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CONTENTS

	Page
Editorial	1
An Alphametric	1
Hopes and Fears	2
A Genetical Application of Homogeneous Coordinates	5
Two Problems on Cubes	9
Fermat's Last Theorem and the Golden Mean	12
The Multiplicator	14
A Shuffling Problem	17
Uniform Curves and Surfaces	19
A Further Examination of the 'Surprise Examination Paradox'	23
John Adams	24
The Archimedeans	25
Movement without Speed	26
The Mathematical Association	29
Polyhedra	29
On Writing Text-Books	33
Problems Drive 1969	34
Book Reviews	36
Solutions to Problems	40

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Editorial

Looking through past copies of Eureka, we were struck by the regularity with which previous editors appealed for contributions. The magazine was at one time published only when sufficiently many articles had been submitted; this practice resulted on more than one occasion in a longer period than was usual between issues. This year, there has been an abundance of articles. The Archimedeans have responded magnificently to one solitary notice placed at the Arts School. As a result, however, lack of space prevents our publishing all the articles submitted; there is scarcely room for the editorial. At the same time we should like to offer our sincere thanks to all who have sent in contributions during the past year.

It is regretted that is has been necessary to increase the price of Eureka from 2s. 6d. —the price for the last nine years—to 3s. 0d. per copy. Last year the cost of producing each magazine was 2d. more than that charged: this really cannot be allowed to continue as we are sure you will agree. Copies of Eureka sent by post will be charged at the rate of 3s. 6d., unless 13s. 0d. or more is sent in advance, in which case accounts will be debited 3s. 3d. per issue. Dollar rates have been increased to 60c. per copy, or 55c. if $2.00 or more is paid in advance. Anyone wishing to take out a subscription should write to the Circulation Manager at the above address. Back numbers are also available on request. Copies for the years 1961 to 1968 may be bought for 2s. 6d. each (plus postage); editions for the years 1949 to 1960 cost 4s. 0d. each (plus postage), while the first 12 editions are available only as xeroxed copies at 12s. 6d. each. Cheques and postal orders should be made payable to 'The Business Manager, Eureka' and crossed.

Finally, the editor would like to thank all those who have advised and encouraged him throughout the past year. In particular, he is indebted to his predecessor, J. J. Barrett, to I. H. Rose (Business Manager), to A. N. Kemmer (Circulation Manager) and to C. J. Myerscough.

An Alphametric

by J. A. H. Hunter

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   IT'S
      A
  GREAT
  GREAT
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In this alphametric, of course, each distinct letter stands for a particular but different digit.
Our Eureka is not odd: remember that. What do you make of it then?
(Solution on page 40)

Hopes and Fears

by P. A. M. Dirac

Retiring Lucasian Professor of Mathematics in the University of Cambridge.

A research worker who is actively following up some idea referring to the fundamental problems of physics has, of course, great hopes that his idea will lead to an important discovery. But he also has great fears—fears that something will turn up that will knock his idea on the head and set him back to starting point in his search for a direction of advance. Hopes are always accompanied by fears, and in scientific research the fears are liable to become dominant.

As a result of these emotions the research worker does not proceed with the detached and logical mind that one would expect from someone with a scientific training, but is subject to various restraints and inhibitions which obstruct his path to success. He may delay taking some step liable to force a rapid show-down, and may prefer first to nibble at side-issues that provide a chance of achieving some minor successes and gaining a little strength before facing the crisis.

For these reasons the innovator of a new idea is not always the best person to develop it. Some other person without the fears of the innovator can apply bolder methods and may make a more rapid advance. In the following there will be some examples that illustrate this situation.

Anyone who has studied special relativity must have wondered why it was that Lorentz, after he had obtained correctly all the equations of the Lorentz transformation, did not then take the perfectly natural step of considering all frames of reference to be on the same footing and so arriving at the relativity of space and time. History does not record just what it was that held Lorentz back, but it can only have been some kind of fear, perhaps a subconscious one. He did not dare to venture out into a domain of thought completely foreign to anything that anyone had ever imagined. He preferred to remain on the solid ground of mathematical transformations, where his position was unassailable. It needed the boldness of a younger man such as Einstein to take the plunge into a new domain.

The innovator of our present quantum mechanics was Heisenberg. At a time when atomic physicists were floundering about with the orbits of the Bohr-Sommerfeld theory and feeling the need for a drastic alteration of basic principles, Heisenberg had the brilliant idea of constructing a new theory entirely in terms of observable quantities, quantities connected with observations on spectra. These are each connected with two atomic states, and the natural way of expressing them is in the form of matrices. Thus Heisenberg was led to consider matrices as dynamical variables.

