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Editorial

Long-standing readers of Eureka will have noticed that a change has come over the magazine in the last few years. Whereas once Eureka was definitely a Cambridge magazine, written by Cambridge men and providing a forum for discussion of Cambridge's own peculiar problems, it has now lost almost all its bias towards Cambridge. This could, of course, change overnight with the inauguration of a traditionalist as editor and I have no doubt that this breath of fresh air owes a lot to a succession of Churchill editors in recent years.

So what sort of magazine should we be amining for? If Eureka were published several times during the year, the original approach would be essential. As it is at the moment there is great scope for broadening its outlook and in recent issues a large number of articles have come from outside Cambridge. To some extent I think this mirrors the mood of the modern Cambridge undergrad who would rather spend an afternoon discussing the introduction of new courses into the Maths Tripos than the intricacies of the Archimedeans' constitution. I should also point out that we now have a great many readers from outside this university to whom its internal affairs mean nothing and on whose financial support we rely very heavily for our existence.

With that I leave it to readers to voice their own opinions. I or my successor would welcome any written comments or suggestions.

Finally, I should like to thank all those who have helped me in the past year in the production of this issue, in particular my predecessor C. R. Prior, I. M. Gardiner, I. F. McCredie and C. J. Myerscough.

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Inertial Navigation

by Professor M. J. Lighthill

Traditionally navigation of ships, and later, aircraft was carried out by reference to external objects: the sun and the stars; topographical features; lighthouses, or their later equivalent, radio beacons. Only in the past twenty years has it become possible for such a vehicle to be navigated without any external reference. This means that its position can be known with precision by someone in a sealed cabin into which no light or other signal from outside the vehicle penetrates.

To see how this is done, consider first a much simpler type of vehicle: a railway train on a straight horizontal track. Even in a sealed cabin of the train, someone in a forward facing seat feels a force pressing him against the back of the seat while the train is accelerating, and a force the other way when the train is slowing down. In mechanics we call this an 'inertial force', and recognise that on any object of mass m in a train whose distance moved from its starting point is $X(t)$ there acts an inertial force $-m\ddot{X}(t)$.

Thus if x measures position in a frame of reference moving with the train, the relation of the ordinary forces F acting on the mass to its total acceleration $\ddot{x} + \ddot{X}$ can be written

$$m\ddot{x} = F - m\ddot{X},$$

if the $m\ddot{X}$ is taken to the right-hand side. This shows that the effect of acceleration of the frame of reference is exactly as if an external force, the inertial force $-m\ddot{X}$, acts on the mass. If this force is measured by the extension it causes in a suitable spring, the acceleration $\ddot{X}$ can be inferred (actually, a well chosen spring-mass combination on these lines can be quite a good 'accelerometer') and a double integration, starting from given initial conditions, can be performed by automatic means to deduce the position X.

Admittedly, no-one would need to use that method for a navigation problem as simple as that of a train on a straight horizontal track! Is there, however, a possibility of extending it to vehicles that pitch and heave and roll as they pursue complicated paths in three dimensions?

At first sight you might think that the same method can apply in three dimensions, except that the vehicle position is now a vector $\underline{X}(t)$ and so is the inertial force $-m\ddot{\underline{X}}(t)$. The sealed cabin might contain three accelerometers in fixed mutually orthogonal directions to measure the components of $\ddot{\underline{X}}$, which could then be doubly integrated to give $\underline{X}$.

You can see, however, that this would be useless if the accelerometer directions were fixed relative to the pitching and rolling cabin, because at every instant they would be measuring components of $\ddot{\underline{X}}$ in directions themselves subject to continual change in a way unknown within the sealed cabin. An important further difficulty is that any mass m is subject to a gravitational force mg, which must be subtracted from the total measured force to give the inertial force $-m\ddot{\underline{X}}$ (now that this is no longer just horizontal), and the direction of $\underline{g}$ also is unknown within the cabin.

Almost all the different solutions to these difficulties, that have been brilliantly thought out (notably by Draper in the U.S.A. and by Ishlinsky in the U.S.S.R) and developed so as to make 'inertial navigation' a reality have one feature in common. They involve a 'space-stabilized platform' in the vehicle, so engineered that it maintains a prescribed orientation in space however much the vehicle may pitch and roll around it. Out of this wide variety of inertial navigation methods, including many diverse ways of engineering the platform, I shall describe one particular method of notable interest and elegance.

In this method the platform is caused always to remain <u>horizontal</u>, and to keep a fixed orientation relative to true north and east. Three accelerometers fixed on the platform measure the components of acceleration in the northward, eastward and upward directions, from which, with known initial conditions, changes in latitude, longtitude and altitude are automatically calculated—provided that a clock is available to give information on how much the earth has rotated beneath the vehicle since the journey started! The subtracting of the force $\underline{mg}$ is straightforward since $\underline{g}$ has magnitude known as a function of position and acts always in the direction of the vertically measuring accelerometer.

It is the wonderful properties of modern gyroscopes that have made it possible to build a platform that does this trick of remaining horizontal and fixed in orientiation although attached to a pitching and rolling vehicle. To understand their role, consider how one would be used to solve a simpler problem: suppose that we have a fixed horizontal turntable in the laboratory and wish to mount it on a horizontal platform in such a way that an arrow marked on the platform will always point north whatever action may be taken to rotate the turntable beneath it with randomly varying angular velocity which somehow is controlled so as to be equal and opposite to that of the turntable!

For this purpose the platform must carry a device which senses if the platform starts to turn at all and immediately sends a signal to the motor to turn it back in the opposite sense. This is the principle of 'negative feedback' that is the staple technique of the control engineer. An excellent choice for the sensing device is the modern single-degree-of-freedom flotation gyroscope. The basic simplicity of its conception has helped in allowing it to be engineered to an extraordinary degree of precision.

The external shape of the 'gyro unit' is of something like cylindrical shape, and in the present application would be mounted with its axis horizontal. Pivoted about that horizontal axis on jewelled bearings inside the external case is the 'gyro element', whose function is to signal, when the platform has turned about its vertical axis, by itself undergoing a small angular deflection α about its horizontal axis. The gap (a small fraction of a millimetre) between the internal gyro element and the external case of the gyro unit is filled with a special oil of extreme purity, free of any minute bubbles or solid particles.

The density of this oil is so chosen to be exactly equal to the mean density of the gyro element; the weight of the gyro element is then exactly balanced by its buoyancy, and therefore puts no load on the delicate pivots. The gyro element can be said to be supported by 'flotation' in this liquid. At the same time the viscosity of the oil is so chosen as to provide an exactly linear damping couple $-D\dot{\alpha}$ resisting any change in the angular deflection α.

Enormous ingenuity has been lavished on the 'gyro element', which contains inside itself the rotor—a rotating mass of metal of complicated shape spinning at an enormous angular velocity such as 400 revolutions per second. The speed is kept very accurately constant by meticulous ball-race design and by the precision of the frequency of the alternating-current supply to the synchronous motor inside the gyro element; however, great care has to be taken to give the electrical leads, which penetrate the flotation fluid to supply the motor, such flexibility that their presence negligibly modifies the couple resisting changes in α. The complexity of shape of the rotor is aimed at making its angular momentum H as large as possible (figures as great as $1 m^2 kg s^{-1}$ are achieved, in grapefruit-sized gyros) while making the whole gyro element as resistant to deformation as possible.

If the platform is being given a varying angular deflection $^1\theta(t)$ about a vertical axis, the rotor's horizontal angular momentum H is being swung round at an angular velocity $\dot{\theta}$, generating angular momentum in a horizontal direction at right angles (that is, about the gyro-unit pivot axis) at a rate $H\dot{\theta}$. The corresponding reaction on the gyro element itself is an equal and opposite couple $H\dot{\theta}$, the so-called 'precessional couple'. It is balanced primarily by the resisting couple $-D\dot{\alpha}$ of the viscous oil, so that if $\alpha = \theta = 0$ initially then α continues to measure departures of θ from zero according to the linear relationship $\alpha = H\theta/D$.

Thus the negative feedback required to ensure that θ is kept as close as possible to zero must be a feedback which causes the motor on the turntable to apply a twisting couple to the platform depending in a negative sense on α; that is, tending to reduce θ when α is positive and increase it when α is negative. To this end, an electromagnetic pick-off on the gyro unit generates an electric signal proportional to α (reflecting changes in the phases of the alternating current, induced in small coils clustered around the gyro-unit pivot by coils opposite them in the gyro-element, when the latter turns about the pivot), and this signal is passed through special control circuitry whence a signal determining the twisting couple emerges. The special control-circuitry is designed according to the interesting 'stability theory' of control systems to provide stably the necessary rapid reduction of α (and therefore also of θ) to zero.

An important feature of the system described above is that α always remains very small indeed, which is most desirable for keeping gyro performance good and correctly describable by simple, tractable linear equations. In fact, when we are posed with the different problem, not of keeping θ as close as possible to zero but of making the angular velocity $\dot{\theta}(t)$ take as closely as possible some externally prescribed form $^1\omega(t)$, then the system described above is unsatisfactory because all sorts of difficulties follow whenever α is allowed to depart significantly from zero. However, a modified form of the system solves this modified problem.

In this modified form, the pivot opposite to the pivot around which the electromagnetic pick-off coils are clustered is surrounded by similar opposing clusters of coils which enable a small couple to be applied tending to turn the gyro-element within the gyro-unit. By making this couple equal to $-H\omega$, which supplements the precessional couple $H\dot{\theta}$ and the viscous damping couple $-D\dot{\alpha}$, and by using the same negative feedback system tending to annul α as before, we see that $\dot{\theta}$ is forced to be close to ω and, indeed, that the difference $\theta - \int_0^t \omega dt$ is equal to $D\alpha/H$ and therefore itself is kept very close to zero.

Another interesting modification of the basic gyro concept is the type of accelerometer called PIG ('pendulous integrating gyro'), which can give more accurate results than the mass-spring type mentioned at the beginning of this article. In a PIG the centre of gravity of the gyro-element has been displaced: although still on the spin axis of the rotor, it lies at a distance l, say, from the axis on which the gyro-element is pivoted; hence the name 'pendulous'. The gyro is accordingly sensitive to acceleration in the direction of the input axis: from the associated inertia force, plus the gravitational force in cases like that discussed above when the input axis is vertical, giving a total force F, there results a couple F supplementing the precessional couple $H\dot{\theta}$. In this case the negative feedback system tending to annual α leaves the angle θ satisfying the equation $H\dot{\theta} + Fl = 0$, so that this angle θ through which the motor turns the system takes a value proportional to the integral of F with respect to the time; hence the name 'integrating gyro'. Often the time integral of an acceleration component (corrected in the case of a vertical component for gravity effect) is a very useful quantity to have measured.

We are now ready to get an idea of how the space-stabilised platform inside a vehicle works. We saw that its function is to carry three accelerometers, responsive to accelerations in the northward, eastward and upward directions. At the same time, it carries three gyros. In particular, in order to maintain its horizontal condition, there are two gyros, responsive respectively to twists about the northward and eastward directions.

