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$a^n b^n = (ab)^n$

by Professor I.N. Herstein, University of Chicago

If G is an abelian group then it is an utter triviality that for every integer n and every $a,b \in G$, $(ab)^n = a^n b^n$. On the other hand, it is quite easy to give examples of non-abelian groups in which, for some fixed n, $(ab)^n = a^n b^n$ for all pairs of elements a and b in the group. For instance, if G is the non-abelian group of order 6 then $(ab)^6 = a^6 b^6$ for all $a,b \in G$ since $x^6 = e$, the identity element of G, for all $x \in G$.

It seems a fairly natural question to ask: how can one describe the groups G in which, for some fixed n, $(ab)^n = a^n b^n$ for all $a,b \in G$? Although the question was first asked a long time ago, and a considerable amount had been written about the matter, a clear cut, definitive answer was given only fairly recently. This was done by Alperin in [1] and Kaluznin in [4].

Why limit the question to groups ? What can one say about the structure of an associative ring in which, for some fixed $n \geq 2$, $(ab)^n = a^n b^n$ for all $a,b \in R$? Here, too, it is possible to give a rather sharp, definitive answer; this was done in [2]. However, unlike the group situation which turns out to be formal and elementary, the case of rings requires some deep ring-theoretic results in its solution. This is not too surprising, for any description of such rings would imply Wedderburn's theorem that any finite division ring is a field. For, in such a division ring having n elements, $a^n = a$ for all a, hence $(ab)^n = a^n b^n$ for all a,b.

We shall address ourselves to these things here. For groups we shall give all the details of the solution. For rings we shall merely state the results, without proof; to go into the proof would require too detailed a development of a sement of ring theory than would be appropriate.

We begin with the story for groups. Clearly $n = 0$ and $n = 1$ are cases which are universally satisfied by all groups G, and give no limitation on the nature of G. We shall therefore suppose that $n \neq 0,1$; n may be positive or negative.

Before starting on this let us consider a problem in our text <u>Topics in Algebra</u> (Problem 4 on p.35 of [3]). Let G be a group and suppose that for some $a,b \in G$, $(ab)^i = a^i b^i$ for three consecutive integers i; then $ab = ba$. We show this now; note that if we insist that $i \geq 0$ then the proof is valid in semigroups with cancellation.

So, suppose that $(ab)^i = a^i b^i$ for three consecutive integers j, j+1, j+2. Thus $a^{j+1}b^{j+1} = (ab)^{j+1} = (ab)^j ab = a^j b^j ab$, cancelling a^j on the left and b on the right yields $ab^j = b^j a$. From our hypothesis, however, this argument is also valid for $i = j+1$; thus we get $ab^{j+1} = b^{j+1}a$. Because $ab^{j+1} = b^{j+1}a = bb^j a = bab^j$, we end up with the required conclusion that $ab = ba$.

It might be of some interest to characterize the triples of integers k, m, n such that whenever $(ab)^k = a^k b^k$, $(ab)^m = a^m b^m$, and $(ab)^n = a^n b^n$ then $ab = ba$ is implied. Some attempts have been made at this but, to the best of our knowledge, no really sharp result has been obtained.

We turn now to the description of n-abelian groups, that is groups in G in which $(ab)^n = a^n b^n$ for all $a,b \in G$. We assume $n \neq 0,1$.

Before trying to prove the result let us see some obvious examples where the condition holds true. To begin with, as we mentioned at the outset, it holds in all abelian groups; call these of type A. A rather obvious class of examples comes from groups in which $a^n = e$, the identity element for all a; call these of type B. Finally, a third class of obvious examples comes from groups in which $a^n = a$ (and so $a^{n-1} = e$) for all a; call these of type C.

If we take the direct product X of a group of type A, of type B, and of type C then clearly the identity $(ab)^n = a^n b^n$ holds in X. In fact, this identity holds for any subgroup of X. Even more, this identity is valid for any homomorphic image of a subgroup of X.

In this way, by the simplest of constructions, one builds up a rather large, but seemingly special, family of examples of n-abelian groups. The remarkable thing is that every n-abelian group arises in this way. To be more precise, given an n-abelian group G we can find groups of type A, B, and C and a subgroup H of their direct product such that G is the homomorphic image of H. This is what we intend to prove now. We follow the very beautiful treatment given by Alperin.

In what follows G will be an n-abelian group.

Lemma 1 If $a,b \in G$ then

(i) $(ab)^{n-1} = b^{n-1}a^{n-1}$,

(ii) $a^n b^{n-1} = b^{n-1} a^n$, and

(iii) $(aba^{-1}b^{-1})^{n(n-1)} = e$.

Proof. We prove the three parts in turn.

(i) Since $(ba)^n = b^n a^n$, cancelling a b on the left and an a on the right gives us the result.

(ii) By (i), $(ab)^{n-1} = b^{n-1}a^{n-1}$, hence $a^n b^n = (ab)^n = (ab)^{n-1}ab = b^{n-1}a^{n-1}ab = b^{n-1}a^n b$. Cancelling a b on the right yields the required result.

(iii) $(aba^{-1}b^{-1})^{n(n-1)} = \{((aba^{-1})b^{-1})^n\}^{n-1} = ((aba^{-1})^n b^{-n})^{n-1}$ (since G is n-abelian) $= (ab^n a^{-1} b^{-n})^{n-1} = (a(b^n a^{-1} b^{-n}))^{n-1} = (b^n a^{-1} b^{-n})^{n-1} a^{n-1}$ (by (i)) $= b^n a^{1-n} b^{-n} a^{n-1} = e$ by (ii).

Let $G^n = \{x^n;\ x \in G\}$ and $G^{n-1} = \{x^{n-1};\ x \in G\}$. Since G is n-abelian it follows that G^n is closed under multiplication, and because $(x^{-1})^n = x^{-n}$ we have that G^n is closed with respect to taking inverses. So G^n is a subgroup of G. Using part (i) of the lemma we see that G^{n-1} is closed under multiplication, and since it is trivially closed under the taking of inverses, G^{n-1} is also a subgroup of G. Since $(xyx^{-1})^i = xy^i x^{-1}$, we have that G^n and G^{n-1} are normal in G. Hence the

Corollary. G^n and G^{n-1} are normal subgroups of G.

Recall that the commutator subgroup, G', of G is the subgroup of G generated by all $xyx^{-1}y^{-1}$ where $x,y \in G$. G' is the smallest normal subgroup N of G such that G/N is abelian.

Lemma 2. Suppose G' is such that G/G' has no elements of finite order. Then $G' \cap G^n \cap G^{n-1} = \{e\}$.

Proof. Let $g \in G' \cap G^n \cap G^{n-1}$; since $g \in G^n$, $g = h^n$ for some $h \in G$. We claim that $h \in G^{n-1}$; for $h = gh^{-(n-1)}$ and since $h^{-(n-1)} \in G^{n-1}$, $g \in G^{n-1}$, and G^{n-1} is a subgroup of G, we do indeed have that $h \in G^{n-1}$. Finally, because $h = y^{n-1}$ and $g = h^n$, we have that $g = y^{n(n-1)}$. In G/G', this tells us that $\bar{y}^{n(n-1)} = \bar{e}$ because $\bar{g} = \bar{e}$ since $g \in G'$. However, G/G' has no elements of finite order, so $\bar{y} = \bar{e}$. But this translates into $y \in G'$. Thus $y = (a_1b_1a_1^{-1}b_1^{-1}) \cdots (a_kb_ka_k^{-1}b_k^{-1})$ for appropriate $a_i, b_i \in G$. Therefore $g = y^{n(n-1)} = ((a_1b_1a_1^{-1}b_1^{-1})^n \cdots (a_kb_ka_k^{-1}b_k^{-1})^n)^{n-1}$ (since G is n-abelian) $= (a_kb_ka_k^{-1}b_k^{-1})^{n(n-1)} \cdots (a_1b_1^{-1}b_1^{-1})^{n(n-1)}$ (by (i) of Lemma 1) $= e$ (by (iii) of Lemma 1). Thus $g = e$ and $G' \cap G^n \cap G^{n-1} = \{e\}$.

We are now ready to prove the theorem. However we need the notion of a free group and some of its elementary properties. We refer the reader to almost any book on group theory. What we shall use are

1. Given any group G then G is the homomorphic image of a free group.
2. If F is a free group and F' its commuter subgroup then F/F' is a free abelian group and has no elements of finite order.

With these things in hand we proceed. Let G be an arbitary n-abelian group. Then G is the homomorphic image of a free group F; let T be the kernel of the homomorphism of F onto G. We know that $G \cong F/T$; moreover, if $x,y \in F$ then, since G is n-abelian, $(xy)^n y^{-n} x^{-n} \in T$ since its image in G is e. Therefore T contains the normal subgroup R generated by all $(xy)^n$ for $x,y \in F$. Let $K = F/R$; then K is n-abelian by its construction. Since $R \subset T$, by one of the basic homomorphism theorems, $(F/R)/(T/R) \cong F/T \cong G$, so that G is a homomorphic image of K. Because F/F' is abelian and hence n-abelian, $F' \supset R$. Furthermore, $K' = F'/R$ and $K/K' \cong (F/R)/(F'/R) \cong F/F'$ is free abelian and has no elements of finite order. By Lemma 2, $K' \cap K^n \cap K^{n-1} = \{e\}$ where $K^n = \{k^n; k \in K\}$ and $K^{n-1} = \{k^{n-1}; k \in K\}$. If $A = K/K'$, $B = K/K^n$ and $C = K/K^{n-1}$ then A is of type A, B is of type B, and C is of type C. Map K into $A \times B \times C$ by sending k to (kK', kK^n, kK^{n-1}); because $K' \cap K^n \cap K^{n-1} = \{e\}$, the map is a monomorphism of K into $A \times B \times C$. Let H be the image of K in $A \times B \times C$. Since G is a homomorphic image of K, G is a homomorphic image of H, a subgroup of $A \times B \times C$. Thus we have proved

Theorem 1. Given any n-abelian group G we can find groups of type A, B and C and a subgroup H of their direct product such that G is a homomorphic image of H.

If G is a finite n-abelian group then the result of Theorem

1 can be sharpened. One can show that the groups of type A, B and C in Theorem 1 can be chosen to be finite also. To get this result requires considerably more effort, and in particular, the establishment of a rather hard combinatorial theorem. We do not do this here, but we refer the interested reader to the paper by Alperin [1].

We shall now discuss analogous situations for associative rings. Let R be an associative ring in which $(ab)^n = a^n b^n$, $n \geq 2$ a fixed integer, for all $a, b \in R$. We first note that this does not force the commutativity of R. For let F be a field and

let $R = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} ; x,y,x \in F \right\}$; then R is a non-commutative ring in which $0 = (ab)^3 = a^3 b^3$ for all $a,b \in R$. In fact it is easy to check that $0 = (ab)^2 = a^2 b^2$ for all $a,b \in R$. (Curiously enough, however, if R should be a ring with unit element in which $(ab)^2 = a^2 b^2$ for all $a,b \in R$ then R must be commutative. For this, and related problems, see p.168 of [3]).

The example above, while special, indicates that the presence of lots of elements which are nilpotent, that is, elements such that $a^k = 0$ for some k, impedes driving through to the conclusion that R must be commutative. What is more relevant is the presence (or absence) of nilpotent two-sided ideals; an ideal I is said to be nilpotent if there is some integer $k > 0$ such that $a_1 a_2 \ldots a_k = 0$ for all $a_1, a_2, \ldots, a_k \in I$.

We state, without proof, what can be shown for rings R in which $(ab)^n = a^n b^n$. The proofs are not formal, but make use of some rather difficult theorems of ring theory [2].