He had not proceeded far in developing this idea before he noticed that his dynamical variables would not satisfy the commutative law of multiplication. This was most disturbing. It was inconceivable to a physicist in those days that dynamical variables could be any other than ordinary algebraic quantities, and with the appearance of non-commutation Heisenberg had grave fears that his whole beautiful idea would have to be given up.

When I read Heisenberg's first paper on the subject, I had the advantage over him in not having his fears, as it was not my own idea that was at stake. I was therefore able to look at the question from a more detached point of view.

I needed only a week or two to realize that the non-commutation which alarmed Heisenberg was really the dominating feature of the new theory. The idea of building up a theory entirely in terms of experimentally observed quantities, although a very pleasing philosophical doctrine, was of only secondary importance for the purpose of establishing a new dynamics.

My early work on quantum mechanics was thus concentrated on the problem of bringing non-commutation into dynamical theory. It was not really very difficult, because the previous atomic theory, the orbit theory of Bohr and Sommerfeld, was based on a form of dynamics, Hamilton's form, which turned out to be specially suitable for adapting to non-commutative algebra.

Heisenberg continued to develop his theory, in collaboration with other people in Göttingen. I worked independently from them, apart from getting the initial idea from Heisenberg. We published papers at about the same time, setting the foundations for quantum mechanics. Our styles were different on account of the different points of view we held, mine being based on non-commutation and Heisenberg's on the use of matrices built up from observable quantities.

Quantum mechanics was discovered quite independently by Schrödinger, working on entirely different lines. He had his own difficulties. He was thinking over the mathematical connection between waves and particles that had been discovered some time previously by de Broglie, and eventually found a way of generalizing it to apply to an electron moving in an electromagnetic field. He then had a very beautiful wave equation, conforming to relativity. He proceeded to apply it to the hydrogen atom and his worst fears were realized. The results did not agree with observation.

We know now that the discrepancy was due to the spin of the electron, which was unknown to Schrödinger at the time, although the experimentalists had begun to suspect it. It was a most depressing situation for Schrödinger, and led him to abandon the work for some months, and eventually to publish it only in the non-relativistic approximation, in which the discrepancy does not show up. The relativistic equation was later rediscovered by Klein and Gordon, who were not afraid to publish an equation in disagreement with observation, while Schrödinger was. So the equation now bears their name. It has some value in describing spinless mesons.

Schrödinger's quantum mechanics was soon found to be equivalent to that originated by Heisenberg, in spite of their first seeming so different. The basic equations of the new mechanics were securely established, and it became necessary to find a physical interpretation for them. With non-commutative algebra it could not be as direct as in the classical theory. The general physical interpretation was found to be only a statistical one. One could calculate probabilities, but could not usually predict an event with certainty.

A difficulty now appeared in connection with the relativistic equation of Klein and Gordon. The theory sometimes gave negative probabilities. It was a satisfactory theory only when it was used non-relativistically. I puzzled over this for some time and eventually thought of a new wave equation which avoided the negative probabilities. I found that it also gave automatically the spin of the electron, a most gratifying result. I proceeded to apply the new equation to the hydrogen atom, taking into account the relativistic corrections only to the first order of accuracy to simplify the calculations, and found agreement with observation.

The natural thing to do at this stage would have been to continue to the higher orders of accuracy, but I did not do so. I was scared that they might not agree with observation. I hastily wrote up a paper with merely the first order of accuracy and published that. In doing so I felt I was consolidating a limited success, and even if the higher orders did go wrong there would still be something to stand on. It was left to Darwin, who did not share my fears, to carry out the calculation to all orders of accuracy and see that the results were alright.

In my first paper on the subject (Proc. Roy. Soc A 117, p610) there occurs the equation

$$F = \left(\frac{W}{c} + \frac{e}{c} A_0\right)^2 + \left(p + \frac{e}{c} A\right)^2 + m^2c^2.$$

The relativist, if he sees this equation nowadays, will say at once—There is a mistake here. The plus signs before the second and third terms on the right should be minus's. He will wonder how such a conspicuous mistake could have remained undetected in the proof-reading. He will wonder still more when he sees the same mistake perpetuated in later equations.

The explanation is that there is really no mistake and things were published as the author intended. The plus signs were the expression of a fear. At that time relativity was still unfamiliar and people had continually to cling to the symmetry of space and time so as not to let it out of their heads. The symmetry becomes perfect only if one uses a time variable which is $\surd(-1)$ times the usual time and makes a corresponding change in all 4-vectors. With this notation there are no mistakes in the paper. This notation was frequently used in those days, and it was not considered necessary to explain it every time it was used, because the context made it clear. The arrival of the new wave equation rather forced one to give it up, as it then became too clumsy.