You might imagine that their function would be to cause any such twists to be annulled, but careful thought shows that for a platform on a vehicle moving over the surface of the globe to stay horizontal, a twisting about the northward axis (for example) is needed at angular velocity equal to v/R, where v is the eastward velocity (directly measured by a PIG accelerometer as the integral of the eastward acceleration) and R is the distance from the centre of the earth. Accordingly, it is the modified type of gyro described above that can make the angular velocity to which it is sensitive follow very closely some externally prescribed form, which is needed to govern the twisting about both the two horizontal axes. However the third gyro needed is of the unmodified type, and its function is simply to prevent twist about the vertical.

In this fully three-dimensional case there are geometrical complications in the arrangement of the three motors needed to drive the platform and maintain its orientation as the vehicle moves. (These motors replace the single motor mounted on the turntable in my simple descriptive example.) The arrangement needed is called a gimbal mounting.

In this arrangement the platform is twisted about the vertical by motor no. 1, which is mounted on a frame that is itself twisted about a perpendicular axis by motor no. 2, which in turn is mounted on a frame twisted about an axis perpendicular to the latter axis by motor no. 3. The negative feedback control system, aimed at ensuring that the angle α for each of the three gyros on the platform remains always very close to zero, is in these circumstances somewhat intricate. There is a direct negative feedback of the α signal from the gyro responsive to twist about the vertical into motor no. 1, but more complicated signals into the other two motors.

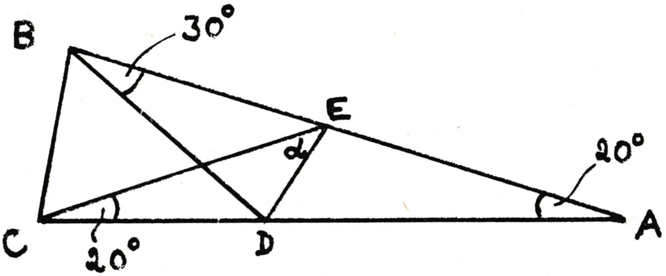
Actually, if the vehicle were keeping fairly close to a straight path, and only pitching and rolling and yawing about that by angles of ten or twenty degrees, it would be possible for the axes of twist of motors nos. 2 and 3 to remain close enough to the north and east directions respectively for a system to be satisfactory in which they derive their negative-feedback signals from the gyros responsive to twists about the north and east directions respectively. The 'control loops' would cooperate to keep the angle α for each of the three gyros very small indeed in this case.

In the actual system, which must operate under a wider range of conditions, the control loops are 'coupled', in the sense that the signals into motors nos. 2 and 3 involve linear combinations of the α angles from the different gyros, the coefficients depending on the angles between the three frames of the gimbal mounting, which are themselves continuously measured by electromagnetic pick-offs. Since however the effectiveness of the negative feedback system is not very sensitive to the exact magnitude of these coefficients, this is not one of the parts of the overall system that has to operate to high precision.

It is, in fact, the gyros themselves whose enormously high precision (in certain cases allowing angular errors to creep in at as low a rate as a thousandth of a degree per hour) has made inertial navigation a reality. Some of the developments for this purpose were very ingenious: for example, the replacement of the jewelled pivot and bearing by electromagnetic suspension, using the same coils to centre the gyro element within its external case as are being used for pick-off and couple-application purposes!

There is great scope for exciting and interesting applied mathematics in inertial-navigation research and development: in problems of the estimation and minimisation of those errors in the gyro response that result from the many effects neglected in the above greatly simplified account, including a full treatment of the rigid-body dynamics and proper account of the effects of elasticity; in problems of estimating errors in the system as a whole, due to many effects such as deviation of earth's gravity from cen central-field-of-force conditions; and, above all, in the subtle and delicate control-engineering mathematics that lies at the heart of the whole technique.

A Triangular Problem



AB = AC

Find α using geometry

answer p. 30

The Marriage Problem

by D.R. Grey

My hypothesis is as follows. In life one is to meet successively a number N of women eligible for Marriage, and at each encounter one must decide whether or not to marry her, knowing her 'marital value'. The marital values of different women are assumed to be independent random variables with a common known density function f(x). The problem is to find the strategy which maximises the expected marital value of one's chosen wife. Divorce and polygamy do not exist and a rejected candidate cannot be reconsidered. Two models are considered here, using different assumptions on the value of N.

Model 1 : N fixed, known

The problem is one of dynamic programming. Let V_n = expected value obtained by choosing optimally from a sequence of n.

Suppose we have a sequence of n + 1 to choose from. Then we will choose X_1, the first, if and only if, X_1 is greater than the expected value obtained by choosing optimally from the remaining n, i.e. the strategy is accept or reject X_1 according as $X_1 >$ or $< V_n$.

The expected value thereby obtained is

$$V_{n+1} = E(\max(X_1, V_n)) \tag{1}$$

This is a recursion formula for the constants V_n, with starting value $V_1 = EX_1$. So for the case where N is fixed and known, the strategy to be adopted is: At stage n accept or reject according as $X_n >$ or $< V_{N-n}$ $(1 \leqslant n \leqslant N-1)$ and the expected value thereby obtained is V_N

As $\{V_n\}$ is increasing (by(1)), $\{V_{N-n}\}$ is decreasing, one becomes less discriminating as one's choices are narrowed.

By way of example, in the case of $f(x) = e^{-x}$ $(x \geqslant 0)$, the recursion formula (1) is

$$V_{n+1} = V_n + e^{-V_n}$$

Model 2 : N a random variable

Regarding the encounter of an eligible woman as a transient recurrent event N will have geometric distribution, i.e.

$$P[N = n] = (1-p)p^n \qquad n = 0, 1, 2, \ldots \qquad (0 < p < 1)$$

where p is a parameter depending on the individual and his circumstances. Because of the 'lack of memory' property of this distribution, the strategy to be adopted at each encounter will be the same, and will take the form : accept or reject according as $X >$ or $< V$ where V is the expected value to be obtained by optimal choice. Because there is a finite probability $1-p$ of not meeting an eligible woman, one must assign a value, w.l.o.g. zero, to bachelorhood.

At the start, there are three possibilities:

a) with probability $1 - p$, no encounter: value zero

b) with probability $pP(X > V)$, encounter and acceptance: expected value $E(X \mid X > V)$.

c) with probability $pP(X > V)$, encounter and rejection, in which the process starts again: expected value V.

Hence V satisfies

$$V = pP(X > V)E(X \mid X > V) + pP(X < V)V$$

i.e. $V = pE(\max(X, V))$, which is to be solved for V.

In the case $f(x) = e^{-x}$ $(x \geqslant 0)$, this equation becomes

$$(1-p)V = pe^{-V}$$

I leave it to the reader to explore other interesting variations, such as uncertainty (on encounter) in marital value, or the possibility of meeting a female mathematician who has adopted a counter-strategy.

Two Problems

1. Transsylvania may be regarded as the complex plane $\underline{C}$. Suicides have been buried at integer points $p + iq$, where $pq = 0$ or $(p, q) = 1$. A vampire travels about $\underline{C}$ by night, with range v/night, terrorizing the inhabitants, but must rest during the day in a suicide's grave. Show that

   1) If $v < \sqrt{2}$ there are graves he cannot visit.

   2) If $v < 2$ there are safe points for the inhabitants.

   3) Show that for no finite v are there no safe points.

2. Complete the magic square using the remaining integers less than 16. (A square is magic if the sums of the entries in each row and column are the same - the diagonals need not possess this property.)

6	1	16	11
.	.	.	2
.	7	.	13
.	.	.	8

answers p. 30

A Moving Story

by K. Andrews and N.J. Davis

Euler's equations with delight
fill all your dreams both day and night
and when you're tiring of these games
refer the lot to moving frames

When dealing with the earth's nutation
you here encounter more frustration
because you're answer isn't quite
what the stars proclaim is right.

To help you all God did arrange
to live a Frenchman called Lagrange.
His holonomics were such great fun,
you q'd all day for one equation.

$\underline{r} \times m\underline{v}$ is a vector;
to prove this takes you one whole lecture;
and when you've got these vector moments,
guess what? Yes, then you take components.

Moving axes are a farce,
and tops rotate about their $\underline{r}$'s;
it's all right when they just precess,
but otherwise we're in a mess.

There is no need to pull a face
at rigid bodies in free space:
to solve you have this simple code -
"Polhode rolls on Herpolhode"

If your body has a cone
it's best to leave it well alone;
but if you're trying to coerce her
stick to principles of inertia.

Now if you think we're tongue in cheek
come and join with us next week.
Then you'll see that these polemics
sum up the lectures on Dynamics.

Double Six

by W.L. Edge

Professor at Edinburgh University

Despite the prevalent opinion that no one nowadays knows any geometry it is perhaps permissible to hope that there are still a few people aware that 4 lines in [3] - [n] is a standard notation for n-dimensional projective space - have 2 transversals. If, then, a, b, c, d, e are 5 lines which all meet a line f', any 4 of them have a second transversal besides f': there are 11 lines.

a	b	c	d	e	
a'	b'	c'	d'	e'	f'

Then T, the theorem of the double-six, states that a', b', c', d', e' have a common transversal f: there are 2 sextuples of lines; each line meets 5 of the complementary sextuple but is skew to every other line. Indeed a, b, c, d, e, f' lie on a cubic surface F. For such a surface - the left-hand side of its equation being a quaternary cubic composed of 20 terms - can be found to satisfy 19 independent linear conditions; and at most 19 ensure that F contains the 6 lines: 4 for f' and thereafter 3 for each of a, b, c, d, e. And F contains 27 lines among which are 36 double sixes. This figure, with its group of 51840 permutations that admits representations of degrees 40 and 45 as well as 27 and 36, must be one of the most thoroughly flogged horses to run in our mathematical literature.

But T itself can be proved, was indeed proved (1) in 1911, without mentioning F or the other 15 of its 27 lines. This proof, depending only on properties of lines and quadrics, was incorporated by its discoverer in a standard text (3, p. 159); but he came to appreciate that a less artificial proof might ensue on projecting a figure onto [3] from [4]. He eventually devised this proof and published it (2) in 1920. Its naturalness and inevitability are consequent on 3 lines in [4] having a unique transversal, so that 4 lines in [4] yield a double-four consisting of the lines themselves and the transversals of their 4 sets of 3.

If lines in [4] intersect so do their projections onto [3] from a point 0; so the projection of a double four V in [4] is a double four v in [3] - no additional intersections arise if 0 is "of general position". But 4 lines in [3] belong to an infinity of double fours, each set of 3 having not merely one but an infinity of transversals. The crux is that a double four in [3] is not, in general, the projection of one in [4]; one half a, b, c, d has transversals e', f'; the other half a', b', c', d' transversals e, f; for v to be a projected double four it is necessary for one of e, f to meet one of e', f': say - a mere matter of labelling - e must meet f'. Then the geometry of those planes (4, p. 123) through 0 that meet the halves of V implies that e' meets f, and this is T.