Theorem 2 Let R be a ring in which, for some integer $n \geq 2$, $(ab)^n = a^n b^n$ for all $a,b \in R$. Then the nilpotent elements form an ideal I; moreover, R/I is commutative.

It can be shown that Theorem 2 is equivalent to

Theorem 3 Let R be a ring with no non-zero nilpotent ideals, and in which $(ab)^n = a^n b^n$, for some integer $n \geq 2$, and for all $a,b \in R$. Then R is commutative.

The essential facts that enter into the proofs of these theorems above (and of that to follow) come from the fact that R satisfies the non-trivial polynomial $p(x,y) = (xy)^n - x^n y^n$ in non-commuting variables x and y. This allows us to make use of the rich variety of theorems known for rings satisfying a polynomial identity.

While $(ab)^n = a^n b^n$ holds in all commutative rings, the identity $(a+b)^n = a^n + b^n$ holds only in very special commutative rings (many high-school students notwithstanding). One might ask: what is the structure of rings R in which, for some $n \geq 2$, $(a+b)^n = a^n + b^n$ for all $a,b \in R$? The answers are exactly as in Theorems 2 and 3; however, there is a little more that follows about the additive order of elements of R.

What if one imposes both conditions $(ab)^n = a^n b^n$ and $(a+b)^n = a^n + b^n$, for all $a,b \in R$? That is, what happens to R if the map $q: R \to R$ defined by $q(x) = x^n$ for all $x \in R$, is a ring homomorphism of R ? The answer is exactly (with some condition on the additive order of elements of R) as in Theorems 2 and 3, no more, no less.

Can one restrict the situation enough to force the commutativity of R ? The answer is yes, but the result is somewhat special, namely

Theorem 4 Let R be a ring, $n \geq 2$ an integer such that the map $q: R \to R$, defined by $q(x) = x^n$ for all $x \in R$, is a ring homomorphism of R onto itself. Then R is commutative.

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Quantum Gravity

by Professor S.W. Hawking

The interactions that one observes in the physical universe are normally divided into four categories according to their botanical characteristics. In order of strength they are, the strong nuclear forces, electromagnetism, the weak nuclear forces and, the weakest by far, gravity. The strong and weak forces act only over distances of the order of 10^{-13} cms. or less and so they were not discovered until this Century when people started to probe the structure of the nucleus. On the other hand electromagnetism and gravity are long range forces and can be readily observed. They can be formulated as classical, i.e. non quantum, theories. Gravity was first with the Newtonian theory followed by Maxwell's equations for electromagnetism in the 19th Century. However the two theories turned out to be incompatible because Newtonian gravity was invariant under the Gallilean group of transformations of inertial frames whereas Maxwell equations were invariant under the Lorentz group. The famous experiment of Michelson and Morley, which failed to detect any motion of the Earth through the lumiferous aether that would have been required to maintain Gallilean invariance, showed that physics was indeed invariant under the Lorentz group, at least, locally. It was therefore necessary to formulate a theory of gravity which had such an invariance. This was achieved by Einstein in 1915 with the General Theory of Relativity.

General Relativity has been very successful both in terms of accurate verification in the solar system and in predicting new phenomena such as black holes and the microwave background radiation. However, like classical electrodynamics, it has predicted its own downfall. The trouble arises because gravity is always attractive and because it is universal i.e. it affects everything including light. One can therefore have a situation in which there is such a concentration of matter or energy in a certain region of spacetime that the gravitational field is so strong that light cannot escape but is dragged back. According to relativity, nothing can travel faster than light, so if light is dragged back, all the matter must be confined to a region which is steadily shrinking with time. After a finite time a singularity of infinite density will occur.

General Relativity predicts that there should be a singularity in the past about 10,000 million years ago. This is taken to be the "Big-Bang", the beginning of the expansion of the Universe. The theory also predicts singularities in the gravitational collapse of stars or galactic nulcei to form black holes. At a singularity General Relativity would lose its predictive power: there are no equations to govern what goes into or comes out of a singularity. However when a theory predicts that a physical quantity should become infinite, it is generally an indication that the theory has broken down and has ceased to provide an accurate description of nature. A similar problem arose at the beginning of the Century with the model of the atom as a number of negatively charged electrons orbiting around a positively charged nucleus. According to classical electrodynamics, the electrons would emit electromagnetic radiation and would lose energy and spiral into the nucleus, producing a collapse of the atom. The difficulty was overcome by treating the electromagnetic field and the motion of the electron quantum mechanically. One might therefore hope that quantization of the gravitational field would resolve the problem of gravitational collapse. Such a quantization seems necessary anyway for consistency because all other physical fields appear to be quantized.

So far we have had only partial success in this endeavour but there are some interesting results. One of these concerns black holes. According to the Classical Theory the singularity that is predicted in the gravitational collapse will occur in a region of spacetime, called a black hole, from which no light or anything else can escape to the outside world. The boundary of a black hole is called the event horizon and acts as a sort of one way membrane, letting things fall into the black hole but preventing anything from escaping. However, when quantum mechanics is taken into account, it turns out that radiation can "tunnel" through the event horizon and escape to infinity at a steady rate. The emitted radiation has a thermal spectrum with a temperature inversely proportional to the mass of the black hole. As the black hole emits radiation, it will loss mass. This will make it get hotter and emit more rapidly. Eventually it seems likely that the black hole will disappear completely in a tremendous final explosion. However the time scale for this to happen is much longer than the present age of the Universe, at least for black holes of stellar mass, though there might also be a population of much smaller primordial black holes which might have been formed by the collapse of irregularities in the early Universe.

One might expect that vacuum fluctuations of the gravitational field would cause "virtual" black holes to appear and disappear. Particles, such as baryons, might fall into these holes and be radiated as other species of particles. This would give the proton a finite lifetime. However it is difficult to discuss such processes because the standard perturbation techniques, which have been successful in quantum electrodynamics and Yang-Mills theory do not work for gravity. In the former theories one expands the amplitudes in a power series in the coupling constant. The terms in the power series are represented by Feynmann diagrams. In general these diverge but in these theories all the infinities can be absorbed in a redefinition or "renormalization" of a finite number of parameters such as coupling constants and masses. However in the case of gravity, the infinities of different diagrams are different and so they would require an infinite number of renormalization parameters whose values could not be predicted by the theory. In fact the situation is not really that much worse than with the so-called renormalizable theories since even with them the perturbation series is only asymptotic and does not converge, leaving the possibility of adding an arbitrary number of exponentially vanishing terms with undertermined coefficeints.

The problem seems to arise from an uncritical application of perturbation theory. In classical general relativity it has been found that perturbation expansions around solutions of the field equations have only a very limited range of validity. One cannot represent a black hole as a perturbation of flat space-time yet this is what summing Feynmann diagrams attempts to do. What one needs is some approximation technique that will take into account the fact that the gravitational field and the spacetime manifold can have many different structures and topologies. Such a technique has not yet been developed but we, at Cambridge, have been approaching the problem by studying the path integral approach formulation of quantum gravity. In this the amplitudes are represented by an integral over all metrics

$$\int D[g] \exp(-\hat{I}[g])$$

where $D[g]$ is some measure on the space of all metrics g and $I[g]$ is the action of the metric g.

If the integral is taken over real physical metrics (that is, metrics of Lorentzian signature $- + + +$), the action I is real so the integral oscillates and does not converge. To improve the eigenvalues one does a rotation of 90° in the complex t plane. This makes the metric positive definite (signature $+ + + +$) and the action I pure imaginary so that the integral is of the form

$$\int D[g] \exp(-\hat{I}[g])$$

where $\hat{I} = -iI$. The Euclidean action $\hat{I}$ has certain positive definite properties.

One is thus led to the study of positive definite metrics (particularly solutions of the Einstein equations) on four-dimensional manifolds. If the manifolds are simply connected, their topology can be classified (at least up to homotopy) by two invariants, the Euler number χ and the Hirzebruch signature τ. One can regard the Euler number as measuring the number of "holes" or "gravitational instantons" and the signature measures the difference between right-handed instantons and left-handed ones. It seems that the dominant contribution to the path integral comes from metrics with about one instanton per Planck volume 10^{-142} cms^{3}secs. Thus spacetime seems to be very highly curved and complicated on the scale of the Planck length 10^{-33} cms. even though it seems nearly flat on larger scales.

However we still do not have a proper scheme for evaluating the path integral. The difficulty lies in defining a measure $D[g]$ on the space of all metrics. In order to obtain a finite answer it seems necessary to make infinite subtractions and these leave finite undetermined remainders. There is a possible way of overcoming this difficulty which may come from an extension of General Relativity called supergravity. In this the spin 2 graviton is related to a spin $^3/2$ field and possibly fields of lower spin by anticommuting "supersymmetry" trans-formations. In these theories there is an equal number of bosons (integer spin particles) and fermions (half integer spin particles). The infinities that arise in the path integral from the integration over boson fields seem to cancel when the infinities that arise from the integration over the fermion fields, raising the hope that one could provide a proper mathematical definition of the path integral, maybe some limiting process.

Supergravity theories have another very desirable feature, they may unify gravity with the other interactions and particles in physics. In 1967 Salem and Weinberg proposed a unified theory of the electromagnetic and weak interactions. This has had considerable success in predicting experimental results though the final confirmation will have to wait for the next generation of particle excelerators. Nevertheless, it has given great stimulus to attempts to unify the strong, the weak and the electromagnetic interactions into a "Grand Unified Theory". A feature of such theories is that the complete unification is seen only at very high energies of the order of 10^{16}Gev. This is nearly as large as the Planck energy, 10^{19}Gev, at which quantum gravitational effects should become important. It may well be therefore that one will be able to achieve the unification only by incorporating gravity as well in a completely unified theory which would describe all of physics. This was the goal to which Einstein devoted the last thirty years of his life, without much success. The prospects look brighter now though it is still probably quite a long way off.

The Irrationality of $\zeta(3)$

by Professor A. Baker

In a lecture given in Marseilles last summer, Professor Apéry astonished the audience with a proof of the irrationality of

$$\zeta(3) = 1^{-3} + 2^{-3} + 3^{-3} + \ldots$$

I was not there myself, but I gather that the exposition was greeted with a good deal of scepticism. Indeed this is not surprising, for it concerned the sort of problem that would have appealed to Euler, and no method of attack whatsoever had been described until then. Furthermore, the manner of presentation, together with the mysterious nature of the formulae used, added to the general feeling of incredulity. After the lecture, a number of mathematicians, in particular Henri Cohen, Alf van der Poorten and Don Zagier, attempted to construct a proof from Apéry's notes. It transpired, after several weeks, that Apéry's basic ideas were quite correct, and, at the Helsinki Congress in August, Cohen was able to give a complete demonstration that removed any vestige of doubt. The proof could be readily checked, but it still depended on some complicated and rather obscure identities. A little later, however, Fritz Beukers, a young Dutch mathematician, obtained a new version of the argument, and this can be described very quickly.