The new wave equation led to a difficulty in that it allowed states of negative energy for the electron. Negative energies are never observed, but they could not be ignored in the theory. I thought of a way of coping with them, namely, to assume that in the physical world all or nearly all of the negative-energy states are occupied, so that ordinary positive-energy electrons cannot jump into them. An unoccupied negative-energy state is a hole which appears as a particle with a positive energy and a positive charge.

Right from the beginning I had the feeling that there would be symmetry between the holes and the electrons. This feeling was strengthened by the knowledge that in the chemical theory of the valency of atoms, there is a considerable amount of symmetry between an electron lying outside the closed shells and a hole in a closed shell. I did not want the symmetry. At that time it was believed that all positive charges were in protons, and the proton was much heavier than the electron. So I struggled with the hope that in some way the Coulomb interaction between the electrons would lead to a dissymmetry between the holes and the electrons, and was afraid that if this hope should fail the whole idea would have to be abandoned. It was left to others, in particular Weyl and Oppenheimer, to make the bold assertion that mathematical symmetry demanded that the holes should have the same mass as the electrons.

With these developments the theory of single particles was put into order. There remained problems concerned with interaction. If one sets up precise relativistic equations one finds that the interaction is so violent that the equations do not have any solutions. The difficulties are still not satisfactorily resolved and point to the need for some further drastic change in the foundations of atomic theory.

A Genetical Application of Homogeneous Coordinates

by A. W. F. Edwards

In last year's Eureka, Professor C. A. B. Smith gave an account of some theorems in the mathematical study of evolution (Mathematics and Evolution, p. 5). The simplest is the Hardy-Weinberg Theorem, according to which the proportions of the genotypes AA, Aa and aa amongst the zygotes arising in any one generation, following random mating in a parental generation in which the gene frequencies are p A and q a ($p + q = 1$), are p^2, $2pq$ and q^2 respectively. In 1926, Professor Bruno de Finetti, now famous for his contributions to the theory of probability, pointed out that if one represents a population with genotypic proportions u, v and w, respectively, by a point in the plane, using homogeneous coordinates and (since $u + v + w = 1$) an equilateral triangle of unit height as the triangle of reference (Figure 1), any population of zygotes which

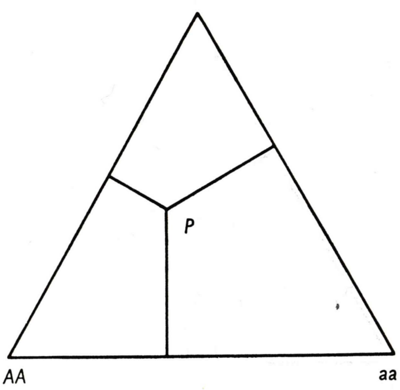

Fig. 1. A de Finetti diagram for a diallelic locus. The lengths of the perpendiculars from the point P to the three sides of the equilateral triangle represent the proportions of the three genotypes AA, Aa and aa in the population.

satisfies the Hardy-Weinberg law will lie on the parabola $v^2 = 4uw$. Haldane and Moshinsky later showed that if a population is inbred to degree F, in which case the genotypic proportions are $p^2 + Fpq$, $2pq(1 - F)$ and $q^2 + Fpq$, it will lie on the parabola $4uw - v^2 = F(2u + v)(2w + v)$. Figure 2 shows this parabola for $F = \frac{1}{2}$ and for $F = 0$, which is the Hardy-Weinberg case.

In this article I indicate how some of the elementary theorems of evolution can be proved and illuminated using homogeneous coordinates. The results were obtained by Dr. C. Cannings and myself, and have been published as Natural Selection and the de Finetti Diagram in the Annals of Human Genetics, 31, 421 (1968).

The 'Hardy-Weinberg parabola' passes through the base vertices of the triangle of reference, and is there tangential to the sides. Now suppose the zygotes of each

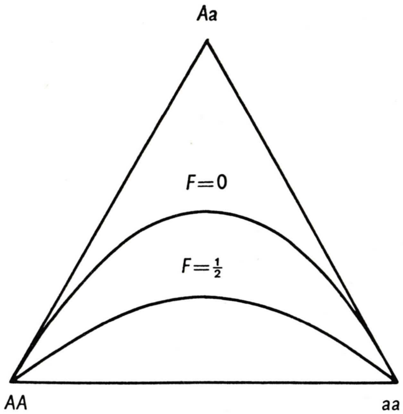

Fig. 2. The loci of populations with genotypic proportions in Hardy-Weinberg equilibrium ($F = 0$) and in equilibrium under inbreeding of degree $F = \frac{1}{2}$.

generation are differentially eliminated by natural selection before they themselves grow to be parents, so that zygotic genotype proportions u, v and w become adult proportions au, hv and bw, for the genotypes AA, Aa and aa respectively. Since, by the Hardy-Weinberg theorem, $v^2 = 4uw$, after selection the point representing the adult population satisfies $abv^2 = 4h^2uw$. This equation is evidently that of a conic symmetrical about the vertical axis of the diagram, $u = w$, and dependent on only one constant, $k = h^2/ab$. By considering the intersection of this conic and the line at infinity $u + v + w = 0$ it is apparent that it takes the following forms:

$h^2 < ab, k < 1$: ellipse,

$h^2 = ab, k = 1$: parabola,

$h^2 > ab, k > 1$: hyperbola.