The main purpose of this article is to signalise the extension, Grace's extension, of this theorem of the double six; and then to ask readers to consider whether, and in what circumstances, his extension is itself extensible. When 6 lines a, b, c, d, e, f in [3] have a transversal t the application of T to the quintuples among the 6 lines produces lines u, v, w, x, y, z that complete double sixes. Grace's extension G says that these last six lines have a common transversal. The originality and brilliance of Grace's paper (6) are proverbial. He proves a theorem about linear complexes, or screws. A screw can be specialised to consist of all lines meeting a given directrix line, and when all the screws of a certain set are so specialised G is a property of their directrices.

The Grace figure is symmetrical in the sense that the 6 lines which emerge by applying T to quintuples among u, v, w, x, y, z are the original a, b, c, d, e, f. Each of these sextuples includes 15 sets of 4 lines having a single transversal accompanying that of the whole sextuple; and this, as the description of T itself shows, meets 2 lines of the other sextuple. So one builds a figure of 12 + 2 + 30 = 44 lines, each of the 12 meeting 16 of the 32 and skew to the other 16 (10, p. 165).

Geometers prefer to discuss properties of lines in [3] by using Klein's mapping: a (1, 1) correspondence, without any exceptional elements, between the lines of [3] and the points of a non-singular quadric in [5]. All these matters - T itself, Grace's extension G and his general result of which G is a special instance - can be described in terms of the geometry in [5]. Many would regard this as a simplification; some details follow p. 58 of (4).

There emerge, by omitting in turn each of 7 lines with a common transversal, 7 of Grace's lines. Do these have a common transversal too? The consensus, if one may so describe the opinions of people so few in number as competent to hold one, is that they do not. But then, if not, are there particular sets of 7 lines for which they do? One has tacitly supposed that the customary field C of complex numbers underlies the geometry: the very fact that 4 lines in [3] have 2 transversals would not be true if a quadratic did not have 2 zeros; in this context particularisation means imposing some geometrical restriction over C. But one can also base the geometry on a different field, say on a finite field GF(q) where q is a power of a prime; is there a Grace extension over GF(q), and does it, if existing at all, extend? Some approach to these matters has been made by a young Australian geometer (7, 8, 9) and, as is to be expected with finite fields, some fascinating figures appear. Of course, since a line consists of only q + 1 points, the lowest values of q are irrelevant; no double six could occur unless $q + 1 \geq 5$.

One may, whatever field lies at the base of the geometry, take 5 planes, no 4 concurrent. This pentahedron P has 10 vertices, each opposite to one of 10 edges; 3 lines join any vertex v to those on the opposite edge, i.e. to the intersections of this edge with the 3 planes meeting at v; these 15 lines lie 3 in each of the 5 planes, those in any one being the diagonals of the quadrilateral in which this plane is intersected by the others. All 15 lines lie on a cubic surface D, Clebsch's diagonal surface. They are indeed those among 27 residual to a double six Δ on D, Δ existing provided that the roots of $x^2 - x - 1$ are in the field; the real field itself will serve. The 10 vertices of P are Eckardt points on D, a point on a cubic surface being so named if it is the concurrence of 3 coplanar lines though, in fact, such points were noted by Cayley so long ago as 1849. Since there is a projectivity imposing any one of the 120 permutations on the planes of P it follows that Δ is invariant under a group of 120 projectivities; it is a Burnside double six (5).

The equation $x^2 = x + 1$ is also soluble in GF(4), the smallest field to permit the existence of a double six; and the corresponding cubic surface H has a perhaps unexpected symmetry. For GF (4), like every field of characteristic 2, provides a geometry in which the 3 diagonals of any plane quadrilateral concur; the 15 lines that were on D afford not 10 but 15 Eckardt points on H. Indeed P is now only one of 6 pentahedra; the residual double six Δ is now invariant under a group of 720 projectivities, and H is a diagonal surface in sextuplicate. The left-hand sides X, Y, Z, U, V of the equations of the faces of P are linearly dependent so that one may take

$$X + Y + Z + U + V = 0$$

Then H, like D, is

$$X^3 + Y^3 + Z^3 + U^3 + V^3 = 0$$

and contains the 15 lines with equations such as

$$X = Y + Z = U + V = 0$$

But, over GF(4), this is the same line as

$$X = (X + Y) + (X + Z) = (X + U) + (X + V) = 0,$$

and the pentahedron whose faces are

$$X = 0, \quad X + Y = 0, \quad X + Z = 0, \quad X + U = 0, \quad X + V = 0$$

has precisely the same properties as P; the left hand sides of these 5 equations sum identically to zero, and the sum of their cubes is, identically over GF(4), $X^3 + Y^3 + Z^3 + U^3 + V^3$.

But all this is far less than half the story, indeed only one thirty-sixth of it. All 36 double sixes on H are projectively equivalent. The 15 lines on H residual to any one of its double sixes lie by concurrent threes in 15 planes, faces of 6 pentahedra every pair of which share a common face. Those projectivities for which H is invariant form a group of order 25920, transitive on the 36 double sixes; H consists of 45 points, every one an Eckhardt point (7, p. 87).

Scrutiny of the geometry over the smaller fields discloses that there is no double six when $q = 5$, no Grace extension when $q = 7, 8$. If $q = 8$ the lines u, v, w, x, y, z do appear but (9, p. 355) they are concurrent, generators of a quadric cone, so that the question of their having a common transversal does not arise. Although the complete Grace figure involves lines met by 16 others this need not inhibit investigations for every $q < 16$; certain coincidences among the 16 points are permissible. For example: if $q = 9$ every line consists of 10 points. The Grace figure does exist; each of the 12 sets of 16 collinear points reduces to a hexad, reckoned twice, and its residual tetrad. This tetrad is harmonic, and over a field of characteristic 3, and so over GF(9) in particular, such a tetrad admits not merely 8 but 24 self-projectivities. Some of the Grace figures may have interesting groups of automorphisms; but beyond all this the outstanding question still poses itself. Over which fields, and in which special circumstances, if any, is there an extension of G?

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The Quadratic Equation

by Dr. E. A. Maxwell

This note is written with a moral: that those of us who cannot produce the great discoveries of mathematics (and we are the majority) may nevertheless both find and give pleasure by surveying the familiar scene from fresh viewpoints. It is unlikely that what I have to say is new, but I have not myself seen it in print.

The quadratic expression

$$ax^2 + 2hx + b$$

can be written in matrix form

$$\underline{x'ax}$$

where $\qquad \underline{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \qquad \underline{a} = \begin{pmatrix} a & h+\lambda \\ h-\lambda & b \end{pmatrix}$

The matrix $\underline{a}$ is usually taken to be symmetrical, so that $\lambda = 0$; this, indeed defines the matrix of the quadratic expression. Suppose, however, that we adopt the alternative convention that $\underline{a}$ is chosen to be singular. Then

$$ab - (h + \lambda)(h - \lambda) = 0,$$

so that $\qquad \lambda = \pm\sqrt{(h^2 - ab)}$

We assume that $h^2 - ab \neq 0$.

The quadratic equation

$$\underline{x'ax} = 0$$

is certainly satisfied if

$$\underline{ax} = \underline{0},$$

or (taking the positive square root)

$$\begin{pmatrix} a & h + \sqrt{(h^2 - ab)} \\ h - \sqrt{(h^2 - ab)} & b \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \underline{0}.$$

The respective rows give the equations

$$ax + \{h + \sqrt{(h^2 - ab)}\} = 0,$$

$$\{h - \sqrt{(h^2 - ab)}\}x + b = 0.$$

Now these two relations for x are the same, since a is singular (or by a simple independent calculation).

Hence $x'ax = 0$ is satisfied by

$$x = \frac{-h + \sqrt{(h^2 - ab)}}{a}$$

and, similarly, by

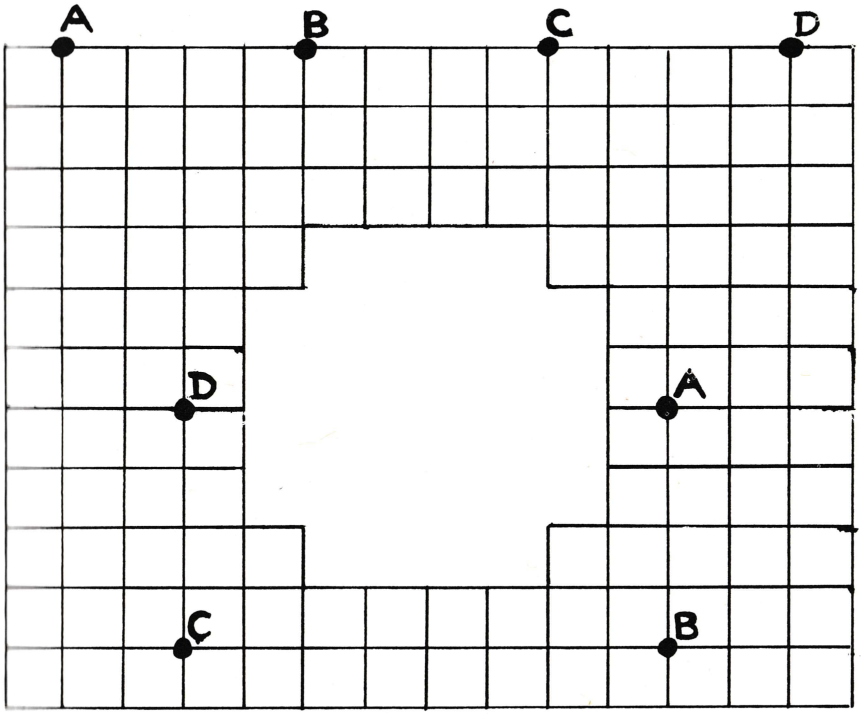
$$x = \frac{-h - \sqrt{(h^2 - ab)}}{a}.$$

Problem for Poultry Farmers

The chicken was twice as old when when the day before yesterday was tomorrow today was as far from Sunday as today will be when the day after tomorrow is yesterday as it was when when tomorrow will be today when the day before yesterday is tomorrow yesterday will be as far from Thursday as yesterday was when tomorrow was today when the day after tomorrow was yesterday. On what day was the chicken hatched out? (answer p. 31)

(acknowledgements to Eureka 1940)

It's a Plot!



Join the pairs of dots marked similarly (i.e. A to A, B to B, etc.) using four lines which do not cross or touch at any point. The routes must follow the lines of the grid, and may not pass through any of the lettered dots.

The Archimedeans

As was expected, the Society again had a very successful year, with good attendance at all meetings. Professor E.C. Cherry's talk on 'Ideas and Images' and Professor R.A. Rankin's address 'Marriage by Permutation in Primitive Societies' were particularly interesting. Dr. B.J. Birch gave us a fascinating account of 'Theorem's I would like to prove' and Professor M.W. Thring's inspiring lecture 'The Design of Robots to Relieve Human Beings from Repetitive Work' was accompanied by a film of some of the robots he had produced. The tea meetings proved very popular and notable talks were given by Dr. A.J. Casson, Dr. J.T. Knight and Dr. J.C.P. Miller.

