team	P	W	D	L	F	A	Pts
Scotland	3	2	1	0	4	0	5
Wales	3	1	2	0	2	1	4
England	3	1	1	1	3	4	3
N. Ireland	3	0	0	3	1	5	0

12. E28.93.

The Society

1981-2 was another successful year for the society, with around 70 events organised either by the Archimedeans or by the college societies. Membership stands at over 500 and interest and involvement have been great. The year began with the introduction of a new constituion, the main effect of which was to reduce the committee to an executive of six, which has proved much more convenient in running the Society

Evening meetings continued to be addressed by speakers from far and wide. They included Professor Formanek from the States, Dr Portous from Liverpool, Dr Woodall from Nottingham and Professor Zeeman from Warwick. There were lively gatherings for coffee after all these meetings where there was further discussion of the talk with the speaker. There was also the Careers Evening, organised with the ever helpful Careers Service and an evening of topological films. The lunchtime meetings moved to the Chetwynd Room in King's, which allowed for the improvement of the snack lunch provided. The tradition of having speakers from outside our own faculty continued and the talks varied from Dr Lear on "Wittgenstein's philosophy of mathematics" through Professor Hahn on "Why mathematics could save us from Mrs Thatcher" to Mr Quadling on "2,4,8,...". The college societies have continued to provide a wide range of meetings, with the TMS and the Adams Society having some outside speakers.

There were improved attendances, especially in the Michaelmas term. This was probably largely due to the newsletter, which was produced on a regular basis this year and has carried a wide variety of articles. New posters, bearing the distictive symbol which appears on the "tomb" of Archimedes, were produced and the lecture card system put on a firm footing. "QARCH" has continued to be produced.

The Society's social events were wide and varied. The Triennial Dinner took place this year and we were addressed by Dr Edward de Bono, Dr John Polkinghorne for the last time before he enters the church and once again by Professor Christopher Zeeman. Greater contact (including more common events) was made with the Invariants of Oxford and with the mathematical societies at King's College London and at Warwick. All three took part in a remarkably successful "Call My Bluff" in which bluff proofs were given to obscure theorems. Other social events included the Problems Drive, a puzzles and games afternoon in Oxford, a trip to Granchester by punt and the ever popular punt joust against the Dampers, with about 30 of us in the Cam watched by a large crowd on the bank.

The standard facilities have continued to be provided: the bookshop, access to the university computer, the telephone answering service for exam results and reference to many publications the Society receives in exchange for "Eureka".

Finally, Dr Taunt must be thanked for all the help he has given the Society as Senior Treasurer in this, his last year in that post. He has throughout over twenty years in the position given his advice to the Society with great patience and audited the accounts with extraordinary care and precision.