In the special case $h^2 = ab$ we say that there is no dominance in fitness on a multiplicative scale, for the 'fitness' h of the heterozygote Aa is then the geometric mean of the fitnesses of the two homozygotes AA and aa. The 'after selection' curve then coincides with the 'before selection' Hardy-Weinberg parabola.

In the general case $k \neq 1$ the conic still touches the sides of the triangle at the base vertices, since putting either $u = 0$ or $w = 0$ induces the double root $v^2 = 0$. It follows that the conics of the family have double contact with each other, and that they do not cross each other elsewhere. If every point on the pre-selection curve is joined to its corresponding point on the post-selection curve, selection may be viewed graphically (Figure 3). Starting at the point representing zygotic proportions before selection (P), the population will move to the corresponding point Q as a result of selection. Random mating will then reassort the genes so that Hardy-Weinberg proportions are restored without changing the gene frequency, the population thus moving vertically back to the Hardy-Weinberg parabola at P'. The cycle is then repeated, the population following a saw-toothed path to its ultimate equilibrium.

Q1: Prove that vertical lines are lines of constant gene frequency, and that such a line divides the base in the ratio $p:q$.

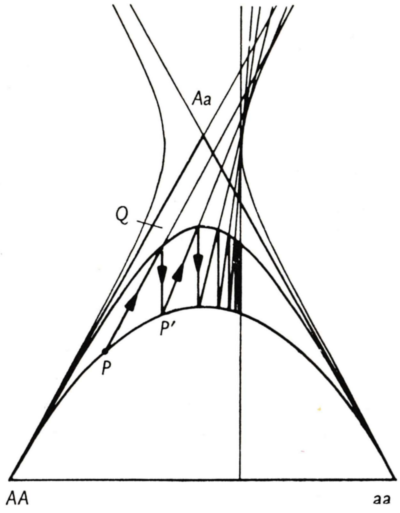

Fig. 3. Selection viewed graphically. The curve through P and P' is the pre-selection curve, and that through Q the post-selection curve. The selective values used in this example are ($\frac{1}{4}$, 1, $\frac{1}{2}$), the equilibrium gene frequency thus being 0.4 for the A gene. See text for further explanation.

In order to construct the lines joining corresponding points on the pre- and post-selection curves, we may find their envelope.

Q2: Prove that the envelope of lines joining (x, y, z) to (ax, hy, bz), where $y^2 = 4xz$, is $v^2 = 4kuw$, in which

$$k = 4(h - a)(b - h)/(a - b)^2.$$

This result shows that the envelopes are also members of the family of conics we have already considered. But note how this time k = 1 when h = (a + b)/2, or when there is no additive dominance in fitness. The joining lines are then tangential to the Hardy-Weinberg parabola. When h = a but h ≠ b, as when gene A is fully dominant to a in fitness, the lines form a pencil through the 'aa' vertex.

We can thus construct the diagram for any selective values a, h and b, and any starting point, if we have available a chart of the family of conics $v^2 = 4kuw$, for all k. This is given in Figure 4, which was drawn by a computer.

Q3: Find a simple proof that $v^2 = uw$ is a circle.

Several of the well-known results of selection theory are now easily derived. First, it is clear that selection is always monotonic in the gene frequency; secondly, there will be equilibrium whenever the line joining (p^2, 2pq, q^2) and (ap^2, 2hpq, bq^2), the pre- and post-selection points, intercepts the base in (p, 0, q), for then selection leaves the gene-frequency unchanged.

Q4: Prove that the condition for the gene-frequency to remain unchanged in this way is p/q = (h − b)/(h − a).