The bookshop is flourishing, the music and bridge groups have met regularly and the puzzles and games ring has continued to thrive. The computer group has had an active year. As well as the usual visit to the Mathematical Laboratory there was a visit to I.C.L.

This year's speakers include Professor E.C. Zeeman, Professor B.H.P.Rivett and Dr.Imré Lakatos. There will be the usual careers meeting and a problems drive against the Invariants has been arranged for February. There will also be a visit to the Rutherford Laboratory, Harwell and a course of Philosophy seminars is being arranged.

Any suggestions of possible changes or improvements in the Society's activities are most welcome and can be made directly to the Secretary or via the suggestion's book in the Arts School.

Oliver Reid, Secretary

Ships and Eigenvalues

by D.Rees

Professor at Exeter University

One of the most characteristic features of good mathematics is the way that arguments belonging to one branch of mathematics are used with effect in quite a different branch. A very good example of this is the original proof by Wielandt and Hoffman of what is now known as the Wielandt-Hoffman inequality in the theory of eigenvalues of real symmetric matrices (or, indeed, normal complex matrices). The basis of this proof is a simple argument first used in the study of routing of cargo ships.

One of the first, if not the first, problems, which are now subsumed under the name of Linear Programming, was the so-called transportation problem. The original form of this was the following. A number of ships are stationed at ports $P_1, \ldots, P_m$, a_i ships being stationed at port P_i. These ships are required at n other ports $Q_1, \ldots, Q_n$, b_j ships being required at Q_j, and by fortunate chance $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. The distance from P_i to Q_j is d_{ij} nautical miles. The problem is to move the ships in such a way that the total distance sailed is a minimum.

Let us now put this problem in mathematical form. Let the variable x_{ij} denote the number of ships moved from P_i to Q_j. The variables x_{ij} must satisfy the equations

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, \ldots, n$$

In addition the values of the x_{ij} should be non-negative integers, but we will forget about the latter part of this statement for the moment. The problem is to minimise the function $\Sigma d_{ij}x_{ij}$, subject to these conditions. Such a minimum exists since range of values of the x_{ij} permitted defines a bounded closed subset of the space of mn dimensions. I want to show that among the points where this minimum is attained is at least one where all the x_{ij} take integer values.

Among the points where $\Sigma d_{ij}x_{ij}$ takes its minimum value choose one with as few of the x_{ij} different from zero as possible. Suppose that N of the x_{ij} are non-zero and that the solution is

$$x_{ij} = u_{ij}.$$

We first note that although we have $m + n$ equations among the constraints, any one of them is a consequence of the rest, and so we can remove one of them. Put all the variables for which the u_{ij} are zero equal to zero. We are then left with $m + n - 1$ equations in N variables. Now suppose that $N > m + n - 1$. Then the system of equations has a family of solutions of the form

$$x_{ij} = u_{ij} + tv_{ij} \quad i = 1, \ldots, m; j = 1, \ldots, n$$

where t varies and, firstly, v_{ij} is zero whenever u_{ij} is zero and secondly, at least one of the v_{ij} is non-zero. Now it is clear that if $|t|$ is small enough, all the x_{ij} will remain non-negative, and further, because $\Sigma d_{ij}x_{ij}$ takes its minimum value at $x_{ij} = u_{ij}$,

$$\Sigma d_{ij}x_{ij} = \Sigma d_{ij}u_{ij} + t\Sigma d_{ij}v_{ij} \geqslant \Sigma d_{ij}u_{ij}$$

and hence $\Sigma d_{ij}v_{ij} = 0$.

Now by letting t increase or decrease from zero we can make one of the x_{ij} for which u_{ij} is not zero, become zero, and at the same time, those x_{ij} for which u_{ij} is zero remain zero and the other u_{ij} remain non-negative. Further the value of $\Sigma d_{ij}x_{ij}$ remains equal to its minimum value, since $\Sigma d_{ij}v_{ij} = 0$. Thus we have made one more x_{ij} equal to zero, contradicting the definition of N. Hence we must have $N \leqslant m + n - 1$.

Now arrange the numbers u_{ij} in m rows and n columns. Since we must have either $N < 2m$ or $N < 2n$, there must be either a row or a column containing only one non-zero u_{ij}. Since the sum of the variables in a row or a column is an integer, the value of this variable is an integer. Remove the variables in this row or column and adjust the equations accordingly. The number N of non-zero u_{ij} and $m + n - 1$ are both decreased by 1. Repeating the argument, or using induction, we see that all the u_{ij} have integer values.

Now suppose that we take $m = n$ and $a_i = b_j = 1$ for all i, j. Then $\Sigma d_{ij}x_{ij}$ will take its minimum value for a set of values of the x_{ij} in which each variable takes the values 0 or 1. Since the rows and columns of $\underline{X} = (x_{ij})$ add up to 1, it follows that this matrix contains precisely one 1 in each row and in each column and so is a permutation matrix.

Now we turn to eigenvalues. The problem we are concerned with is an eminently practical one. We wish to calculate the eigenvalues of a real symmetric matrix $\underline{A}$. The entries in the matrix $\underline{A}$ are subject to various sources of error, such as rounding-off errors, etc., and the matrix actually presented is the symmetric matrix $\underline{B} = \underline{A} + \underline{E}$, where E is the error matrix and presumably has small entries. The problem is to estimate the errors in the eigenvalues which arise from these errors. The inequality of Wielandt and Hoffman stated that, if we arrange the eigenvalues of $\underline{A}, \underline{B}$ in ascending order as $\lambda_1, \ldots, \lambda_n$ for $\underline{A}$ and $\mu_1, \ldots, \mu_n$ for $\underline{A}$ and $\mu_1, \ldots, \mu_n$ for $\underline{B}$ then

$$\sum_{i=1}^{n} (\lambda_i - \mu_i)^2 \leqslant \sum_{i,j=1}^{n} e_{ij}^2$$

To prove this result we turn the problem round. We suppose that $\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, \mu_n$ are given, $\underline{A}, \underline{B}$ are matrices with these eigenvalues and $\underline{E} = \underline{B} - \underline{A}$, and we seek the minimum value of Σe_{ij}^2, subject to the conditions on $\underline{A}$ and $\underline{B}$.

We first make the simple and easily checked observation that, if $\underline{U}, \underline{V}$ are orthogonal matrices, then the sum of the squares of the entries in the matrices $\underline{E}, \underline{UE}, \underline{EV}, \underline{UEV}$ are all the same. Now, by the principal axes theorem we can find orthogonal matrices $\underline{U}, \underline{V}$ such that

$$\underline{L} = \underline{U}^{-1}\underline{AU}, \underline{M} = \underline{VBV}^{-1}$$

where $\underline{L} = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$, $\underline{M} = \text{diag}(\mu_1, \ldots, \mu_n)$.

Then if $\underline{W} = \underline{VU}$

$$\underline{MW} - \underline{WL} = \underline{VEU}$$

and hence we have to minimise the sum of the squares of the entries of $\underline{MW} - \underline{WL}$ as $\underline{W}$ ranges over all orthogonal matrices. But this sum of squares is

$$\Sigma(\mu_j - \lambda_i)^2 w_{ij}^2$$

Write $x_{ij} = w_{ij}^2$, so that

$$\sum_i x_{ij} = 1$$

$$\sum_j x_{ij} = $$

and $x_{ij} \geqslant 0$. Then the minimum value of $\Sigma(\mu_j - \lambda_i)^2 x_{ij}$ is attained in particular for $\underline{X} = (x_{ij})$ equal to a permutation matrix, and hence this minimum is also attained for the original problem with $\underline{W}$ equal to the same matrix, which is orthogonal.

It follows that the minimum value of Σe_{ij}^2 is equal to the minimum value of $\Sigma(\mu_{i'} - \lambda_i)^2$ over all permutations $i \to i'$ of $1, \ldots, n$. This minimum is attained for the identity permutation since otherwise, for some $i, \mu_{(i+1)'} < \mu_{i'}$ and by interchanging $\mu_{(i+1)'}$ and $\mu_{i'}$, we decrease $\sum_{i=1}^{n} (\mu_{i'} - \lambda_i)^2$ by

$$(\mu_{i'} - \mu_{(i+1)'})\,(\lambda_{i+1} - \lambda_i).$$

This concludes the proof. We may note incidentally that if we have equality, i.e.,

$$\Sigma e_{ij}^2 = \Sigma(\mu_i - \lambda_i)^2$$

the $\underline{W}$ can be taken to be the identity, and hence we can reduce $\underline{A}, \underline{B}$ simultaneously to diagonal form.

Ohm Sweet Ohm

1. Each vertex of a (3-dimensional) cube is joined to every other vertex by a 1Ω resistance.

What is the total resistance between any two vertices?

2. Each vertex of a tesseract (a 4-dimensional cube) is joined to every other vertex by a 1Ω resistance.

What is the total resistance between any two vertices?

(answers p.31)

The Optimum Size for an Establishment

by H. W. O. Petard

(The Editor would like to thank Professor M. V. Wilkes who discovered this paper quite recently and very kindly passed it on to Eureka. As older readers will know the author was famed in mathematical circles for his incredible article on the Mathematical Theory of Big Game Hunting which appeared in the American Mathematical Monthly during 1938 and later in Eureka no. 16. The article below was probably written some time during the war)

1. Introduction

It is a commonplace that the amount of work done by an Establishment of Civil Servants is not directly proportional to its size; indeed, cases may readily be quoted in which an increase in staff has impeded rather than furthered the objects for which an establishment exists.

In the present paper I shall examine the problem analytically; and derive formulae by means of which the optimum size for an establishment may be determined. I shall make use for this purpose of the Kinetic Theory of Civil Servant Swarms which has been developed by Ponticelli. As this theory is not as well known as it deserves, I propose in the next section to give a brief summary of the most important results.

2. The Kinetic Theory of Civil Servant Swarms

The fundamental concept in this theory is 'pressure', which may be defined as the force that the establishment as a whole can bring to bear in order to achieve its objects, for example, to overcome the obstruction of other establishments or departments, or itself to obstruct some design originating from without. At first sight it might be thought that a Civil Service establishment, by virtue of the random motions of its members, would exert a pressure of precisely zero; a moment's consideration of the analogous case of a gas consisting of molecules moving with random velocities, will, however, show that this is not the case, and that in fact a finite pressure is always exerted. This pressure is, of course, less than the pressure of a hypothetical pseudo-establishment in which the energies of the staff are all oriented in the same direction.

I will not here enunciate all the results of the Kinetic Theory, for which reference should be made to the original sources; two results, however, are of particular importance:

(a) In order to do work an establishment must continually expand.
(b) The larger the establishment becomes the less is the pressure it exerts.