Let n be a positive integer, and consider the integral

$$I = \int_0^1\int_0^1\int_0^1 \frac{f^n \, dx\, dy\, dz}{1-(1-xy)z}, \quad \text{where} \quad f = \frac{xyz(1-x)(1-y)(1-z)}{1-(1-xy)z}.$$

On integrating partially n times with respect to y we obtain

$$I = \int_0^1\int_0^1\int_0^1 \frac{(1-x)^n(1-z)^n P_n(y)}{1-(1-xy)z}\, dxdydz, \text{ where}$$

$P_n(y) = \frac{1}{n!}\frac{d^n}{dy^n}(y^n(1-y)^n)$; thus $(-1)^n P_n(\frac{1}{2}(y+1))$ is the familiar Legendre polynomial. The substitution $w = \frac{1-z}{1-(1-xy)z}$ then gives

$I = \int_0^1\int_0^1\int_0^1 \frac{(xyw)^n(1-x)^n P_n(y)}{(1-(1-xy)w)^{n+1}}\, dxdydw$, and on integrating partially n times with respect to x we obtain $I = \int_0^1\int_0^1\int_0^1 \frac{P_n(x)P_n(y)}{1-(1-xy)w}\, dxdydw.$

Now plainly we can remove one of the variables; we have namely

$I = -\int_0^1\int_0^1 \frac{P_n(x)P_n(y)\log(xy)}{1-xy}\, dxdy.$ Further the polynomials $P_n(x)$ and $P_n(y)$ have integer coefficients; thus I is a linear combination with integer coefficients of the integrals

$I_{rs} = \int_0^1\int_0^1 \frac{x^r y^s \log(xy)}{1-xy}\, dxdy$ $(0 \leqslant r,s \leqslant n)$. But I_{rs} can be obtained from the integral $J_{rs} = \int_0^1\int_0^1 \frac{x^{r+t}y^{s+t}}{1-xy}\, dxdy$ by differentiating with respect to t and then putting t = 0. Furthermore, on writing

$(1-xy)^{-1} = 1 + xy + (xy)^2 + \ldots$, we get $J_{rs} = \sum\limits_{1}^{\infty} (r+j+t)^{-1}(s+j+t)^{-1}$.

Hence, if $r = s$, we see that $I_{rs} = -2(\xi(3) - \sum\limits_{1}^{\infty} j^{-3})$.

If $r > s$, then J_{rs} can be written as a finite sum, namely $J_{rs} = (r-s)^{-1}\sum\limits_{1}^{r-s} (s+j+t)^{-1}$; thus we have $I_{rs} = -(r-s)^{-1}\sum\limits_{1}^{r-s} (s+j)^{-2}$. and a similar formula holds for $s > r$. We conclude that $I = (a_n + b_n\zeta(3))/d_n^3$, where a_n and b_n are integers, and d_n denotes the lowest common multiple of $1,2,\ldots,n$. From the original expression we see that $I > 0$, and it is easily verified that $f \leq (\sqrt{2}-1)^4$ for all x,y,z, whence $I < 2\ (3)(\sqrt{2}-1)^{4n}$. Moreover, by prime number theory we have $d_n \leqslant 3^n$. Hence $0 < a_n + b_n\zeta(3) < 2\zeta(3)c^n$, where $c = 27(\sqrt{2}-1)^4 < 1$. This implies that $\zeta(3)$ is irrational; for if $\zeta(3)$ were rational, say a/b with a,b positive integers, then $a_n + b_n\zeta(3)$ would be at least $1/b$ which is impossible for n sufficiently large.

Apéry's original proof, or at least the version of it given by Cohen in Helsinki, depended on the derivation of a continued fraction expansion for $\zeta(3)$, namely

$$\zeta(3) = 6/(5 - 1^6/(117 - 2^6/(535 - \ldots$$

The general term here is n^6/c_n, where $c_n = n^3 + 4(2n+1)^3 + (n+1)^3$. The analysis leads to asymptotic formulae for the integers a_n and b_n referred to above, and hence to a measure for the degree of précision by which $\zeta(3)$ can be approximated by rationals; in fact for all rationals p/q $(q > 0)$ and any $\varepsilon > 0$, we have

$$\left|\zeta(3) - \frac{p}{p}\right| \geqslant \frac{c(\varepsilon)}{q^{\theta+\varepsilon}}, \text{ where } c(\varepsilon) > 0 \text{ and } \theta = 13.417\ldots .$$

Of course, if $\zeta(3)$ behaves like almost all numbers, then the inequality holds with the best possible value $\theta = 2$. A peculiar expression for $\zeta(3)$ of a rather different kind was also given by Apéry, namely

$$\zeta(3) = \frac{5}{2}\sum_{1}^{\infty} \binom{2n}{n}^{-1}\frac{(-1)^{n-1}}{n^3},$$

and I understand that this was in fact the original starting point for his researches.

The theory just described is plainly still in its infancy; indeed no one knows how to generalise the work so as to establish for instance, the irrationality of $\zeta(5)$, $\zeta(7)$, ..., or moreover the transcendence of $\zeta(3)$. Nevertheless several morals can, I think, be drawn at once from the discovery. First it destroys the widepsread belief that it is no use trying an old problem by old methods that must have been second nature to the past masters in the subject; there may always be a new twist that had previously been missed. Secondly it provides a counter-example to the theorem that the only really original mathematics is done by relatively young people (according to Hardy, mathematics is a 'young man's game'); I do not think Professor Apéry will mind if I say that he is of rather senior years. And thirdly it goes against the strong movement that has taken place in relatively recent years towards more and more axiomatisation in mathematics, as epitomised by the Bourbaki school; clearly Apéry's work owes nothing to this. It shows that ingenuity alone is quite enough to solve difficult mathematical problems, and it should give us all, I think, much encouragement.

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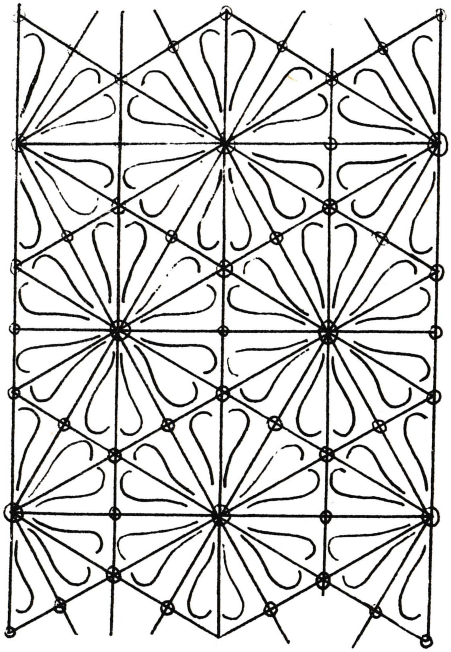
The work of Apéry is to appear in Acta Arithmetica; the paper of Beukers is to appear in the Bulletin of the London Math.Soc. There is an interesting article on Professor Apéry in La Recherche 10 (1979), 170-172.



On Wallpaper

by Dr. S. J. Patterson

My interest in wallpaper dates from my earliest days. Most of the rooms in the house of my childhood were papered with florated patterns, which not only involved the shifts and half-shifts of school art, but which could be rotated, or reflected with a change in colour, and so on. Unravelling these patterns was one of the ways of passing long hours in bed with measles or some other childhood illness. As it turns out, the patterns are full of symmetries; but there are only seventeen essentially different wallpapers. Thus the eye of the adult sees the world of the child.

However, having started, I will go on to describe, from the point of view of the mathematician, wallpaper patterns. Better, we shall look at certain groups of isometries of the plane. The general isometry of the plane has the form $\underline{x} \to A\underline{x} + B$ where $A \in O(2)$, the group of 2×2 orthogonal matrices (which includes reflections), and $B \in \mathbb{R}^2$. We shall consider a group G of such mappings; if $g \in G$ we shall write $g(x) = A(g)\underline{x} + B(g)$. If we check $g_1(g_2(\underline{x})) = (g_1g_2)(\underline{x})$ then $A(g_1g_2) = A(g_1)A(g_2)$, and $B(g_1g_2) = B(g_1) + A(g_1)B(g_2)$. In particular, $A: G \to O(2)$ is a homomorphism. Let $G_o = \mathrm{Ker}(A)$. If then, $g_1, g_2 \in G_o$ we have that $B(g_1g_2) = B(g_1) + B(g_2)$, so that $B: G_o \to \mathbb{R}^2$ is also a homomorphism.

The time has come to put the 'wallpaper' assumption on G. This is that the image of G_o under B should be a <u>lattice</u>, henceforth denoted by L. (A lattice is a subgroup of $\mathbb{R}^2$ of the form $\{mu+nv: m,n \in \mathbb{Z}\}$, where $u,v \in \mathbb{R}^2$ are linearly independent.) This is the set of shifts which is so characteristic of wallpaper.

If now $g \in G$, $h \in G_o$ then $(ghg^{-1})(\underline{x}) = \underline{x} + A(g)B(h)$; in particular, if $l \in L$ then $A(g)(l) \in L$. It follows from this (nor is it too hard) that $A(G)$ is a <u>finite</u> subgroup of $O(2)$. It turns out that the fact that $A(G)$ preserves the lattice L puts very strong restrictions on what $A(G)$ and L can be. If $A(G) = \{I\}$, or $\{I,-I\}$ then there is no restriction on L. If $A(G) = \{I,R\}$ or $\{\pm I, \pm R\}$, where R is a reflection, then we can take R to be $R: (x,y) \to (x,-y)$ and L to be generated by $(1,0)$ and $(0,t)$ or $(\frac{1}{2},t)$ where $t > 0$. Otherwise $A(G) \cap SO(2)$ is of order 3, 6 or 4; then the lattice L is generated by $(1,0)$ and $(\frac{1}{2}, \sqrt{3}/2)$ in the first two cases, and by $(1,0)$ and $(0,1)$ in the third.

Now we have a good idea of the possible $A(G)$ and L; we must now try to see how these can be fitted together to form the group G. This is rather more subtle than one first supposes. Let the order of $A(G)$ be n, and let $g_1, \ldots, g_n \in G$ be such that the $A(g_j)$ $(1 < j < n)$ make up $A(G)$. We shall look at $b = \frac{1}{n}\sum_{j=1}^{n} B(g_j)$. Note that if $g \in G$ then gg_j must be of the form $h(j,g)g_{\sigma j}$ where $h(j,g) \in G_o$ and σ is a permutation of $\{1,2,..,n\}$. Then

$$A(g)b = \frac{1}{n}\sum_j A(g)B(g_j) = \frac{1}{n}\sum_j (B(gg_j) - B(g)) = \frac{1}{n}\sum_j B(h(j,g)g_{\sigma j}) - B(g)$$

$= b - B(g) + \frac{1}{n} \sum B(h(j,g))$. The last term on the right-hand side is of the form $n^{-1}L(g)$, with $l(g) \varepsilon L$. Thus $B(g)=b - A(g)b+n^{-1}L(g)$. If we now replace $\underline{x}$ by $\underline{x}+b$ (changing the origin in the wall-paper) we see that we may assume that $B(g)$ has the form $n^{-1}L(g)$. In general we cannot assume that $B(g)$ is in L.

This essentially allows us to reduce the problem to a finite one. To see this we shall consider the quotient group L/nL which is finite, of order n². Let $c \varepsilon A(G)$; then $c = A(g)$ for some $g \varepsilon G$ and we let $l_o(c) = l(g)$; this, mod nL, is well defined. So the l_o satisfy $l_o(c_1c_2) = l_o(c_1) + c_1 l_o(c_2)$ (mod nL). A similar argument to the one above shows that there is at most one G which gives rise to an L_o with this property. The converse question, as to whether, given $G_1 \subset O(2)$, L a lattice preserved by G_1 and l_o as above, there exists a wallpaper group G with this lattice, $A(G) = G_1$ and the given l_o (up to a suitable notion of equivalence), is a little more difficult but it can also be solved.

When all is said and done, one finds 17 essentially different patterns, which are reproduced here. In these, the repeated motive is a simple ; even with this one finds, as in a kaleidoscope, attractive patterns. The centres of rotations are marked with • .

In Moorish art, where no representation of nature was allowed (for it was blasphemous to vie with the creator) all of the seventeen patterns were known and (reputedly) used in the Alhambra in Granada.

Finally, according to the modern ethos, I should add that all this can be generalised to 3 (and more) dimensions; then the groups are called crystallographic groups. They play a major role in crystallography which is one of the chief areas of application of group theory. There are 219 of them in dimension 3, and 4783 in dimension 4. Beyond this little is known.