This result establishes the equilibrium gene frequency. Thirdly, the stability of the equilibrium is easily determined: for p/q lies between its value in the previous gene-

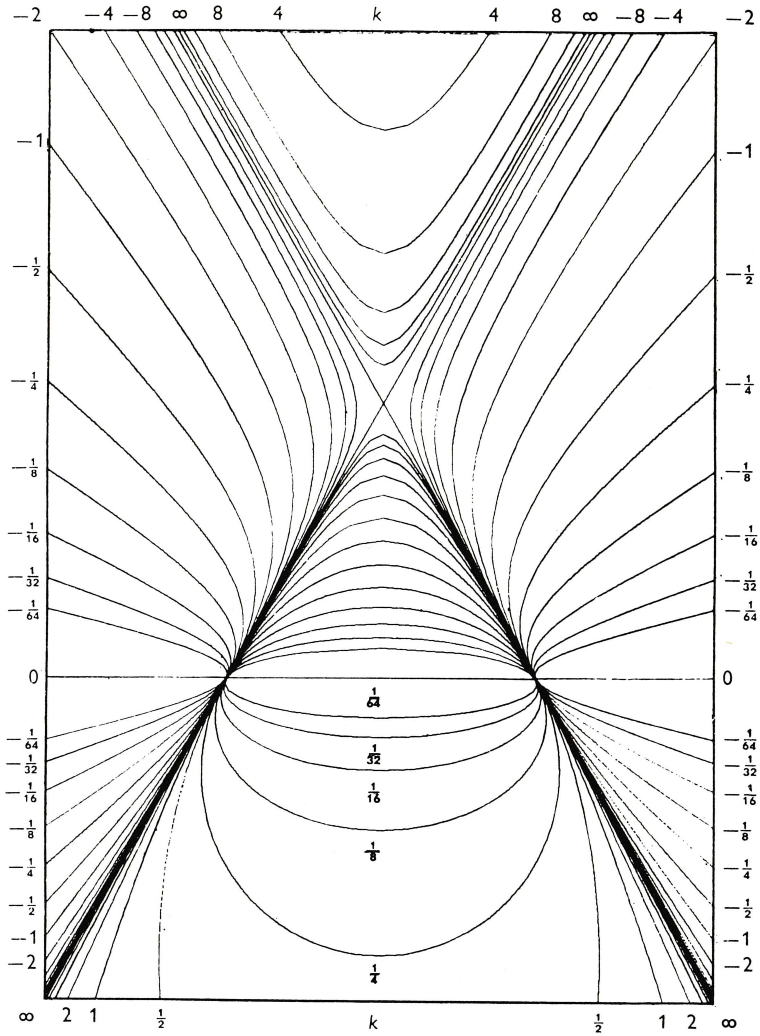

Fig. 4. The general diagram of post-selection curves and envelopes, being the family of conics $v^2 = 4kuw$.

ration and $(h - b)/(h - a)$, being their geometric mean. Thus all the lines through the corresponding pre- and post-selection points incline towards the vertical through the (non-trivial) equilibrium point, being closer to it at the top of the diagram. It follows that the equilibrium will be stable if the post-selection conic is above the Hardy-Weinberg parabola, and unstable if below. But we already know that these events correspond to $k > 1, h^2 > ab$ (hyperbola) and $k < 1, h^2 < ab$ (ellipse) respectively, thus establishing the well-known rules for finding the stability of the equilibrium.

Fourthly, we may prove the theorem mentioned by Professor Smith, that the mean zygotic fitness increases from generation to generation, or remains stationary at equilibrium. For the mean fitness, taken before selection, is given by

$$V = au + hv + bw,$$

where $v^2 = 4uw$. For different V, this is evidently a family of parallel lines on the diagram, and since no two values of V give rise to the same line, it changes monotonically as the lines are traversed. For the equilibrium point to be a stationary value of V, it is only necessary to show that the tangent to the Hardy-Weinberg parabola at the equilibrium point is one of the family of parallel lines:

Q5: Prove that the tangent to $v^2 = 4uw$ at $(p^2, 2pq, q^2)$ where $p/q = (h - b)/(h - a)$, is one of the family $au + hv + bw = V$.

It is easy to show that the mean fitness increases rather than decreases as the equilibrium is approached.

Finally, the diagram enables us to see why it is that if there is either no additive or no multiplicative dominance in fitness, the mean fitness of the zygotes of one generation is equal to the mean fitness of their parents in the previous generation: that is, mating does not change the mean fitness. For in the former case it is easy to show that the lines of constant V are vertical, and in the latter case the post-selection, or parental, curve coincides with the Hardy-Weinberg parabola. In either event, the vertical return to the Hardy-Weinberg parabola does not change the mean fitness.

The theorem about the mean fitness never decreasing under selection is true for any number of alleles, and not just two, but, as Professor Smith showed, the proof is not trivial. One might be able to find a simpler proof using homogeneous coordinates with a simplex of reference in many dimensions. I would welcome hearing from any reader who achieves it.