3. The Optimum Size for an establishment.

We shall suppose that the establishment consists of n members; this number is only to include those members of staff in positions of responsibility, as it is not my intention in this paper to examine in detail the relations existing, for example, between an officer and his secretary. Let a fraction k of each officer's time (office hours only being considered) be devoted to the work of the establishment*, and let a fraction m be devoted to productive work; the difference, namely (k-m), will then be devoted to internal liaison with other members of the establishment. We will suppose for simplicity that the quantities k and m are the same for all members. Normally, k^2 and higher powers may be neglected. In what follows it will be convenient to take the working day as the unit of time.

Since each officer has to liaise with each of his (n—1) colleagues, we have, assuming he spends the same amount of time with each,

$$k - m = a(n - 1)$$

where a is a constant.

The total time spent on liaison by all the officers taken together is then

$$a\,n\,(n - 1)$$

while the total time spent on useful work is

$$n\,m$$

It will thus be seen that as n increases, the time spent on liaison increases much more rapidly than the time spent on useful work.

If we assume that the size of the establishment is optimum when these two quanities are equal, we have

$$a\,n\,(n - 1) = n\,m^{**}$$

or

$$n = 1 + m/a$$

Since exactly half the time k is spent on useful work, we must have

$$m = \tfrac{1}{2}\,k$$

so that the above equation for n becomes

$$m = 1 + k/(2a)$$

---

* We shall not consider what use is made of the remaining fraction (1—k), as it is felt that this could better be dealt with elsewhere, and by other methods.

** We reject, regretfully, the solution n = 0

A Numerical Example

To illustrate the foregoing results in a practical case, let us suppose that the daily relation between officers is limited to the writing of one short minute, or the making of one short telephone call, the total duration being on the average 3 minutes. If the length of the working day is five hours, we have

$$a = 3/(5 \times 60) = 0.01$$

If we now take for k the value $\frac{1}{2}$, we get for n

$$n = 1 + \tfrac{1}{2}/(2 \times 0.01) = 26$$

Thus the optimum size for an establishment under these conditions is 26.

What Price Virtue?

The castle of Sir Theobald The Prudent contains six rooms and twelve corridors, and may be regarded as an octahedron, each room connecting with just four others. Sir Theobald has a daughter, Elaine the Fair, much admired by the wicked Sir Jaspar. Conventional entreaty proving vain, Sir Jaspar determines to win his objective by base means. Elaine is virtuous—She will yield only with probability $(r-1)/2(r+1)$, where $r > 1$ is the price offered in rubies. To seize Elaine by force Sir Jaspar will require the services of some henchmen—enough to prevent her eluding them for ever in the castle. Lady Elaine and her pursuers move with equal speeds down the corridors: she is captured if she finds herself in the same room as a pursuer. Sir Jaspar risks capture and incarceration for non-payment of S.E.T. on henchmen before he can enjoy the fruits of his labours, with probability $(n-1)/n$, n being the number of henchmen employed. Sir Jaspar will not assist in his dirty work.

The castle doorkeeper has payed Sir Jaspar 4 rubies protection money and will co-operate. Sir Jaspar has at present no other ready cash, his estates being already heavily mortgaged, but he regards money as being of no object up to the limit of his possession. He tosses an unfair dud coin to decide his course of action—the probability is 1/3 for the henchmen, 2/3 for pecuniary appeal.

What premium (to the nearest ruby) should Isaac charge Sir Theobald to insure his daughter's virtue for 20 rubies if he requires an expected gain of 10 per cent of the premium charged? (answer p. 31)

The Football League Eigenvector

by J. R. Gillett

The usual method of compiling an 'order of merit' after a tournament is to compare the total number of points scored by the different teams (ties being separated by some arbitrary method such as 'goal average'). This is not entirely fair, however, since a good team should surely be given more credit for beating a good team than for beating a poor one. A fairer method is to give each team a 'second order score' equal to the sum of the scores of all the teams they beat. A fairer order still is given by the 'third order scores', obtained by summing the second order scores of the teams beaten. This method can be continued indefinitely. However, for it to be acceptable in practice the vector of scores (when suitably normalized) must converge rapidly, so that the final order quickly becomes apparent and is not too different from the order as usually compiled. The following refinement of this technique has these properties.

Let $A = (a_{ij})$ be the n x n matrix (n being the number of teams) in which a_{ij} is team i's score against team j. (The usual scheme is 2 for a win, 1 for a draw and 0 for a loss, so in an 'all-play-all-twice' tournament we have $a_{ij} + a_{ji} = 4$ whenever $i \neq j$. To ensure convergence, however, we give each team a technical draw against itself, so that A has 1's down its principal diagonal). Let $\mathbf{v}$ be the column vector of n 1's. Let $p_i(k)$, team i's 'k-th order score' ($i = 1, 2, ---, n; k = 1, 2, ----$), be the i-th coordinate of the vector $\mathbf{p}(k) = A^k\mathbf{v}$. Thus

$$p_i(1) = \sum_{j=1}^{n} a_{ij}$$

is the total number of points scored by team i, plus one, while

$$p_i(k) = \sum_{j=1}^{n} a_{ij}\, p_j(k-1) \qquad (k = 2, 3, -----)$$

gives greatest credit for teams with high scores against teams which themselves have high $(k-1)$-th order scores. Now let $\| . \|$ denote a suitable norm, such as the sum of the coordinates, and let $\boldsymbol{\pi}(k) = \mathbf{p}(k)/\|\mathbf{p}(k)\|$. Then $\boldsymbol{\pi}(k)$ always converges as $k \to \infty$, usually very rapidly, and to the eigenvector of A corresponding to the eigenvalue of greatest absolute value.

The applied mathematician's proof of this, which is the well known 'Power Method' for finding eigenvectors, is very simple. Let A have eigenvalues $\lambda_1, \lambda_2, -----, \lambda_n$ ($|\lambda_1| > |\lambda_2| \geqslant ---- \geqslant |\lambda_n|$) and let $\mathbf{x}_1, \mathbf{x}_2, ----\mathbf{x}_n$ be the corresponding eigenvectors. If

$$\mathbf{v} = a_1\mathbf{x}_1 + a_2\mathbf{x}_1 + ------- + a_n\mathbf{x}_n,$$

then $\quad \lambda_1^{-k}A^k\mathbf{v} = a_1\mathbf{x}_1 + \lambda_1^{-k}\lambda_2^k a_2\mathbf{x}_2 + ------- + \lambda_1^{-k}\lambda_n^k a_n\mathbf{x}_n \to a_1\mathbf{x}_1$ as $k \to \infty$.

Hence $\quad \boldsymbol{\pi}(k) = A^k\mathbf{v}/\|A^k\mathbf{v}\| \to \mathbf{x}_1$ as $k \to \infty$.

This proof makes three basic assumptions: first, that A has n linearly independent eigenvectors; second, that A has a simple eigenvalue λ_1 which strictly exceeds all other eigenvalues in absolute value; and thirdly, that $a_1 \neq 0$, i.e. $\mathbf{v}$ is not orthogonal to $\mathbf{x}_1$ with respect to the basis of eigenvectors. However, D. R. Woodall (in a private communication) has shown that all three of these assumptions can be removed and that $\pi(k)$ always converges. (The only properties of the matrix A assumed in Woodall's proof are that all the entries are non negative real numbers and that those on the principal diagonal are positive—the latter only being needed to eliminate the second of the three assumptions).

More important in practice, however, is that the convergence should be rapid. The 'Power Method' for eigenvectors is notoriously slow in converging. However, in our special case of a matrix of non negative entries, whose row sums do not in general differ by too large a factor, the convergence is quite rapid, as can be seen from the following application (by computer) to the 1969-70 English Football League Division 1. The teams are listed in the official order, with ties separated by goal average. $\mathbf{p}(1)$ gives the actual number of points scored by each team, plus one. The sum of these is 946, and the vectors $\pi(2)$, $\pi(3)$, $\pi(4)$ have been multiplied by this for easy comparison. It will be seen that the final order emerges as early as $\pi(3)$ (with all ties separated) and is very close to the original order. There are, however, some interesting reversals!

	$\mathbf{p}(1)$	$946\pi(2)$	$946\pi(3)$	$946\pi(4)$
Everton	67	68.83	68.63	68.58
Leeds United	58	60.33	60.10	60.06
Chelsea	56	56.42	56.18	56.17
Derby County	54	55.28	55.44	55.43
Liverpool	52	51.92	51.88	51.85
Coventry City	50	48.31	48.21	48.25
Newcastle United	48	48.90	48.90	48.89
Manchester United	46	45.83	45.65	45.66
Stoke City	46	44.20	44.10	44.15
Manchester City	44	44.63	44.63	44.60
Tottenham Hotspur	44	43.81	43.57	43.60
Arsenal	43	42.68	42.83	42.88
Wolverhampton Wanderers	41	40.18	40.02	40.02
Burnley	40	38.87	38.94	38.98
Nottingham Forest	39	38.84	38.91	38.91
West Bromwich Albion	38	36.81	37.13	37.14
West Ham United	37	36.94	37.05	37.03
Ipswich Town	32	30.51	30.53	30.55
Southampton	30	31.68	31.85	31.80
Crystal Palace	28	28.04	28.25	28.23
Sunderland	27	28.43	28.55	28.52
Sheffield Wednesday	26	24.55	24.62	24.67

The Numerical Range of an Operator

by F.F. Bonsall

Professor at Edinburgh University

Most recent developments in functional analysis need too much background material to be described in a short self-contained article, but happily much of the theory of the numerical range is of interest even when restricted to operators on finite dimensional spaces, and it is then easily accessible.

The history of the numerical range has a clear cut beginning in a short article by O. Toeplitz (1918). Toeplitz considered the space $\mathbf{C}^n$ with its usual inner product $\langle x, y\rangle = \sum_{k=1}^{n} x_k \bar{y}_k$ and Euclidean norm $\|x\| = \langle x, x\rangle^{1/2}$, and defined the numerical range $W(T)$ of a linear operator $T : \mathbf{C}^n \to \mathbf{C}^n$ by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\} \tag{1}$$

It is clear that $W(T)$ is a set of complex numbers that contains the spectrum $Sp(T)$ (i.e. the set of eigenvalues of T), for if λ is an eigenvalue we can choose an eigenvector x with $\|x\| = 1$, and then

$$\lambda = \lambda \langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle \in W(T)$$

Let $w(T) = \sup\{|z| : z \in W(T)\}$, and let $\|T\|$ denote the norm of T defined as usual by

$$\|T\| = \sup\{\|Tx\| : \|x\| \leqslant 1\}$$

It is easy to see that $w(T) \leqslant \|T\|$, for the Cauchy-Schwartz inequality gives $|\langle Tx, x\rangle| \leqslant \|Tx\| \,.\, \|x\| \leqslant \|T\| \,.\, \|x\|^2$; and Toeplitz proved the non-trivial inequality

$$\|T\| \leqslant 2w(T).$$

The constant 2 in this inequality is best possible; for if we take $n = 2$ and T the operator $T(x_1, x_2) = (0, x_1)$, then

$$W(T) = \{x_1\bar{x}_2 : |x_1|^2 + |x_2|^2 = 1\} = \{z \in \mathbf{C} : |z| \leqslant \tfrac{1}{2}\},$$

so that $w(T) = \frac{1}{2}$, but $\|T\| = 1$. These results of Toeplitz already show that the numerical range is related to both the algebraic and the metric properties of the operator T and this has been characteristic of recent work. Toeplitz nearly proved that $W(T)$ is convex, and the proof was completed by F. Hausdorff (1919). This is the best known theorem on the numerical range but I believe it has been somewhat of a red herring which may well have delayed the development of a satisfactory general theory.