The analogous problem can be posed for any space on which a large group of transformations acts. Thus one considers tesselations of the hyperbolic plane, which are closely related to Reimann surfaces, and of higher dimensional spaces, which touch on some of the deepest problems of number theory and of mathematics in general.

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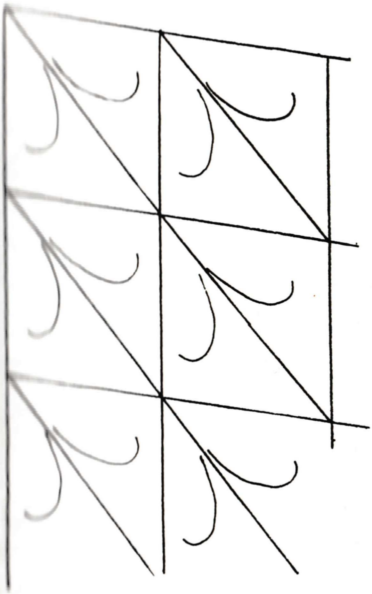
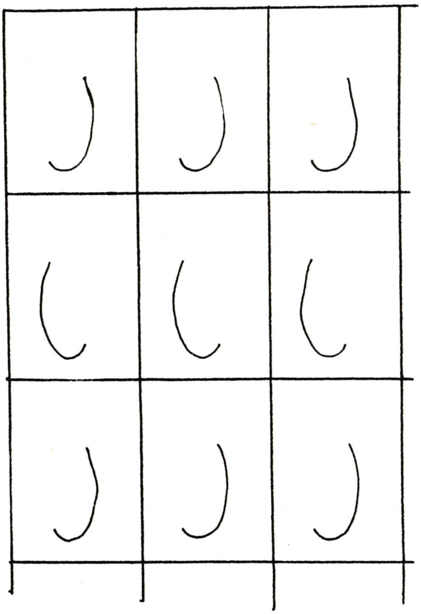
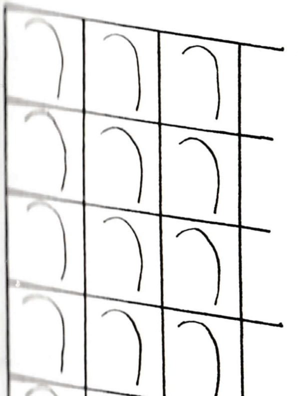
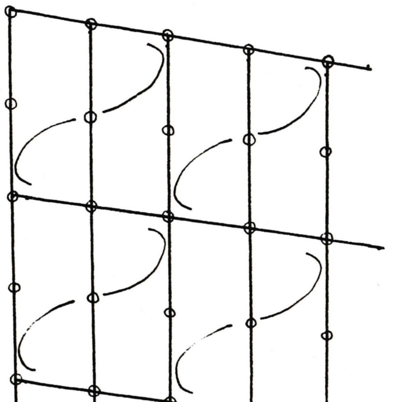
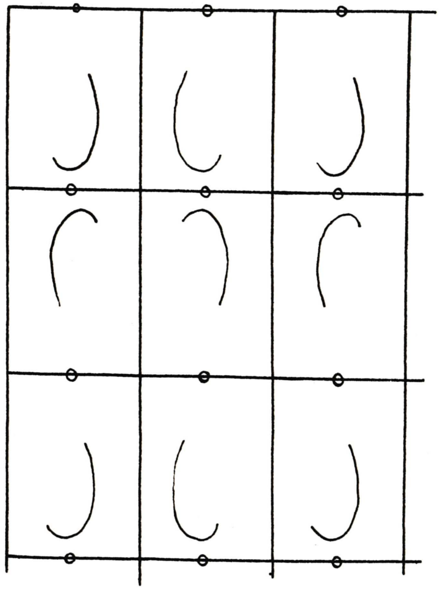
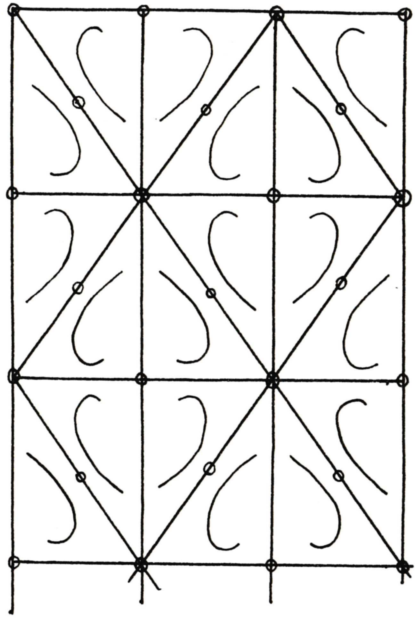