Two Problems on Cubes

by H. T. Croft

1. Baston [2] considered the following problem of Bagemihl [1] at length: how many disjoint (solid) tetrahedra can there be in 3 dimensions such that each touches each of the others, in the sense that they have a positive common facial area? Bagemihl had already showed a configuration of 8 tetrahedra, and this is presumed to be best possible; Baston showed that the greatest number was ≤ 9. If we replace 'tetrahedra' by 'cubes', we get a very much simpler problem, metrical and not affine; it is interesting to observe that the extremal configuration here shares with the conjectured extremal of the tetrahedral problem the property that all the bodies touch one plane.

The maximum number of cubes that touch pairwise in the above sense is 6.

First, such a configuration of 6 is certainly possible: let all the cubes touch a common plane, 3 on each side. Let the faces in contact with this plane be as in fig. 1, full lines representing faces of those on one side of the plane, dotted lines those on the other. This clearly satisfies the conditions.

Now, suppose if possible that there were a set S of cubes satisfying the above conditions but not all touching a common plane. First we remark that they must all have the same orientation. If not, take two with different orientations; since their common plane π is not a common plane for the whole set, some other cube touches each of them along parts of faces that are each perpendicular to π, which is impossible if the orientations are different.

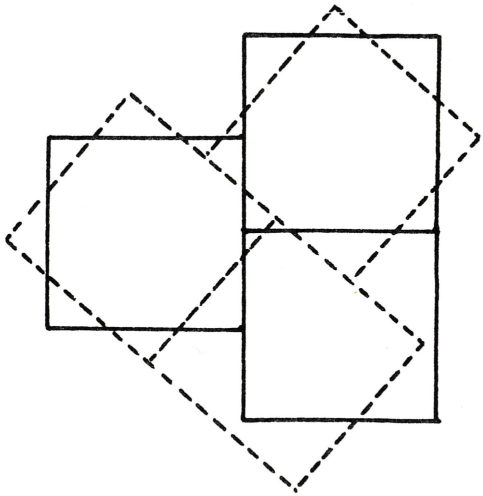

figure 1

Take now a common plane π of any 2 cubes of S. (In case there is one common plane of S, let π be that one). Consider the sections of the whole configuration by 2 planes, parallel to π a small distance away on either side. Each is seen to cut the configuration in squares; and a little consideration shows that these must each touch each of the others along a segment of positive length. In a plane no more than 3 such squares are possible. Hence the greatest possible numbers of cubes is 6; and that only when the 2 sections cut different triples of cubes, i.e. when the set of 6 touches a common plane.

A similar argument gives immediately an upper bound of $3 . 2^{n-2}$ in $\underline{n}$ dimensions; however it is not clear that an analogue of figure 1 can be found, and so whether this number can actually be achieved.

2. How many disjoint translates of a 3-dimensional cube can touch it, in the sense described above, i.e. along a positive facial area?

The answer is 14.

First, this number is easy to achieve: let the central cube C be Max $(|x|, |y|, |z|) = 1$, then we may place 4 disjoint cubes to touch C on $x = 1$, another 4 on $x = -1$, then 'between' these sets, 2 to touch on $y = 1$, and 2 on $y = -1$, and finally one to cover the whole of each of the z-faces of C.

We now need to show that no more than 14 is possible.

Suppose a cube C' touches C in such a way as to 'overlap' a corner X of C: more exactly we mean that the line joining the centre O of C to X pierces the interior of C'; then we say that X is a 'designated' corner of C. Clearly only one C' can designate any particular corner, and no C' can designate more than one corner. Another possible way of touching is for an edge <u>e</u> of C, but no corner, to be 'overlapped' by a C', again in the sense that plane Oe pierces the interior of C'. We then say the edge is 'designated'; again only one C' can designate any particular edge, and no C' can designate more than one edge. The final possibility is that a C' overlaps no corner or edge, but touches the whole of one face of C. We say the face is 'designated'. Hence we see every touching cube designates <u>one and just one</u> object, a corner or an edge or a face.

Our next aim is to show that the number of designated corners plus the number of designated edges ≤ 12. This will follow from combinatorial arguments plus the trivial geometrical fact that 2 incident edges and their common corner cannot all be designated (or else the corresponding cubes C' are not disjoint). Let there then be <u>c</u> corners designated and <u>e</u> edges. Then there are 2e 'ends of edges' to be accommodated amongst 8 vertices, so 2e—8 'excess ends' of which any vertex can only accommodate 2 at most (only 3 edges in total at any corner), so the number of corners that are unable to be designated since they are the meets of incident designated edges is at least $\frac{1}{2}(2e-8)$. So $\frac{1}{2}(2e-8)+c \leq 8$; as desired.

Thus if more than 14 cubes touch C, there must be 3 designated faces of C, or more. The 'or more' possibility is disposed of by counting, observing that such a designated face can only be in contact with one cube in total, and any other face with at most 4 disjoint cubes. Finally in the case of 3 designated faces, whether all adjacent or not, we see from the previous remark that we can only have a total of 15 if all the 3 non-designated faces touch 4 cubes each. In either case this would imply a pair of adjacent faces touching 4 cubes each; and this is impossible (the common corners would be 'designated from both sides').