After 1919 the numerical range was stuck in a rut and little progress was made until G. Lumer (1961) and F.L. Bauer (1962) independently introduced different concepts of numerical range for operators on general normed linear spaces where the norm is not given by a scalar product, so that the Toeplitz definition (1) is not available. Lumer's construction depends on generalizing the concept of inner product, and I shall not give it here. Bauer's construction uses the linear functionals on the space in the following way.

Let E be a finite dimensional linear space over the complex field, and let $\|\ .\ \|$ be a norm on E, i.e. a non-negative real valued function on E such that

$$\|x+y\| \leqslant \|x\| + \|y\|, \quad \|ax\| = |a| \ . \ \|x\| \quad (x, y \in E, a \in \mathbf{C}),$$

and such that $\|x\| = 0$ only when $x = 0$. A linear functional on E is a linear mapping $f : E \to \mathbf{C}$, and the norm of f, $\|f\|$, is defined by

$$\|f\| = \sup\{|f(x)| : \|x\| \leqslant 1\}$$

Let S denote the unit sphere, $S = \{x \in E : \|x\| = 1\}$, let S' denote the set of all linear functionals f with $\|f\| = 1$, and let

$$\Pi = \{(x, f) : x \in S, f \in S', f(x) = 1\}$$

Finally, given a linear operator $T : E \to E$, we define the numerical range V(T) and the numerical radius v(T) by

$$V(T) = \{f(Tx) : (x, f) \in \Pi\}, \quad v(T) = \sup\{|z| : z \in V(T)\}$$

It is clear that V(T) is a set of complex numbers, since T maps E into E and f maps E into C. Also, since

$|f(Tx)| \leqslant \|f\| \ . \ \|Tx\| \leqslant \|f\| \ . \ \|T\| \ . \ \|x\|$, we have $v(T) \leqslant \|T\|$.

Example 1. If the norm is given by an inner product, i.e. $\|x\| = \langle x, x\rangle^{1/2}$, then for each $y \in S$ the functional f defined by $f(x) = \langle x, y\rangle$ belongs to S' and satisfies $f(y) = 1$, and it is the only functional with these properties. Thus in this case $V(T) = W(T)$ given by (1).

To understand the construction of V(T) the reader may like to work out the following particular example.

Example 2. Let $E = \mathbf{C}^2$, $\|x\| = \max(|x_1|, |x_2|)$ $(x = (x_1, x_2))$. Each linear functional f on E is of the form $f(x) = a_1x_1 + a_2x_2$ with a_1, a_2 complex constants, and it is easy to check that $\|f\| = |a_1| + |a_2|$. Then $(x, f) \in \Pi$ if and only if $\max(|x_1|, |x_2|) = |a_1| + |a_2| = a_1x_1 + a_2x_2 = 1$.

(i) If $|x_1| = 1 > |x_2|$, $(x, f) \in \Pi$ if and only if $a_1 = \bar{x}_1, a_2 = 0$.

(ii) If $|x_1| < 1 = |x_2|$, $(x, f) \in \Pi$ if and only if $a_1 = 0, a_2 = \bar{x}_2$.

(iii) If $|x_1| = |x_2| = 1$, $(x, f) \in \Pi$ if and only if $a_1 = r\bar{x}_1$, $a_2 = s\bar{x}_2$ with $r \geqslant 0$, $s \geqslant 0$ and $r + s = 1$.

Consider the operator T given by $Tx = (ix_1 + x_2, -x_1 - ix_2)$; corresponding to the subsets of Π given in (i), (ii), (iii), we have $V(T) = V_1 \cup V_2 \cup V_3$, where

$$V_1 = \{\bar{x}_1(ix_1 + x_2) : |x_1| = 1 > |x_2|\} = \{i + z : |z| < 1\},$$

$$V_2 = \{\bar{x}_2(-x_1 - ix_2) : |x_1| < 1 = |x_2|\} = \{-i + z : |z| < 1\},$$

$V_3 = \{r\bar{x}_1(ix_1 + x_2) + s\bar{x}_2(-x_1 - ix_2) : |x_1| = |x_2| = 1, r \geq 0, s \geq 0, r + s = 1\}$

$= \{(r - s)i + re^{\theta i} - se^{-\theta i} : r \geq 0, s \geq 0, r + s = 1, \theta \in \mathbf{R}\}.$

By taking $t = r - s$, we obtain

$$V_3 = \{t \cos\vartheta + i(t + \sin\vartheta) : 1 \leq t \leq 1, \theta \in \mathbf{R}\}.$$

Note that V_3 contains the points $(1, i), (1, -i)$ but that

$V(T) \cap \mathbf{R} = V_3 \cap \mathbf{R} = \{t \cos\theta : t = -\sin\theta\} = \{-\tfrac{1}{2}\sin 2\theta : \theta \in \mathbf{R}\}$

Thus $V(T) \cap \mathbf{R}$ is the closed interval $[-\tfrac{1}{2}, \tfrac{1}{2}]$, and in particular the point $(0, 1)$ does not belong to $V(T)$. Since $(1, i), (1, -i) \in V(T)$ but $\tfrac{1}{2}(1, i) + \tfrac{1}{2}(1, -i) \notin V(T)$, this shows that $V(T)$ is not convex.

In fact $V(T)$ is shaped like 

Although Example 2 shows that $V(T)$ need not be convex, it was proved in 1968 that it is always connected. We have seen that $V(T)$ is bounded, and in the finite dimensional case which we are considering here $V(T)$ is a closed set, and these simple facts remain our entire knowledge of the geometry and topology of $V(T)$ for a general T and a general norm. Examples suggest that $V(T)$ is always a very smooth and simple set - no hairy examples are known - and I hope that geometers and topologists will find this situation a challenging one. Is $V(T)$ pathwise connected? Is $V(T)$ simply connected, i.e. is $\mathbf{C} \setminus V(T)$ connected?

It is again easy to prove that $Sp(T) \subset V(T)$, but using an ingenious argument Chr. Zenger (1968) has proved the deeper result that

$$\text{co } Sp(T) \subset V(T),$$

where, given a set A of complex numbers, co A denotes the smallest convex set containing A.

If we want to know all about the algebraic properties of an operator T we must determine its classical canonical form, and this means that we must determine the eigenvalues and their ascents, where the ascent of an eigenvalue λ is the least positive integer k such that $(\lambda I - T)^{k+1}$ and $(\lambda I - T)^k$ have the same null space, I being the identity operator. Let ∂A denote the topological boundary of a set A of complex numbers. N. Nirschl and H. Schneider (1964) proved the following theorem.

Theorem 1. Each eigenvalue belonging to $\partial\text{co } V(T)$ has ascent 1.

The proof of this theorem is easy, but it may be more interesting to exhibit an application to stochastic matrices.

Example 3. Let T be an operator on $\mathbf{C}^n$ given by a stochastic matrix (a_{ij}), i.e. $a_{ij} \geq 0$ for all i, j and $\sum_{j=1}^{n} a_{ij} = 1$ for all i. We prove that every eigenvalue λ of T with $|\lambda| = 1$ has ascent 1.

To prove this, let $\|\cdot\|$ be the norm on $\mathbf{C}^n$ given by

$\|x\| = \max(|x_1|, \ldots, |x_n|)$. If $y = Tx$, we have $y_i = \sum_{j=1}^{n} a_{ij} x_j$, and so

$$|y_i| \leqslant \sum_{j=1}^{n} a_{ij} |x_j| \leqslant \sum_{j=1}^{n} a_{ij} \|x\| = \|x\|.$$

Thus $\|Tx\| \leqslant \|x\|$ for all x, and so $\|T\| \leqslant 1$. It follows that $v(T) \leqslant 1$, and so $V(T)$ is contained in the unit disc $D = \{z \in \mathbf{C} : |z| \leqslant 1\}$. Since D is convex, we have co $V(T) \subset D$, and therefore each eigenvalue λ with $|\lambda| = 1$ belongs to ∂ co $V(T)$. Therefore, by Theorem 1, each such λ has ascent 1.

Very recently, M. J. Crabb has improved Theorem 1 by showing that every eigenvalue belonging to $\partial V(T)$ has ascent 1. In fact he has proved the following theorem.

Theorem 2. Let $u \in S$ and let $(\lambda I - T)^2 u = 0 \neq (\lambda I - T)u$.
Then $V(T)$ contains the disc $\{z \in \mathbf{C} : |z - \lambda| \leqslant (3 - \sqrt{8})\|(\lambda I - T)u\|\}$

This theorem seems to lie deeper than Theorem 1, its proof depending on a generalization of the Brouwer fixed point theroem due to S. Kakutani (1941).

Example 4. Suppose that $V(T)$ has void interior. Then

$$T = \alpha I + e^{i\beta} R,$$

where α, β are real numbers and R is an operator that can be represented by a real diagonal matrix. To see this, note that by Zenger's theorem (2), co Sp(T) has void interior and so the eigenvalues of T lie on a straight line. By Theorem 2 all the eigenvalues have ascent 1. Therefore there exist real numbers α, β such that the operator $e^{-i\beta}(T - \alpha I)$ (= R say) has all its eigenvalues real and of ascent 1. We may therefore take a basis consisting of eigenvectors and then R is represented by a real diagonal matrix.

I do not know any example in which $V(T)$ has void interior without it being a straight line segment. Can this occur?

Generalizing the Toeplitz inequality (2), it is known that

$$\|T\| \leqslant e v(T)$$

and that e is the best possible constant. In 1969, M. J. Crabb proved that

$$\|T^n\| \leqslant n!(e/n)^n (v(T))^n \quad (n = 1, 2, \ldots)$$

where again the constant is best possible. Since $v(T^n) \geqslant \frac{1}{e}\|T^n\|$, this shows that the ratio $v(T^n)/(v(T))^n$ can be arbitrarily large; which makes even more remarkable the inequality

$$w(T^n) \leqslant (w(T))^n \quad (n = 1, 2, \ldots),$$

which was proved by C. A. Berger (1966) for spaces in which the norm is given by an inner product.

In this short article I have mentioned only a few of the recent results on the numerical range and have said nothing about its application to Banach algebra theory, where it ties together the algebraic and metric structures. That is another story.