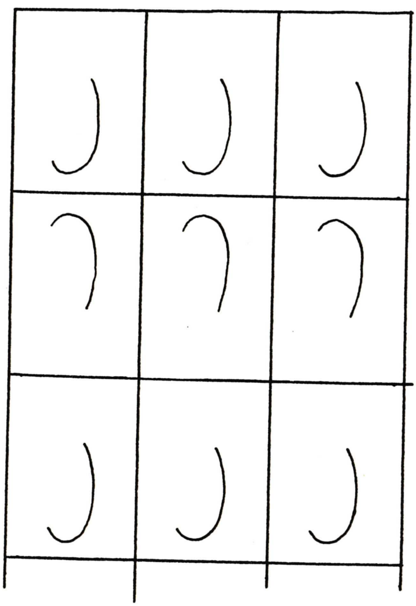


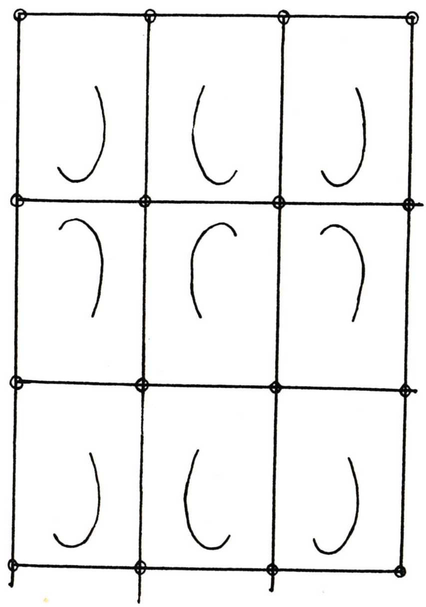


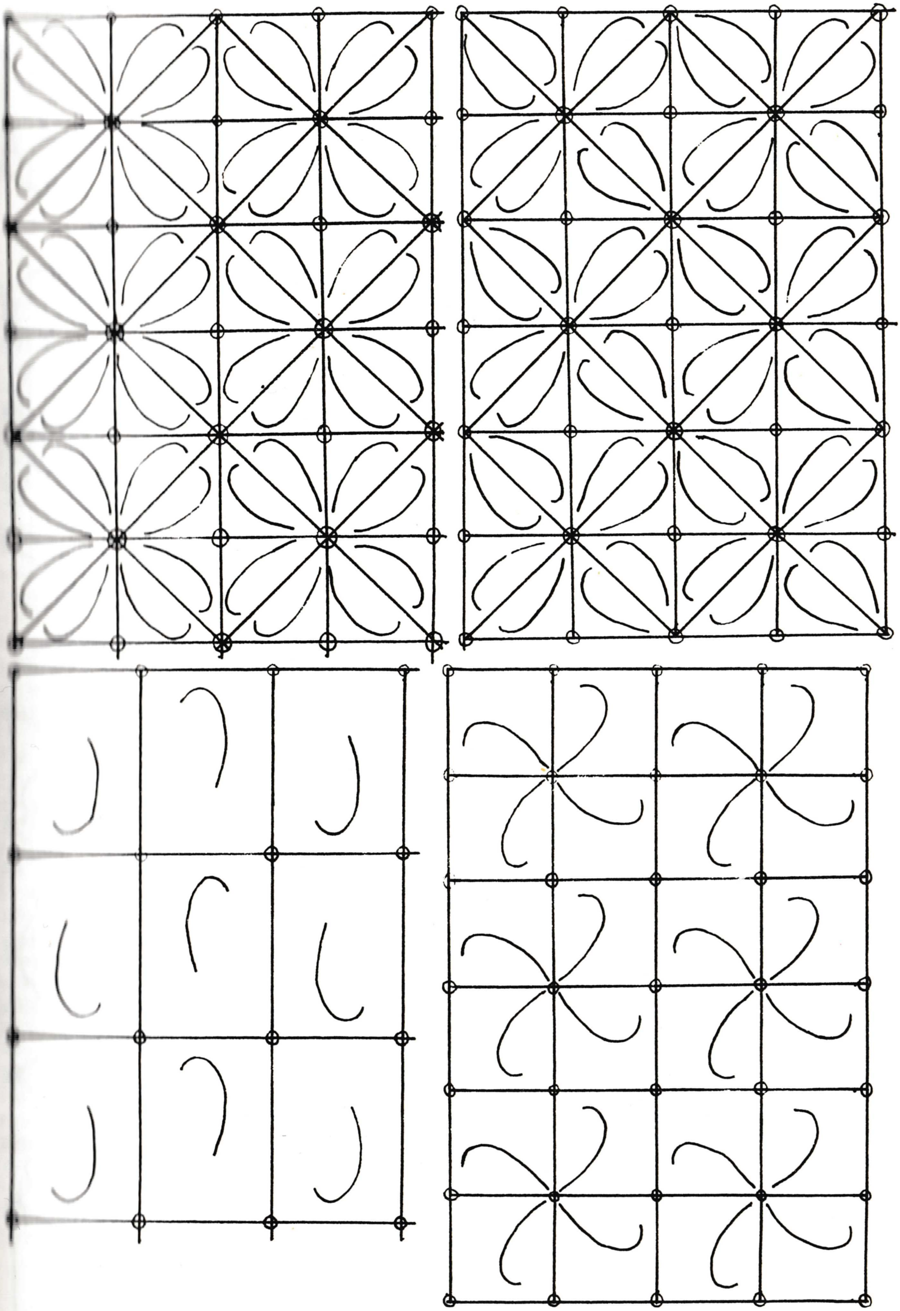


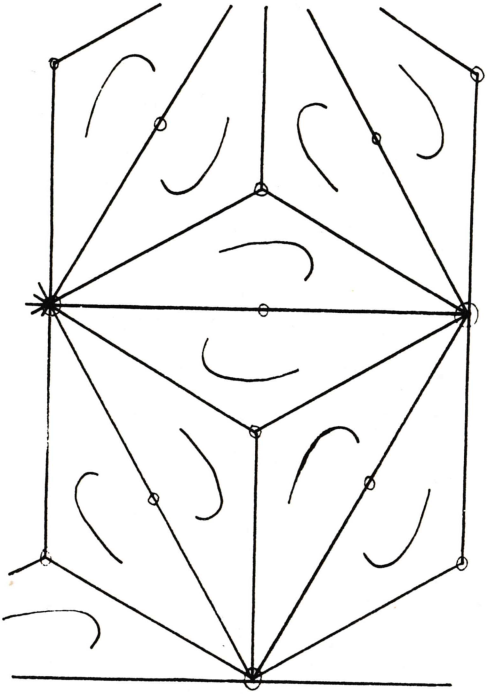
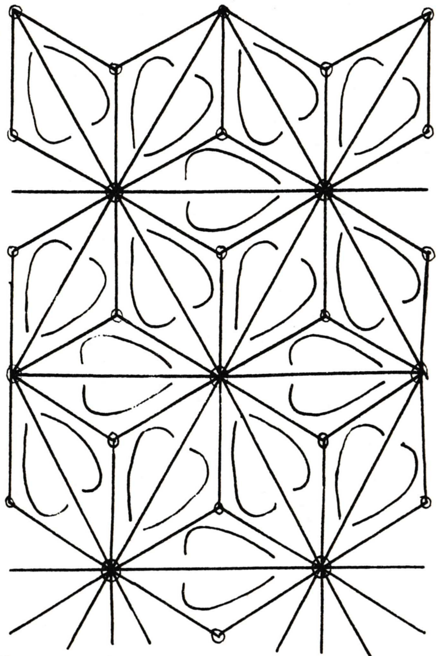
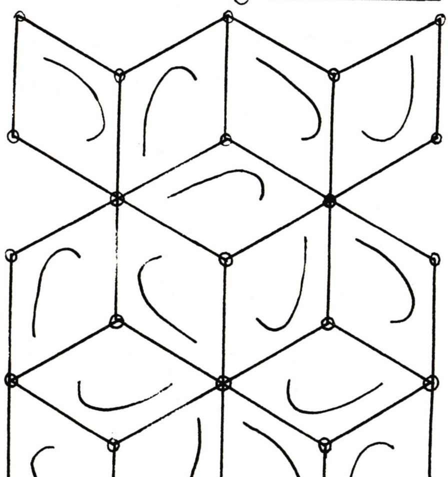
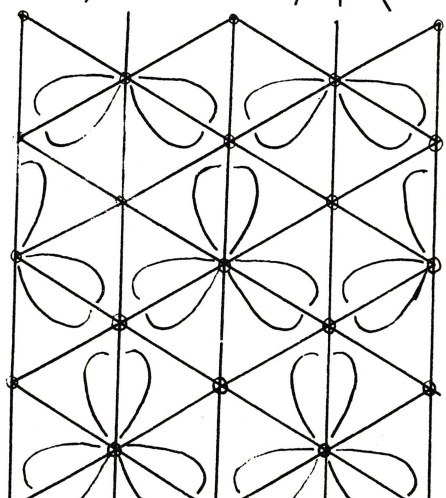










The Unification of Weak and Electromagnetic Interactions

by Professor J.C. Polkinghorne

Elementary particle physicists conventionally divide the forces of nature into four classes: the strong interactions of very short range which hold nuclei together; the weaker but long range electromagnetic interactions; the very weak interactions responsible for such phenomena as beta-decay; and finally the gravitational interaction which has an intrinsic strength so infinitesimal compared to the others that we are content to forget about it. The division appears a sensible one, since the different types of interaction really do have such contrasting properties. Nevertheless one of the most exciting recent developments in what has been a heady period of advance for particle physics is the growing conviction of a unity between weak and electromagnetic interactions. Something like this has happened in physics before. Electric and magnetic phenomena appeared very different from each other at the beginning of the nineteenth century. Yet the researches of Faraday and Maxwell led to their combination in the theory of electromagnetism, the creation of which was one of the great triumphs of nineteenth century physics. The successful unification of weak and electromagnetic interactions would be a comparable achievement for the twentieth century!

In high energy physics interactions are pictured as mediated by the exchange of particles. These particles transfer energy and momentum from the object that emits them to the object that absorbs them, which is just what we mean by the action of a force. The carrier of electromagnetism is the photon, the quantum of light. It is what we call a vector particle, that is it behaves as though it were spinning with one unit of angular momentum. If the weak interactions are also mediated by a particle it too must have this vector character. This point of coincidence in the properties of the two interactions - that they are both mediated by vector particles - is the germ from which the grand synthesis can grow. The difference in apparent strengths of the interactions is to be attributed to the different properties of the vector particles which are exchanged. The photon is massless, but the intermediate vector boson W (as the weak interaction mediator is called) would have to be very heavy, almost a hundred times the mass of the proton. This makes it impossible to create at currently accessible laboratory energies. At those energies where it could be made the weak interactions would have increased their strength to be comparable with electromagnetic processes. Their weakness as we know them is simply an attribute of the energy scale over which we can presently probe them.

A successful unification scheme requires much more than simply making the W heavy. There are subtle requirements which have to be satisfied. Some of these are physical. The weak interactions do not conserve parity. By this we mean that they are found to have an intrinsic 'handedness', so that if we observed an experiment in a mirror we should perceive an apparently different law operating from that obtained by direct observation. Electromagnetic interactions do not have this handedness; they appear unchanged in character if we study their reflections. Clearly it is no easy matter to reconcile these contrasting behaviours in a single theory. Some other problems are more mathematical in character. Relativistic quantum mechanics is a notoriously tricky theory with which to calculate, for it has a tendency to produce intractable infinities to spoil its sense. This is particularly likely to occur when vector particles are around. Only very carefully chosen formulations will be free from this disaster.

In the late 1960's Steven Weinberg (of Harvard) and Abdus Salam (late of St.John's) realised independently that there was an elegant and economic scheme which overcame all these problems. The clue to its discovery lay in the notion of gauge theories. This powerful idea has underlain many developments in particle physics in recent years. We shall have to attempt to convey its flavour whilst eschewing matters of detail.

Let us start with the idea of a Lie group. It is a group of transformations depending on one or more continuous parameters. The paradigm example is three dimensional rotations, with the Euler angles being the parameters in this case. Such groups have a natural place in theoretical physics if only because we are obviously concerned with honest-to-goodness physical rotations in honest-to-goodness three-dimensional space. However it turns out that Lie groups are much more widespread that that, involving all sorts of Pickwickian 'rotations' with nothing to do with ordinary space, such as that which turns a proton into a neutron. This latter is called isotopic spin and the fact that strong interactions treat protons and neutrons similarly can be expressed by saying that they are invariant under isotopic spin 'rotations'. So far so good. These fictitious 'rotations' like isotopic spin are completely independent of ordinary spatial properties. If we 'rotate' a proton into a neutron we do so everywhere; all protons make the change wherever they are located. However some people (Yang and Mills and a Cambridge contemporary of mine called Ron Shaw) asked the question of whether one could construct theories where isotopic spin 'rotations' could be made differently at different points, so that here a proton turned into a neutron but there it did not. Invariance under this sort of transformation is a much more stringent requirement on a theory. It can only be done if the theory contains vector particles (aha!) and the resulting formulation is very tightly knit and highly symmetric. Such theories are called gauge theories.

Salam and Weinberg produced an elegant gauge theory which united weak and electromagnetic interactions. However in physics elegance is nothing compared to correctness (though the two often go together). So the question is whether their theory is right. Muchexcitement has been generated in the last three or four years by the accumulation of evidence to suggest that it may well be so.

The predictions of the theory can be divided into two classes: those testable now and those for whose verification we have to wait. Let us consider the first category. Here the most striking thing was the prediction of a totally new type of weak interaction. All the examples known were of charge-changing type. For example the neutron decays by a weak interaction and changes its charge by becoming a proton. We can express this by saying that the intermediate vector bosons W must themselves be charged, to carry off the missing electric charge. Now the gauge theory will not work unless there is also a neutral intermediate vector boson which we call Z^o. This implies that there must be a new type of weak interaction, mediated by the Z^o, which does not change charge. These interactions are called 'neutral currents'. They are much harder to detect than the 'charged currents' mediated by the W's, but if Salam and Weinberg are right they must be there. The way to find them is to use neutrinos, elusive particles which can pass right through the earth with ease. Their detection is difficult but possible with skill, patience and money. Experiments at CERN and elsewhere have triumphantly shown the existence of these neutral currents in just the form the unified theory expects. In fact the theory has passed all the tests to which it can be subjected with present experimental facilities. It looks good.

The acid test, however, is yet to come. Unfortunately it falls in the class of experiments not presently possible. It involves, of course, studying weak interactions at energies high enough for the W and Z^o bosons actually to materialise. With the aid of money from the taxpayers of developed countries these experiments will be performed, in all probability in the early 80's. Everyone expects that nature will do what Salam and Weinberg tell it to, but there is nothing like actually seeing for oneself. Even then there is a final test to come, for the theory involves another particle (called the Higgs after its proposer). The role of this particle is as difficult to explain in the lay terms of this note as the Higgs itself is likely to prove hard to identify. Let us just say that its discovery would be the final piece in place in the weak interaction-electromagnetic jigsaw.

Why stop there ? Needless to say people are thinking about mixing in strong interactions also, and gravity for good measure. However at present such notions are just gleams in the eyes of speculative theorists.

Chatterton's Folly

by M. C. Davies

It is generally acknowledged, at least among the writers of donnish novels, that the proper way for academics to occupy the hour or so after dinner is to temper the mellowing effect of imbibing a good port with a little malicious conversation. Tonight, however, Dr. Miller had so far distanced himself from these canons of good taste as to be sitting, and worse, working, on the fifth floor of the University library, exactly as if he were one of those many undergraduates whose minds, concentrated for most of the year on rugby or politics or acting, are forced, in consequence, to concentrate for a few weeks of the summer term until very late at night on anthropology or history or classics. But the distance was not so great, for Dr. Miller was venting his vexation at missing his port with a little malicious book reviewing. The sun was just beginning to set and, while it was still quite light outside, and there was no need for artificial illumination of the desks at the windows, the book stacks were now profoundly gloomy: this is the perfect time of day for gazing out on the sort of green and peaceful prospect facing Dr. Miller, and Dr. Miller was indeed gazing out at it. But his mind was resolutely fixed on the engrossing problem of singular space-times, and the obstinately non-singular patch visible through the window did not distract him. It was at least a quarter of an hour later that he leant back in his chair, stretched, and peered about him. He seemed to be alone, except for a very old, frail and rather dowdy looking man in a corduroy jacket and steel rimmed spectacles, who was poking around on a nearby bookshelf. Dr. Miller would probably have returned immediately to the task of undermining the reputation of his Oxonian colleagues, had he not happened to notice that the poking around took a rather strange and systematic form. He was taking the books one-by-one from the shelf, holding them by the ends of the spine, and shaking them vigorously. Mostly, this process gave no visible result; from a few volumes fluttered bookmarks or bus tickets which he did not retrieve, and one book half detached itself from its cover, at which he had the grace to look shamefaced and replace the book rapidly in its place. After a few minutes he noticed Dr. Miller's curious gaze, turned, pushed his head pugnaciously forward and snapped ferociously and self importantly, "I, young man, am Professor Chatterton. What you are staring at rudely is true scholarly endeavour, an activity for which I doubt you will ever be competent, and something which is sadly lacking in this university nowadays." Dr. Miller mumbled apologetically and was summoning up the courage to make some sort of reply, when he realised that the Professor had already turned back to his rifling of the bookshelves, so he readdressed himself crossly to his work, his enthusiasm for the verbal assassination of his fellow scientists considerably increased. He did not look up again until the closing-bell rang, and by this time Professor Chatterton seemed to have disappeared, though Dr. Miller had not heard the door to his wing open.

When Dr. Miller entered the library tea-room the next afternoon, he spotted the Dean of his college sitting alone over a stale bun, reading the Daily Telegraph. Normally he would quickly have averted his eyes and attempted to give the impression that he was too engrossed in comtemplation to notice a colleague, but Mr. Prynne had a reputation for knowing (and knowing the latest scandal about) everyone of interest in Cambridge, and Dr. Miller had something particular to ask him, so he took his tea across to the Dean's table. As he approached it, the Telegraph lowered itself. "Good Afternoon, Miller." The Dean's voice and hair were clipped equally short, and the resemblance he bore to the archetypal dean of so many Oxford and Cambridge novels was so close that his less charitable colleagues were inclined to suggest that it could not be entirely coincidental. "Ah, Chatterton", he mused, in anser to Dr. Miller's query, "a depressing story I am afraid". The Dean did not sound unduly depressed. "He was a brilliant scholar at one time, a mathematical historian, but his brain began to soften up while he was still quite young - he was teetotal you see - and he became obsessed with Fermat's last theorem." The Dean's voice was slightly warmed by the thought that he, Tobias Prynne, was not the sort of man who developed obsessions, and his mind was certainly as sharp as it ever had been.