This completes the proof.

But we may regard the result in another light. For centuries it was a problem how many disjoint spheres could touch one sphere, all being the same size. It has now long been known that the answer is 12. See e.g. [4], [5]. We may ask, for any convex body C, how many disjoint translates of it can be made to touch C. Presumably the minimum of this (over varying C) is 12, the maximum 27 (for the cube, or, since the problem is affine-invariant, the parallelepiped). Here 'touch' is interpreted in the usual sense of 'having a point of contact in common'. However, in view of the above result, we may expect to get closer bounds if we re-interpret it more strictly. Any reasonable definition will yield 12 for the minimum, but we may ask for the maximum under any one of the following restrictions: (i) if the body C must be smooth (i.e. have a unique tangent plane at any point), (ii) if at every point of contact between C and a C', there is a <u>unique</u> separating plane, (iii) C and C' touch along a plane face of positive area.

Let us call the maximum number of C' under these conditions, α, β, γ, respectively. We can ask the same questions restricting consideration to centrally symmetric bodies: denote the corresponding numbers by $\alpha^*, \beta^*, \gamma^*$. There are various trivial inequalities between these 6 numbers; also it seems clear by making suitable small changes to the shapes of the extremal bodies that $\alpha = \gamma$ and $\alpha^* = \gamma^*$. Perhaps they are all 14?

The first problem can also be generalized to convex bodies; leaving aside the variations on the definition of touching discussed above, at least initially, we can ask how many congruent bodies of a given shape in 3 dimensions can all have pairwise contact, and in particular the minimum and maximum of this taken over the various bodies. The minimum would seem to be 4. (Indeed, presumably it is true that any 4 convex objects in 3 dimensions can be moved so as to be pairwise in contact—perhaps even by translation.) The maximum is at least 7, since it is known that 7 congruent right circular cylinders of appropriate shape can all touch one another [3]. This question can be extended to unbounded (congruent) convex bodies; Professor Littlewood has asked it for infinite-both-ways right circular cylinders—a question that may be put in the following form: how many lines can one have in 3 dimensions with the shortest distance between any pair of them being the same?

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Fermat's Last Theorem and the Golden Mean

by R. W. M. Wedderburn

Theorem If $a^n + b^n = c^n$ where a, b, c and n are positive integers, and if

$$n > \frac{1}{\tfrac{1}{2} - \log r}, \text{ then a, b and c} > \frac{n}{1/n + \log r}, \text{ where } r = \frac{\sqrt{5}+1}{2}.$$

(The quantity r is familiar as the 'golden mean' or 'golden section'.)

Proof We assume throughout the proof that $a < b < c$, so that our assertion becomes

$$a > \frac{n}{1/n + \log r}.$$

First we prove two lemmas.

Lemma 1 (i) For a fixed $n > 0$, the equation $x^n + (x+1)^n = (x+2)^n$ has one and only one positive root, α_n say.

(ii) For $0 < x < \alpha_n$, $x^n + (x+1)^n < (x+2)^n$, and for $x > \alpha_n$, $x^n + (x+1)^n > (x+2)^n$.

(iii) $a \geqslant \alpha_n$.

Proof Consider

$$\frac{(x+2)^n - (x+1)^n}{x^n}$$

If we expand the numerator by the binomial theorem, and then divide by the denominator, we obtain a linear combination of negative powers of x with positive coefficients. Hence the expression is strictly decreasing for positive x. The expression tends to ∞ as x tends to 0 from above, and tends to 0 as x tends to ∞. Also the expression is continuous. So for some positive α_n, the expression is $>, =$ or < 1 according as $0 < x < \alpha_n$, $x = \alpha_n$ or $x > \alpha_n$. Hence (i) and (ii).

Now $a \leqslant b - 1$ and $c \geqslant b + 1$, and so $a^n + b^n = c^n$ implies $(b-1)^n + b^n \geqslant (b+1)^n$. Hence by (ii), $b \geqslant \alpha_n + 1$. Then by noting that $(x+1)^n - x^n$ is an increasing function of x (expand and note positive coefficients), we have

$$a^n \geqslant (b+1)^n - b^n \geqslant (\alpha_n + 2)^n - (\alpha_n + 1)^n = \alpha_n^n.$$

Hence (iii).

Lemma 2 For $x > 0$, (i) $(x+2)^n > x^n.\exp\left(\frac{2n}{x+2}\right)$.

(ii) $(x+1)^n < x^n.\exp(n/x)$.