The Mathematical Association

150 Friary Street, Reading RG1 1HE

President: Professor M. J. Lighthill, F.R.S., F.I.M.A.

The Mathematical Association was founded in 1871 as the Association for the Improvement of Geometrical Teaching, so that 1971 is the Centenary Year. The Association aims now to bring within its purview all branches of elementary mathematics.

At the moment of writing, the subscription is 2 guineas per annum, with junior membership at 10s. 6d.; but these rates will be increased next year.

The journal of the Association is The Mathematical Gazette, published 4 times a year and dealing with a variety of topics of general interest. There is a powerful Reviews section. The present editor is Dr. E. A. Maxwell. It is intended to enlarge the scope of publications, starting in 1971.

Equivalence Relations and Scholarship Problems

by β

Everybody knows that a set with n distinct elements has 2 distinct subsets. A less trivial problem is 'How many distinct equivalence relations can be defined on a set with n distinct elements?' If we have a set S_n with n distinct elements and we let R_n denote the number of equivalence relations which can be defined on S_n, then by trial and some error we find that $R_0 = 1, R_1 = 1, R_2 = 2, R_3 = 5, R_4 = 15, R_5 = 52$.

If anybody bothers to perform this verification they will soon come to the conclusion that either a formula or a computer is called for, and so as to make this article longer and to provide a somewhat tenuous justification for the title we consider the former possibility.

When in doubt with a sequence of integers it is always worthwhile to try and produce a recurrence relation. If we look at the known values for R_n we observe that $R_1 = R_0$; $R_2 = R_1 + R_0; R_3 = R_2 + 2R_1 + R_0$

$$R_4 = R_3 + 3R_2 + \frac{3.2}{1.2} . R_1 + R_0; R_5 = R_4 + 4R_3 + \frac{4.3}{1.2} . R_2 + \frac{4.3.2}{1.2.3}R_1 + R_0$$

$$\text{(W)} \quad R_{n+1} = R_n + nR_{n-1} + \frac{n(n-1)}{1.2}R_{n-2} + \dots + nR_1 + R_0$$

Once (W) has been written down the reader will easily prove the conjecture by induction and a 'wordy' argument, so it will be left as an exercise.

So now we are able to calculate R_n for any n, but being pure mathematicians and disliking numerical computations we fly off at a tangent and look at

$$\text{(J)} \qquad \sum_{k=0}^{\infty} \frac{k^n}{k!} = K_n e$$

This will no doubt be a fimiliar sight to collectors of old examination papers and the associated problem of showing that K_n is an integer for each integer n will be left as a problem for the reader.

If we calculate K_n for a few n, we find $K_0 = 1, K_1 = 1, K_2 = 2, K_3 = 5, K_4 = 15$, which looks rather interesting and worth investigating further. As this is a supposedly mathematical article, and in order to quell twinges of conscience we will juggle with some symbols. We have from (J)

$$e \sum_{n=0}^{\infty} \frac{K_n}{n!} x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{k^n x^n}{k! n!}$$

$$\therefore\ e \sum_{n=0}^{\infty} \frac{K_n x^n}{n!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = \sum_{k=0}^{\infty} \frac{1}{k!} e^{kx} = \exp(\exp x)$$

$$\text{(E)} \qquad \therefore\ y = \sum_{n=0}^{\infty} \frac{K_n x^n}{n!} = \exp\{\exp(x) - 1\}$$

$$\therefore\ K_n = \left(\frac{d^n y}{dx^n}\right)_{x=0}$$

Now by logarithmic differentiation and Leibnitz's formula we obtain

$$\frac{d^{n+1} y}{dx^{n+1}} = \left(\frac{d^n y}{dx^n} + n \frac{d^{n-1} y}{dx^{n-1}} + \frac{n(n-1)}{1.2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + n \frac{dy}{dx} + y\right) e^n$$

$$\therefore\ K_{n+1} = K_n + n K_{n-1} + \frac{n(n-1)}{1.2} K_{n-2} + \ldots + n K_1 + K_0$$

and as $R_0 = K_0, R_1 = K_1$ we have $R_n = K_n$ all n.

It would be nice, instead of using (W) to obtain an asymptotic formula for R_n, since as is easily seen R_n increases rapidly with n. We have the two useful relation (J) and (E) and having carefully satisfied our consciences above we leave it as an exercise for the stout-hearted to show that

$$\frac{\log R_n}{n} = \log n - \log \log n - 1 + \frac{\log \log n}{n} + \frac{1}{\log n} + \tfrac{1}{2} \left(\frac{\log \log n}{\log n}\right)^2 + O\left(\frac{\log \log n}{\log^2 n}\right)$$

The reader who is more interested in the number-theoretic properties of the R_n may care to note and prove the following curiosities: (p is a prime and n a positive integer)

(1) $R_n \equiv 2 \pmod p$ (2) $R_{p+n} \equiv R_{n+1} \pmod p$ (3) $R_n + R_{n+1} + R_{n+2} \equiv 0 \pmod 2$

(4) The sequence $\{R_n\}$ has a congruence period of $\frac{p^p - 1}{p - 1}$ places i.e.

$$R_{n + \frac{p^p - 1}{p - 1}} \equiv R_n \pmod p \quad (n = 0, 1, \ldots)$$

(5) The sum of $\frac{p^p - 1}{p - 1}$ consecutive R_n's is $\equiv 0 \pmod p$.

Solutions to Problems

(The editor would like to thank the Archimedeans and Invariants for several of the problems drawn from the Problems Drive, 1970)

A Triangular Problem: $\alpha = 30°$

Two Problems:

(1.3) Consider the congruences below ($p_1, \ldots p_{n^2}$ are the first n^2 primes)

$$P \equiv 0\ (p_1)\ (p_{n+1}) \ldots (p_{n(n-1)+1})$$
$$P \equiv 1 \quad . \quad . \quad \ldots$$
$$\vdots$$
$$P \equiv n\ (p_n) \ldots (p_{n^2})$$

and

$$Q \equiv 0\ (p_1)\ (p_2) \ldots (p_n)$$
$$Q \equiv 1\ (p_{n+1}). \ldots \quad .$$
$$\vdots$$
$$Q \equiv n\ (p_{n^2-n+1}) \ldots (p_{n^2})$$

The Chinese remainder theorem asserts that these congruences are simultaneously soluble. Hence clear nxn non-grave points.

2. Only solutions are:

6	1	16	11
15	14	3	2
4	7	10	13
9	12	5	8

6	1	16	11
15	14	3	2
9	7	5	13
4	12	10	8

6	1	16	11
15	12	5	2
10	7	4	13
3	14	9	8

Problem for Poultry Farmers: Friday

Ohm Sweet Ohm: $1/4$ and $1/8$

What Price Virtue: (hint—show that only two henchmen needed). Answer 9 rubies Isaac is a Jew)

What is it? A 2-dimensional representation of ϵ_{ijkl}—shading varies according as signature of corresponding permutation is ± 1.

Book Reviews

VECTOR ANALYSIS

by B. Spain (Van Nostrand).

This is a very readable and useful book. There is very little in it that would not be of great service to a first year mathematician. (Especially one who finds the 'Methods of Mathematical Physics' course difficult in the first term). The treatment is clear and concise and takes you step by step through the material. Certainly a book I would recommend to any freshman.

K. McCoy

THE MANY BODY PROBLEM IN QUANTUM MECHANICS.

by N. H. March, W. H. Young and S. Sampanthar (C.U.P)

In quantum mechanics as in other branches of physics, approximate methods are generally needed for calculating the behaviour of systems of more than two interacting particles. The authors here have aimed to produce a comprehensive but detailed survey of the subject, suitable as an introduction for research students. Topics include nuclear matter, superconductivity and Green function methods. The style and presentation are clear, and a large number of complicated Feynman diagrams are reproduced. There are comparatively few books on this particular subject, and the present one would seem very welcome.

P. J. Bussey

ELEMENTARY CLASSICAL HYDRODYNAMICS

by B. H. Chirgwin and C. Plumpton (Pergamon)

The material covered in this book corresponds closely to the syllabus of the Part 1B Fluid Dynamics course at Cambridge. The text is, on the whole, easy to follow, once one is used to the authors' unconventional, if logically justifiable, choice of the symbol d/dt to represent differentiation following the motion. This could be a useful buy for someone who is not intending to study fluid mechanics beyond Part 1B and does therefore not feel justified in buying Batchelor's rather more detailed work.

J. J. Barrett

SOME TOPICS IN COMPLEX ANALYSIS

by E.G. Philips (Pergamon)

This book forms a sequel to the author's well known volume 'Functions of a Complex Variable' and a knowledge of this volume is assumed in the new book. The first two chapters give an introduction to elliptic and Jacobean elliptic functions and includes a section on elliptic integrals. Among other topics treated are conformal transformation which includes a short discussion of the biquadratic and generalised Joukowski transformations; there are short chapters on the Maximum modulus principle; integral functions; infinite series and Gamma, Bessel and Legendre functions are introduced in the final chapter. There are also extensive sets of exercises at the end of each chapter. As an introduction to the variety of topics it covers this book is excellent; however it has the disadvantage of not going into depth with any of the topics.

B.P. McGuire

101 BRAIN-PUZZLERS

by E.R. Emmet (Macmillan)

From a versatile author, a collection of deductive problems requiring no specialised mathematical knowledge other than a basic 'feel' for numbers. The problems come in various categories—miscellaneous, football matches, missing digits, cross numbers etc., and in varying degrees of difficulty, but within each section there is rather a lack of variety. An unusual feature is that over half the volume of the book is devoted to the solutions and full description of the logical steps leading up to them.

D.R. Grey

THEORY OF BEAMS

by T. Iwinski, translated by E. P. Bernat (Pergamon)

This modest little book should provide anyone with an interest in Laplace transforms with the opportunity to see how the theory is applied to solving problems related to beams. A wide variety of complex cases are considered, which makes the book interesting, if concentrated, reading.

P.O. Gershon

METRIC SPACES

by E.T. Copson (C.U.P)

This book based on a course of third year lectures given in the University of St. Andrews, covers about one third of the material supposedly covered in the Analysis IV course given to second year students at Cambridge. This aside, the material included is dealt with in a sensible amount of detail, and the treatment is leisurely without being verbose. This leads to a degree of clarity unusual for an analysis textbook. The printing is up to the high standard customary of the Cambridge University Press, which helps to make this a very useful introduction to the subject.