"He found a letter from Fermat, y'see, among the Huygens' papers in Leyden, which gave a highly cryptic and confused account of a piece of mathematics which Chatterton was convinced was a proof of his last theorem. Unfortunately, the letter was lost, and Chatterton conceived the idea that he had left it tucked inside a book he had borrowed from the University library, which is apparently something he was always doing with important papers. So he started an obsessive search for the damn thing, which he combined with writing a flood of poisoned-pen letters to those second-rate historians who had the temerity to dispute his theories about the letter and the proof, which he had published at length in a previously reputable journal which he edited. Unfortunately, he only ever searched two floors, and found nothing of interest except the notes for his own inaugural lecture which he had lost ten years previously."

"I think you are wrong there,Dean" said Dr. Miller, "I saw him yesterday and he is still going strong."

What passed for a smile flickered at the corners of Mr. Prynne's mouth. "It is you who are mistaken, Miller", he said, crisply, "Professor Chatterton collapsed and died in the library one evening five years ago."

Some Applications of Probability Theory

by Dr. D.J.H. Garling

Right from the beginning (Probability and its Applications) it is stressed at Cambridge that probability theory is a practical down-to-earth subject, with myriads of useful applications. Unfortunately, it is rather difficult (and in the case of the limit theorems literally impossible) to find realistic and interesting examples which illustrate what probability is all about, while not getting involved in statistics (a different and much more difficult subject) or in the philosophical problems of applying what is essentially a mathematical theory to the real world. Fortunately, there is one field of applications where these difficulties disappear - namely pure mathematics itself. In this article, I shall describe a few of these applications.

One of the easiest results in probability is Chebyshev's inequality. This can be used to give a very easy and beautiful proof of Weierstrass' theorem that a continuous function f on $[0,1]$ can be approximated uniformly by polynomials. We can suppose that $|f| \leq 1$ Suppose that $\varepsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ if $|x - y| < \delta$. Choose n large enough so that $n\varepsilon\delta^2 > 1$. Now if $0 \leq t \leq 1$, let $X_1, \ldots, X_n$ be independent random variables, each taking the value 1 with probability t, and 0 with probability $1-t$, let A be the average $(X_1+\ldots+X_n)/n$, and let $p(t) = \mathbb{E}(f(A))$ ($\mathbb{E}$ as usual denotes the expectation). It is easy to see that as t varies, $p(t)$ is a polynomial in t. Let B be the event $\{|f(A) - f(t)| \geq \varepsilon/2\}$. Chebyshev's inequality shows that $P(|A - t| \geq \delta) \leq 1/4n\delta^2 < \varepsilon/4$, so that $P(B) < \varepsilon/4$. But then

$$|p(t) - f(t)| = |\mathbb{E}(f(A) - f(t))| \leq$$
$$\leq \mathbb{E}(|f(A) - f(t)|\,|B)P(B) + \mathbb{E}(|f(a) - f(t)|\,|B^c)P(B^c) \leq \varepsilon .$$

This little argument illustrates the fact that a large part of analysis consists of finding good inequalities, and exploiting them sensibly. One important example is Khinchine's inequality: suppose that $a_1, \ldots, a_n$ are real numbers; let v be the average (over all choices of signs) of $|\Sigma \pm a_j|$; then $v \leq (\Sigma a_j^2)^{\frac{1}{2}} \leq e^{\frac{1}{2}}v$. The left-hand inequality is rather trivial; let us give a probabilistic proof of the right-hand one. By homogeneity, we can suppose that $\Sigma a_j^2 = 1$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be independent random variables taking values ± 1 with probability $\frac{1}{2}$ (such as arise with coin-tossings). Then $\mathbb{E}((\Sigma\varepsilon_j a_j)\Pi(i + \varepsilon_j a_j)) = 1$, while $|\Pi(i + \varepsilon_j a_j)| = \Pi(1 + a_j^2)^{\frac{1}{2}} \leq e^{\frac{1}{2}}$: the rest is duality. The right constant is not $e^{\frac{1}{2}}$, but not $2^{\frac{1}{2}}$ (which occurs when $n = 2$): this was an open problem for a long while, which has been very recently resolved by a young Polish mathematician. Interestingly there is another easy proof, which gives $3^{\frac{1}{2}}$: Littlewood gave this argument, missed a third of the terms, and obtained $2^{\frac{1}{2}}$! A case of intuition defeating accuracy.

Sums of squares are always easier to deal with than absolute values: they lead on to Hilbert space, which has good geometric properties. It follows from Khinchine's inequality that $\Sigma \pm a_j$ converges for almost all choices of signs if and only if $\Sigma a_j^2 < \infty$: thus sums of squares can arise naturally, rather than as a matter of convenience. In a similar vein, if $\Sigma a_j^2 < \infty$, $\Sigma \pm a_j \cos jt$ converges for almost all t for almost all choices of signs, while if $\Sigma a_j^2 = \infty$, the series diverges for almost all t for almost all choices of signs. On the other hand, no choice of signs is actually known for which $\Sigma \pm j^{-\frac{1}{2}} \cos jt$ diverges for almost all t. This illustrates the fact that it is often easier to show that almost every thing has a certain property than it is to find an actual example!

So far, the basic limit theorems have not been mentioned. A classic application of the strong law of large numbers is that if a number is picked at random between 0 and 1, then with probability 1, whatever expansion is used (decimal, binary,...), all the digits occur with the right frequency. Probability is very useful in number theory, although there is the tiresome difficulty that, as there are infinitely many integers, they cannot be given equal probability. Instead, one must consider the first N integers, and then let N go to infinity. In this way, sense can be made of statements like 'the probability that an integer is divisible by p is 1/p' and 'divisibility by a prime p and divisibility by another prime q are independent events'. Now let $\nu(n)$ be the number of distinct prime divisors of n: ν is the sum of 'independent' summands, and so one might expect the central limit theorem to apply. This is indeed so: in the limiting sense that I have suggested, $(\nu(n) - \log\log n)/(\log\log n)^{\frac{1}{2}}$ is normally distributed with zero mean and unit variance.

The final application of probability theory that I would like to mention is the use of Brownian motion, which is the continuous time analogue of random walk. Brownian motion is isotropic: this means that a conformal mapping sends Brownian paths to Brownian paths (with a local change of time scale). This fact means that Brownian motion has proved to be a very powerful tool indeed in the study of harmonic functions and in complex analysis; at present, many new results are being obtained. Let me however mention an old result obtained in this way: Picard's theorem states that a non-constant entire function takes all values except perhaps one. Burgess Davis has given a proof of this using Brownian motion, which is interesting in that it depends upon the topology of the plane: a Brownian path will repeatedly wind and unwind itself about a single point, but will become inextricably tangled about two points.

I hope that these examples show that it is good for the pure mathematician to know some probability: first, because there are powerful results which can give very precise information, and secondly, and more importantly, because probabilistic intuition can give new insights into a problem.

Problems Drive

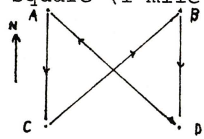
by J. R. Rickard and J. J. Hitchcock

1 X,Y,Z each have 3 digits and contain between them all the digits from 1 to 9. X + Y = Z and Z is a power of a prime. Each digit of X is less than the corresponding digit of Y. Find X,Y,Z.

2 A cube has the letters A,B and C written on its faces (each appears on two faces). From three different directions it appears as

Draw a net of the cube with the letters correctly oriented on it.

3 Four ships A,B,C,D are at the corner of a square (1 mile square). They are going north at 1 knot. A boat goes at 5 knots from A to C to B to D to A, going in a straight line along each segment. When it returns to A, how far have the ships moved?



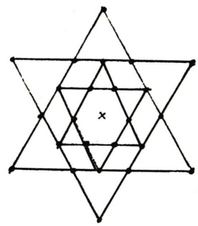
4 What are the next two numbers of the following sequences?

(i) 1,2,4,6,10,12,16, ...

(ii) 1,1,2,3,10,13,23,41,...

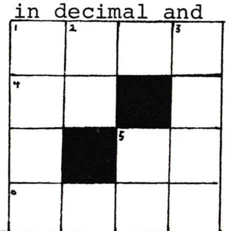
(iii) 0,1,4,9,6,5,6,...

5 How many different closed circuits around x are there, which follow the paths shown, and pass through only 12 of the marked points?



6 Find the volume and the surface area of a regular octahedron with unit side.

7 Solve the cross-number. All numbers are in decimal and there are no leading zeros.



Across		Down	
1.	A cube	1.	2 down times 5 down
4.	A prime	2.	A prime ($\neq$ 4 across)
5.	Sum of two squares	3.	A square
6.	$n! + n + 3$, n an integer	5.	A square

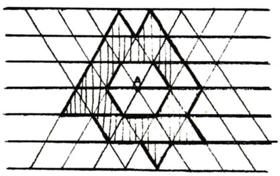
8 In Archimeda, all prices are an integral number of crowns. While waiting in the greengrocers to buy a plum, a peach, an orange and an apple, I notice one person buy 3 apples, 3 peaches, a plum and an orange for 41 crowns, another buy 5 peaches, 2 apples and a plum for 45 crowns, and a third person buy 2 peaches, 2 applies and an orange for 23 crowns. How much do I expect to pay?

9 I ride into a circular desert at 12 o'clock noon. At 1 o'clock I find a small oasis. Someone there tells me that B had been there at 8 o'clock, having entered the desert two hours before. At 2 o'clock I met C. He had started at 11 o'clock in the morning and had met B at 11 o'clock. B, C and I all ride at the same speed and keep to a straight line (all stops are of negligible time). When shall I get out of the desert?

10 Find the largest number such that any two consecutive digits form a two-digit prime, and all these primes are different.

11 8 people each have a hat, and each is wearing a hat, possibly his own.
A is wearing the hat belonging to the person wearing B's hat.
D " " " " " " " " " E's "
G " " " " " " " " " H's " .
B is wearing the hat belonging to the person wearing the hat belonging to the person wearing C's hat.
E is wearing the hat belonging to the person wearing the hat belonging to the person wearing the hat belonging to the person wearing F's hat.
Whose hat is each person wearing?