J.J. Barrett

TOPICS IN ALGEBRA

by Miss H. Perfect M.A. (Pergamon)

This book is intended by the author 'to provide interesting supplementary reading for the sixth form mathematician'. The scope of the text is very well judged to bridge the gap between A-level study and early university work. Topics dealt with are: polynomial algebra; determinants and matrices; vector spaces; complex numbers; congruences and finite fields; rings and fields; and classes and Boolean algebras. The text also attempts to encourage the potential autodidact to break from the formal tuition to which he will have been used. This means that the text is rather long, but in spite of this, most of the interdependencies of the subject matter are relegated to subnotes. The profusion of such notes (27 in the 27 pages of the first chapter) and the constant need to make cross-references to maintain clarity of thought on the text, seem to me the deficiencies of the work. Miss Perfect reveals an excellant judgement of how much to do on these topics' to both satisfy and stimulate the reader. I fear, however, that the explanatory notes will deter all but the most enthusiastic and determined students. The text is hardly suitable for formal class use. Those with the courage to carry it through to the end will have a most useful preliminary survey to early university algebra.

M. A. Lewis

COLLECTIVE OSCILLATIONS IN A PLASMA

by A. I. Akhiezer et al (Pergamon)

The first three chapters of this book contain most of the standard theory of collision-less plasma oscillations, microinstabilities, etc., in the development of which the Russian school, led by Landau, has played so great a part. It is rather less turgid than most Russian books. The last two chapters contain more recent material on scattering, fluctuations, and related phenomena.

C. J. Myerscough

ADVANCES IN PROGRAMMING AND NON-NUMERICAL COMPUTATION

edited by L. Fox (Pergamon)

This book represents an important advance in the theory and application of programming, in that it is the first successful attempt to combine a number of papers on different aspects of programming for non-numerical uses. The whole is well collated, both by the authors and the editor, and forms a good introduction to all the important work in this field in recent years. The two main sections, entitled 'Advances in programming' and 'Non-numerical applications', differ greatly in their value, the first, to my mind, being far more useful, the second far too vague and general. This, however, is a criticism of the subjects, not of the authors. Despite having redundent verbiage in chapters (i), (vi) and (ix), this book can thoroughly be recommended to anyone who wishes to become familiar with the basic ideas and notation of this very important branch of computing. I am told there is an error in the proof of one of the theorems of chapter (iv). The typesetting and layout of the book are excellent.

J. L. Dawson

A SECTION OF PROBLEMS IN THE THEORY OF NUMBERS

by W. Sierpinski (Pergamon)

This little book, written by the most eminent member of the Polish school of mathematics, consists of three chapters. The first: 'On the borders of geometry and arithmetic', the second: 'What we know and what we do not know about prime numbers', and thirdly: 'One hundred elementary but difficult problems in arithmetic'. Chapter 1 is confined to problems connected with the two-dimensional lattice of integral points in the plane. The problems posed are all fairly 'natural' and their solutions are all quite elementary (If the reader is tempted to think the problems are trivial I recommend him to try and provide solutions as elegant as Sierpinski's). Several unsolved problems superficially similar to some of the solved problems are mentioned, with some references to the literature. Chapter 2 contains a wealth of miscellaneous properties and conjectures about prime numbers, but it certainly does not exhaust the subject, as the title might suggest. The problems posed in the third part of the book are nearly all unsolved at the moment, and are likely to remain so far a considerable time to come. In this section the problems are not very difficult, for example, p 14: 'Do there exist any infinity of primes of the form $x^2 + y^2 + z^2 + 1$, x, y, z integers ?' is quite easy. The book as a whole makes fascinating reading (the odd typographical error and quaint translation from the Polish, do not detract from the enjoyment of the book), and can be warmly recommended to anybody with an interest in Arithmetic.

W. J. Ellison

ALGEBRA VOLUME 1

by L. Redei (Pergamon)

With 823 + xviii pages this book covers a great deal of ground—approximately 60 square inches if laid flat. It is an English translation of a German translation of the original Hungarian edition; nevertheless the style is readable. The main topics dealt with are groups, rings, and fields, with the emphasis on the latter two. Much of the development of groups and rings is carried out in parallel. Various unusual items are included, many of them the work of the author. A minor curiosity is the 'list of symbols' on pp. xvii-xviii which is precisely what it claims to be—no page numbers are indicated and is consequently about as useful as a table of even prime numbers. The typography is remarkably free of errors. The price is rather prohibitive, and I would not recommend it as an undergraduate text; on the other hand it is not quite deep enough for research requirements. Despite this, any self-respecting mathematics library should possess a copy.

I. N. Stewart

MODEL ANSWERS IN APPLIED MATHEMATICS FOR A-LEVEL STUDENTS VOLUME 1: DYNAMICS

by J. D. Sweetenham and D. M. Esterson (Pergamon)

This is a valuable book for most A-level students. It does not provide solutions with full explanations—a student who is very uncertain would do much better to look up a text—but the sort of answers which would get full marks on a paper. As is necessary for all those not possessed of a superhuman writing speed, a student must decide what detail to put in, what to leave out—a task easy in retrospect, but not necessarily at the A-level stage. In this, the book is a great help—one hopes that as long as one has to put up with exams, books such as this will exist to help the student.

A. Kaletzky

VECTOR MEASURES

by N. Dinculeanu (Pergamon)

The most noticeable feature of this book is its persistent generality. The principal example of this is that, as the title indicates, the book studies set functions taking their values in any Banach space. The approach adopted is the classical one, rather than the approach via the Daniell integral. The first two chapters give a straightforward and thorough account of the theory of set functions, their extensions to larger rings of sets, integration, spaces of measurable functions, and the Radon-Nikodym theorem. They also include a discussion of the lifting problem for L, i.e. the problem of choosing a member f of each equivalence class $f \in L$, in such a way that the set $f : f \in L$ is closed under all the algebraic operations, defined pointwise. However, such topics as product measures, and real variable integration and differentiation are not included. The third and final chapter gives a good explanation of measures on regularity conditions and disintegration of measures. In summary, I cannot recommend this book as an introduction to measure theory, but it may be of some use as a reference work for certain topics, particularly the lifting problem, Borel measures, and, of course, the special problems of vector measures.

P. G. Dixon

GAUSSIAN QUADRATURE FORMULAE

by A. H. Stroud and D. Secrest (Prentice-Hall)

This book might be called the 'bible' of Gaussian Quadrature. Until recently, Newton-Cotes methods were more commonly used for numerical integration, having the advantage of being based on equally spaced data when so many tables of special functions were given in this form. With the advent of the automatic digital computer this is no longer such an important consideration in the choice of a method of numerical integration. For one thing we are no longer put off by having to use data at points with irrational values. Gaussian quadrature formulae give a higher order of precision and converge under conditions which are almost always realisable in practice, whereas Newton-Cotes formulae are usually asymptotically divergent with the number of points used.

To a large extent this book has only been made possible with the development of high speed automatic computers—the nodes, weights and error coefficients given in the tables were calculated, checked and printed by a CDC 1604. Triple precision was used in the calculations, allowing the carrying of 39 decimal digits, and the values are given to 30 significant figures! Yet it might be argued that book format for tables is out of date and it would be better to produce them on punched paper tape or cards, to minimise errors in transcription.

In the first chapter the authors survey the basic properties of Gaussian quadrature formulae, discussing orthogonal polynomials, properties of coefficients and the convergence of Gaussian formulae. FORTRAN programs for calculating the zeros of the orthogonal polynomials and the coefficients employed in Jacobi, Laguerre and Hermite formulae are described in the second chapter. A welcome feature is the chapter which points out various uses of the tabulated formulae, including product formulae for multiple integrals and the solution of integral equations. In a chapter on error estimates the author's comment that 'it is commonly conceded that Gaussian formulae are the best for use for integrands with high order derivatives. But it is also widely thought that they are not appropriate for functions having only low order derivatives'. They demonstrate that the latter opinion is erroneous. Before presenting the 16 tables that are the 'rason d'etre' of the book, they survey other tables and give a 65 entry bibliography.

D. C. Joyce

SERIES EXPANSIONS FOR MATHEMATICAL PHYSICISTS

by H. Meschkowski, translated by R. Schlapp

CALCULUS OF VARIATIONS

by J. C. Clegg (both: Oliver and Boyd)

It is a shame that the publishers have found it necessary to price these new additions considerably higher than previous in the series of University Mathematical Texts. There is no longer sufficient difference between the prices of these books and others on the same subjects to make them the obvious choice for an undergraduate. Someone for instance who was sufficiently interested in the Calculus of Variations to buy a book solely on that subject would probably be interested enough to afford Pars' treatise; while someone simply wanting to know enough for the Part 1B Mathematical Methods course would be satisfied with the relevant chapter of Jeffreys and Jeffreys. In style and readability, however, these books are of the same standard as others in the series.

J. J. Barrett
P. O. Gershon

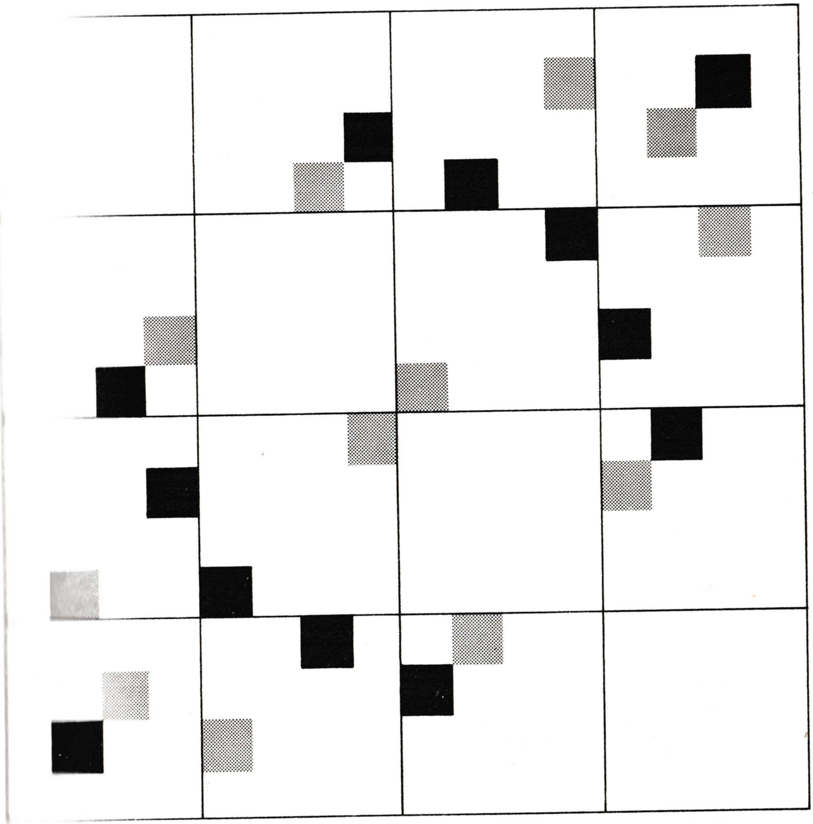
S.M.P. ELEMENTARY TABLES

(C.U.P)

This is the second edition of the only three figure tables I have ever seen. If you are in need of tables of sines, cosines, tangents, logarithms, squares and square roots to such an accuracy then this is for you. But where are the anti-logarithms? However, these tables, with their useful collection of formulae, should prove ideal for school use.

C. D. Evans

What is it?



Clue: a signature. Answer p. 31

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