12 How far is the centre of gravity of this shape from the point A? The parallel lines are all at 5 mm intervals.



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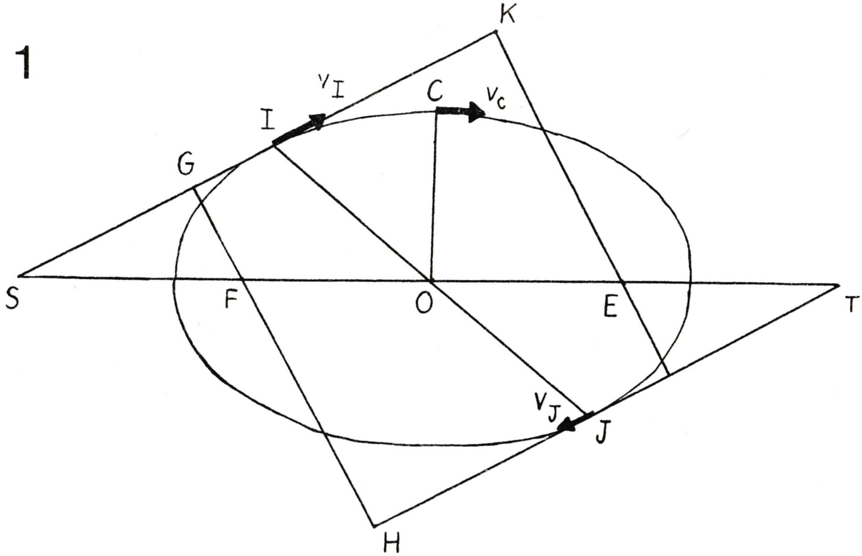

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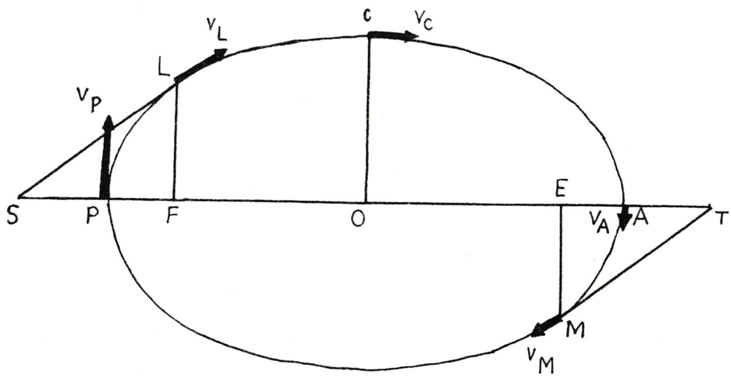

Two Theorems on the Kepler Ellipse

by Professor A. Tan, University of Alabama

In this article, we shall derive two theorems on the velocities of a planet in its orbit around the sun. The first theorem gives the product of the velocities at the ends of any diameter of its elliptic orbit and the second theorem gives the ratio of these two velocities. From the two theorems, one can derive the velocity of the planet at any point on its orbit in terms of its velocity at the end of the minor axis.

We will assume that the sun is infinitely more massive than the planet. Kepler's first law states that the orbit of the planet is an ellipse with the sun at a focus. In Fig.1, F is the focus occupied by the sun, while E is the empty focus. IJ is any diameter of the ellipse and OC is the semi-minor axis. SK and HT are the tangents to the ellipse at I and J respectively. The rest of the construction in the figure is self-explanatory.

Let v_I, v_J and v_C be the velocities of the planet at I, J and C respectively. By the second law of Kepler, the rate of area swept out by the radius vector is constant. Hence, by symmetry,

$$v_I \cdot FG = v_J \cdot FH = v_J \cdot EK = v_C \cdot OC. \quad (1)$$

Also there is an important property of the ellipse which states that the product of the two focal perpendiculars on the tangent at any point I is constant and equal to the square of the semi-minor axis (Cf. [1] and [2]),

$$FG \cdot EK = OC^2 \quad (2)$$

Eliminating OC between (1) and (2),

$$v_I \cdot v_J = v_C^2 \quad (3)$$

Otherwise stated, this gives the following theorem:

THEOREM 1 The velocity of a planet at the end of the minor axis is equal to the geometric mean of the velocities at the ends of any diameter.

This theorem has been derived for two particular diameters. Freeman [3] derived the theorem for the special case when the diameter IJ is the major axis of the ellipse, while Tan [4] derives the same when I and J are at the ends of the latera recta of the ellipse.

It is also instructive to find out the ratio of the velocities at the ends of a diameter. From the similar triangles FGS and FHT,

$$FH / FG = FT / FS. \quad \text{Therefore from (1)}$$

$$v_I / v_J = FH / FG = FT / FS. \quad (4)$$

Thus we have the second theorem:

THEOREM 2 The velocities at the ends of a diameter are inversely proportional to the distance between the focus and the points where the tangents to the ellipse meet the major axis extended.

From (3) and (4), we can find out the velocities at the ends of any diameter in terms of the velocity at the end of the minor axis,

$$v_I = (FT / FS)^{\frac{1}{2}} v_C \quad (5), \qquad v_J = (FS / FT)^{\frac{1}{2}} v_C \quad (6)$$

The velocities can be worked out explicitly in terms of v_C and the eccentricity of the orbit, e, for special points of the orbital ellipse such as the aphelion, the perihelion and the ends of the latera recta. In Fig.2, A denotes the aphelion and P the perihelion of the orbit. FL is the semi-latus rectum and EM the semi-latus rectum of the empty focus. We have the well-known relations

$$AF = a(1 + e), \quad PF = a(1 - e),$$

$$OC = b = a(1 - e^2)^{\frac{1}{2}}, \qquad LF = EM = a(1 - e^2),$$

a and b being the semi-major and semi-minor axes respectively. Clearly, $v_P / v_A = AF / PF = (1 + e) / (1 - e)$, a result cited by Freeman [3]. Also,

$$v_P = (AF / PF)^{\frac{1}{2}} v_C = (1+e)^{\frac{1}{2}} / (1 - e)^{\frac{1}{2}} v_C . \quad (7)$$

$$v_A = (PF / AF)^{\frac{1}{2}} v_C = (1-e)^{\frac{1}{2}} / (1 + e)^{\frac{1}{2}} v_C . \quad (8)$$

(7) and (8) give the velocities of the planet at its perihelion and aphelion respectively.

Furthermore, we know that the slope of the tangent at the latus rectum is equal to the eccentricity of the ellipse (Cf. [5]). Therefore, in triangle LSF, $FS = FL / e = a(1 - e^2) / e$. Also, $FT = FE+ET = FE+FS$. After a little algebra, $FT = a(1+e^2)/e$. Hence, $v_L / v_M = FT / FS = (1 + e^2) / (1 - e^2)$.

Also, for the velocities at the ends of the latera recta, we have

$$v_L = (FT / FS)^{\frac{1}{2}} \cdot v_C = (1 + e^2)^{\frac{1}{2}} / (1 - e^2)^{\frac{1}{2}} \cdot v_C, \quad (9)$$

$$v_M = (FS / FT)^{\frac{1}{2}} \cdot v_C = (1 - e^2)^{\frac{1}{2}} / (1 + e^2)^{\frac{1}{2}} \cdot v_C . \quad (10)$$

Equations (7) to (10) exhibit strikingly symmetrical relationships between the velocities of the planet and its positions in its orbital ellipse.

References

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[2] E.H. Lockwood, A Book of Curves, p.15, Cambridge University Press (1961).

[3] I.M. Freeman, Amer. J. Phys., 45, 585-586 (1977).

[4] A. Tan, to appear in Amer. J. Phys., 47, (1979).

[5] P. Van De Kamp, Elements of Astrodynamics, pp.13-15, Freeman, San Francisco (1964).

[6] The Geometry Problem Solver, p.700, Research and Education Association, New York (1977).

The Archimedeans

by M.R. Kipling, President 1977-8

Returning from long vacations (mis)spent in the United States, the President, Vice-President and Secretary attempted to inject new enthusiasm into the Society's bloodstream. This was manifested in the Cocktail Party, attended by over seventy members and several dons, held in the first week of term.

Professor Roger Penrose instructed the first evening meeting in how to tile the plane aperiodically with fat and thin chickens; "Puzzle Pieces", an autographed set of which are now in the Society's possession. After the Careers Evening the committee was introduced by the Speakers to the advantages of expense accounts, in Trinity Hall bar. A Film Evening was held, showing such educational masterpices as "Donald Duck in Math-magic Land". Professor Sir Herman Bondi gave a relatively general talk entitled "Gravitation".

Seven lunch meetings took place, including Professor Sir Peter Swinnerton-Dyer on "The History of the Mathematical Tripos", and Nobel Laureate Professor Brian Josephson on his new field of research, "Intelligence".

Visits were organised to the Old Royal Observatories at Greenwich,to Oxford - comfortably winning the problems drive - to the Mullard Radio Astronomy Lab. here in Cambridge, and to the Greene King Brewery at Bury St. Edmunds. The traditional Archimedes' Bath day party was revived to end the Lent Term, the President duly displacing his own volume of water from Trinity fountain.

The five college societies had variable fortunes over the year, high points being Dr. Mees' elucidation of "Catastrophe Theory" to an overfull Quintics' audience at Newnham, and the nail-biting, last over victory of Trinity Mathematical Society in the cricket match against the Adams Society.

The punt jousting took place as usual, and all those who had the misfortune to enter the Cam were rewarded with a Chelsea bun baked by the Vice-President herself. After the exams the ramble wound its way through Haslingfield to Great Evesden, where "The Hoops" was drunk dry, then returned to Cambridge via Toft and Coton. Again reviving almost lost tradition, the President engaged in arborial and subaquatic activities during the punt trip, on which we were accompanied by Drs. Skilling and Tennison and a bottle of Pimms No.1.

My thanks are due to Paul Verschüeren for dealing with all the paperwork, especially the frequent newsletters, to Jonathan Hitchcock for his tireless search for the Society's missing past minutes books, and to the rest of the committee for their friendship and support throughout a very successful year.

Knottiness

by Dr.W.B.R.Lickorish

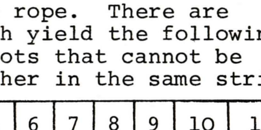
Everybody is aware that knots can occur in pieces of string, though, even in this era of ecology, Nature seems to be unaware of knotting phenomena. If man invented knots he did so some long time ago. A contender for the title of oldest knot is a bowline existing in the remains of a fishing net made of willow that was found in Finland and dated, by pollen analysis, at about 7000 B.C. Conversely, sharing the front page of 'The Times' (a former national newspaper), on 6th October, 1978, with a picture of the Master of Trinity knocking on his front door, was an announcement of the invention by Dr.E.Hunter of a fresh knot, Hunter's bend. In practice, knots used for binding and tying depend on friction for their efficacy. In mathematical knot theory, friction is irrelevant, and in order to avoid the trivial theory, knots are tied in a rope without any ends (though do not ask how that knot ever occurred). Thus a knot is just a simple closed (differentiable) curve in 3-dimensional space $\mathbb{R}^3$, both the space and the curve being oriented.

To specify a particular knot it is accepted that a picture suffices and that to be more precise would be easy, confusing and a waste of time. There is one knot, the trefoil, with a picture containing three cross-overs, one with four and two with five:

Two knots are considered to be the same if there is a homeomorphism of the whole of $\mathbb{R}^3$ to itself sending one simple closed curve to the other. This corresponds to the intuitive idea of moving a knotted loop of rope from one position to another; the movement moves the air in $\mathbb{R}^3$ as well as the rope. There are tables of knots with lists of pictures which yield the following information concerning prime knots (i.e. knots that cannot be regarded as two knots tied one after the other in the same string)

Number of Cross-overs	0	1	2	3	4	5	6	7	8	9	10	11
Number of prime knots	1	0	0	1	1	2	3	7	21	49	165	543 to 550

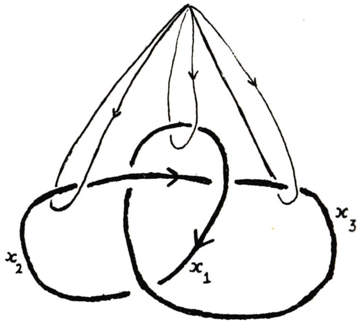
These enumerations neglect orientation, so that changing the direction of the arrow on a knot or reflecting the diagram in a mirror, or both, may yield different knots. Taking orientations into account there are, for example, two trefoil knots each the mirror image of the other; changing the arrow gives the same knot. The four cross-over knot is however fully symmetric, mirrors and arrow changes having no effect upon it. This latter fact can easily be verified in a few moments with the aid of a piece of string.



though plastic 'poppet' beads excel for practical work, a blackboard and coloured chalk being a good second best. It is quite difficult to prove that the two trefoil knots are indeed distinct. For that matter it is not easy to decide that any two pictures represent distinct knots. There is no algorithm for such a decision but there is a whole armoury of algebraic invariants associated with a knot which may be called into service. If calculation shows that an invariant differs on two knots then they are indeed distinct. As may be inferred from the table, the whole range of invariants is not infallible. It is true, and not difficult to guess, that there is a countable infinity of knots, and it seems a prohibitively difficult task to list them, in fact, to derive a list of knots (as of prime numbers) seems perverted. Knot lists are amusing and informative for low numbers of cross-overs, but work gets very much harder as the number of cross-overs increases, so that it seems unlikely that anyone will have the courage to attack twelve cross-over knots or the skill to make a machine attack instead.

Mathematical interest in knots centres around the techniques for associating invariants with knots, so as to develop the power to distinguish a couple of knots when necessary. The first task is to prove that a given picture does not represent the unknot, the 'knot' with zero cross-overs. There is an algorithm for doing this but of such complexity that it is reputed to be in mint condition. Workable methods usually follow from invariants associated with the fundamental group of the complement of the knot. The fundamental group of a space is the group consisting of equivalence classes of closed paths in the space all of which start and stop at the same given base point. Two such paths are equivalent if they are 'homotopic relative to their end points' (at the base point); this means that one path can be continuously slid to the other keeping the base point fixed. Two closed paths at this base point compose by first traversing one path and then the other. The equivalence classes of paths then form a group under this composition, the group identity being represented by the path that never moves at all from the base point, the inverse of a path being the same path traversed in the reverse direction. A fundamental group is associated with any topological space and details may be found in standard introductions to algebraic topology (e.g.[5]). The important thing to note is that a homeomorphism (i.e. a continuous bijection with continuous inverse) between spaces induces in an obvious manner an isomorphism between their fundamental groups. The group of a knot k in $\mathbb{R}^3$ is, by definition, the fundamental group of $\mathbb{R}^3 - k$; then the definition of equality of knots ensures that if two knots are the same their groups are ismorphic.

It is not too hard to gain some insight into the group of a knot. Imagine the knot is made of a loop of rather stiff thick orange wire, and that paths representing elements of the group of the knot consist of thin thread that starts near your eye, winds around the wire and returns to the start. Every time the thread goes round the back of the wire it may as well return to the base point before embarking on another trip around the back of another piece of the wire. The group then needs a generator corresponding to a thread looping around the back of every overpass section of the wire. There are relations amongst these group generators, one corresponding to each cross-over, and no more.



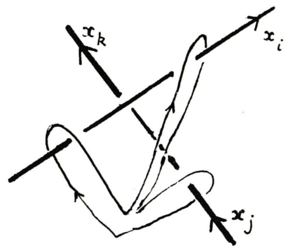
<u>Theorem</u> The group of a knot has a presentation

$(x_1, x_2, \ldots, x_n \,;\, r_1, r_2, \ldots, r_n)$

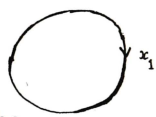
where the x_i are generators one allocated to each over-pass, the r_i are relations, one for each cross-over, of the form

$x_i^{-1} x_j \, x_i = x_k$.

It should be clear from the diagram that there is such a relation for each cross-over, though a precise proof that these are all the relations requires technicalities which will not be pursued. A moment's thought however convinces one that the relation r_n is not required it being a consequence of the other n-1 relations.



Once one has a presentation for a group it feels a little less abstract, though this area of combinatorial group theory can be hazardous. An easy observation, however, is that the group of the unknot has one generator and no relator. It is thus the infinite cyclic group. Of course this is an abelian group, and it can be shown that the unknot is the only 'knot' with an abelian group. To show then that a knot is inequivalent to the unknot it suffices to prove that its group is non-abelian. Here the trefoil knot provides an easy example, the group G being given by

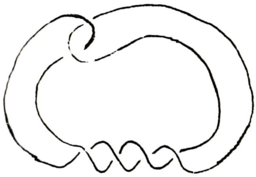


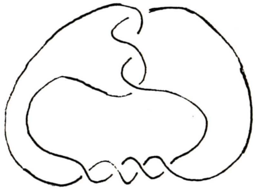
$$G = (x_1, x_2, x_3 \,;\, x_1^{-1} x_2 \, x_1 = x_3 \,,\, x_2^{-1} x_3 \, x_2 = x_1).$$

The permutation group S_3 is generated by the transpositions (1,2), (2,3), and (3,1), and of course (2,3)(1,3)(2,3) = (1,2) etc. Thus $f(x_i) = (j,k)$ defines a homomorphism $f : G \to S_3$ which is onto. As S_3 is not abelian neither is G and so we have proved

<u>Theorem</u> The trefoil is not unknotted.

The fact that the function f gives a homomorphism of groups followed from the fact that the image under f of a relation in G is a relation in S_3. In S_3 the conjugate of one transposition by a second is the third; there are only three transpositions denoted r,g and b. If one can colour the strings in the knot picture red or green or blue so that at any cross-over all three colours, or just one, occur then there is a homomorphism of the knot group to S_3, all the x_i coloured red map to r, the green to g and the blue to b. Provided that the colouring is not monochrome, this is a surjection and so the knot is truly knotted.





This method works for the first of the above two knots but not for the second. If the colours are replaced by numbers 0,1 and 2, then the colouring condition is that at a cross-over the number on the overpass should, modulo 3, be the average of the other two numbers at that cross-over. This can now be generalised to the idea of n-colouring a knot as follows: An n-colouring of a knot is a function c from the over-passes (the x_i) to the integers modulo n such that, at the prototype cross-over depicted earlier,

$$2c(x_i) \equiv c(x_j) + c(x_k) \text{ mod.} n .$$

Now for the generalisation of S_3 take D_{2n} , the dihedral group of symmetries of the regular n-gon ;

$$D_{2n} = (a,b \ ; \ a^n = b^2 = 1, \ ba = a^{-1}b) .$$

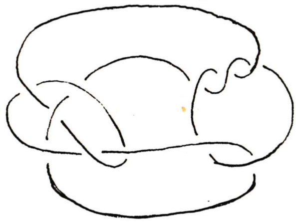
The relation in D_{2n} implies immediately that

$$(ba^p)^{-1}(ba^q)(ba^p) = ba^r$$

if and only if $2p \equiv q + r \text{ mod } n$. Thus $f(x_i) = ba^{c(x_i)}$ defines a group homomorphism $f : G \to D_{2n}$ which is surjective if 0 and 1 appear as values of the $c(x_i)$. With a little amplification this leads to

Theorem The maximal number n for which a knot has an n-colouring featuring 0 and 1 is an invariant of the knot.

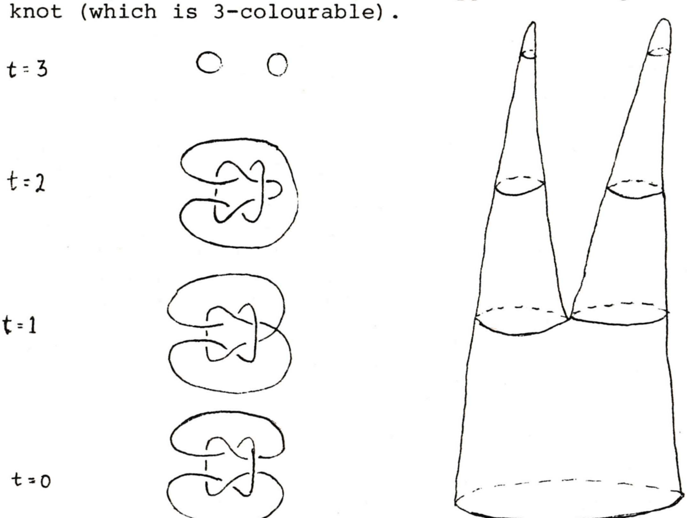
Use of this theorem is the fast way to distinguish knots. A moment's figuring shows that the seven cross-over knot above has a colouring with n = 13 and hence it is distinct from any knot mentioned so far. Unfortunately n-colouring does not always do any good to a knot. The knot C, below, only has an n-colouring when n = 1 which is unhelpful in that it does not distinguish C from the unknot, and non-triviality has to be established with an ad hoc construction of a homomorphism of the knot to a non-abelian subgroup of S_5 .



C

There is a considerable body of knowledge concerning knot groups (see [6]). One point to note is that as abstract groups they are rather particular. If one adds in relations to make all the generators commute with each other, then each original relation changes to the form $x_j = x_k$, so that the generators all become the same. That is a simple way of saying that the group G, quotiented by its commutator subgroup, is infinite cyclic. One last remark is to the effect that the reef knot and the granny knot are known to be different knots, yet they have the same groups.

A simple closed curve in $\mathbb{R}^3$ is unknotted if it is the boundary of a disc in $\mathbb{R}^3$. In recent years a great deal of thought has focussed on the question of which knots in $\mathbb{R}^3$ bound (smooth) discs that are allowed to go into $\mathbb{R}^4_+$, the upper half of 4-space with $\mathbb{R}^3$ as its boundary. A knot that does bound such a disc is called a <u>slice knot</u> because the disc and its reflection into the lower half of 4-space form a sphere (probably knotted) in $\mathbb{R}^4$ which is sliced by $\mathbb{R}^3$ in the given knot. The motivation behind this is to gain some insight into the way surfaces can exist in 4-manifolds and hence into the mysterious nature of 4-manifolds themselves. It turns out to be quite easy to give a description of a specific disc in $\mathbb{R}^4_+$ whose boundary is a knot in $\mathbb{R}^3$. Consider the fourth dimension as time t, and consider how the disc intersects the copy of $\mathbb{R}^3$ at time t. When t = 0 this intersection is the knot, when t is large it is empty. As t increases from zero, the intersections of the copy of R^3 with the disc give a pattern of contour lines on the disc, the interesting times being when the disc gets a minimum, a saddle point or a maximum. The diagram shows how this can happen starting with the reef knot (which is 3-colourable).



The sections of the disc between times zero and one all look much the same except that two points on the knot approach each other coming together at t = 1 which is a saddle point. After the saddle point the knot is divided into two components which are unlinked and unknotted, these components slowly move apart and disappear completely at maxima . In general there may be n saddle points and (n+1) maxima. The introduction of minima into the picture is also permitted in general; this just introduces a little round circle unlinked from the knot. Such minima never seem to do any good, and it is a long standing conjecture that they are unnecessary. If one tries to put a smooth disc on an arbitrary knot by these means one will probably fail, for only 22 out of the first 250 knots are slice. Most knots fail to be slice because, when split into two or more components at saddle points, the various pieces are linked. It is only unlinked unknotted components that are permitted to disappear at maxima. The trefoil knot is not slice nor is the 4-cross-over knot. The eight cross-over knot shown here is slice, the proof being left as an exercise. Poppet beads really come into there own for such an exercise, for they can be unpopped as a saddle point approaches and repopped after it. Nobody knows if the knot C is a slice knot, though quite a number have tried to find out, so, proving that it is, or is not, slice is also left as an exercise for the future writer.

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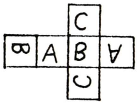
[6] L.Neuwirth. Knot groups. Ann. of Math. Studies 56, Princeton U.P., 1965.

Solutions to Problems Drive

1 X = 143, Y = 586, Z = 729.

2 

3 1 mile.

4 (i) 18, 21. (nth prime - 1).
(ii) 114, 210 (sum of previous two terms, in base 5).
(iii) 9, 4 (last digit of n^2).

5

6 Volume = $\sqrt{2}/3$, Area = $2\sqrt{3}$.

7

1	7	2	8
8	3		1
2		2	0
5	0	5	0

8 23 crowns.

9 9.12 p.m.

10 619737131179.

11 A is wearing D's hat.
B is wearing E's hat.
C is wearing H's hat.
D is wearing B's hat.
E is wearing G's hat.
F is wearing A's hat.
G is wearing C's hat.
H is wearing F's hat.

12 0.