Proof The proof depends on the following inequalities:-

For $x > 0$, $\left(\frac{x+1}{x}\right)^x < e < \left(\frac{x+1}{x}\right)^{x+1}$.

This result is fairly well known, but I give the following proof:-

Let $P(x) = \left(\frac{x+1}{x}\right)^x$ and $Q(x) = \left(\frac{x+1}{x}\right)^{x+1}$.

$$\text{Then } P^Q = Q^P = \left(\frac{x+1}{x}\right)^{\frac{(x+1)^{x+1}}{x^x}}, \text{ and so } \frac{\log P}{P} = \frac{\log Q}{Q} \tag{1}$$

$$\text{Now } \frac{d}{dy}\left(\frac{\log y}{y}\right) = \frac{1 - \log y}{y^2} \begin{cases} = 0, \text{ for } y = e \\ < 0, \text{ for } y > e \\ > 0, \text{ for } 0 < y < e \end{cases}$$

So $\frac{\log y}{y}$ increases strictly from $y = 0$ to $y = e$, and decreases strictly for $y > e$. Then for $x > 0$, we have $0 < P < Q$, which taken together with (1) gives $P < e < Q$. This is the result mentioned above. Raising the first of the inequalities to the power $\frac{n}{x}$ gives (ii), and raising the second to the power $\frac{2n}{x+2}$ gives (i).

Proof of Theorem From Lemma 2, for $x > 0$,

$$(x+2)^n - (x+1)^n - x^n > x^n\left[\exp\left(\frac{2n}{x+2}\right) - \exp\left(\frac{n}{x}\right) - 1\right]$$

$$= x^n\left[\exp\left(\frac{-4n}{x(x+2)}\right)\exp\left(\frac{2n}{x}\right) - \exp\left(\frac{n}{x}\right) - 1\right] \tag{2}$$

Write $k = \frac{n}{1/n + \log r}$. This gives $\exp\left(\frac{n}{k}\right) = \exp\left(\frac{1}{n}\right).\ r$

Now suppose that $n > \frac{1}{\frac{1}{2} - \log r}$. This gives $\frac{n}{2} - n.\log r > 1$

i.e. $\frac{n}{2} > 1 + n.\log r = \frac{n^2}{k}$, i.e. $k > 2n$.

Then $\exp\left(\frac{-4n}{k(k+2)}\right) > \exp\left(\frac{-1}{n}\right)$. Now substitute for x in (2).

$$(k+2)^n - (k+1)^n - k^n > k^n\left[\exp\left(\frac{-1}{n}\right).r^2.\exp\left(\frac{2}{n}\right) - r.\exp\left(\frac{1}{n}\right) - 1\right]$$

$$= k^n\left[\exp\left(\frac{1}{n}\right).(r^2 - r - 1) + \exp\left(\frac{1}{n}\right) - 1\right]$$

$$> 0 \text{ (since } r^2 - r - 1 = 0)$$

So by Lemma 1 (ii) and (iii), we have

$$a \geqslant \alpha_n > k = \frac{n}{1/n + \log r}. \text{ Q.E.D.}$$

Some numerical results are $\dfrac{1}{\frac{1}{2} - \log r} \simeq 52.7; \dfrac{1}{\log r} \simeq 2.07795.$

I am grateful to Professor Davenport for his comments on an earlier draft of this work.

The Multiplicator

by W. F. A. Chambers

Advances in engineering in the future will demand from Mathematics a finer degree of accuracy than any hitherto attained. The Multiplicator, a new automatic method of multiplying, is capable of giving 100% accuracy.

It has already been shown by Tractenberg that it is possible to devise new methods of multiplication. Although that system has the disadvantage that it requires a good deal of working in the head, it does show that the principle of Archimedes 'First get your facts, then frame your propositions' is still valid.

The great advantages of the Multiplicator are:
(1) No need to learn any tables. Method learnt in five minutes.
(2) Absolute accuracy.
(3) No carrying forward nor calculations in the head.
(4) Any figure in the product can be determined instantly.
Facilitates rapid checking.
(5) Saves space in working.
(6) Easy to produce new tables and very cheap.
(7) Can be adapted to other radices than 10.
(8) No conversions as with logarithms, and no approximations as with slide rules.

The basis of this method can be stated as follows:

Take a given multiplier, say ×3, and multiply by it successively numbers between 1 and 10.

$3 \times 1 = 03$	$3 \times 6 = 18$
$3 \times 2 = 06$	$3 \times 6.666\dot{6} = 20$
$3 \times 3 = 09$	$3 \times 7 = 21$
$3 \times 3.333\dot{3} = 10$	$3 \times 8 = 24$
$3 \times 4 = 12$	$3 \times 9 = 27$
$3 \times 5 = 15$	$3 \times 9.999\dot{9} = 30$

Note that the second figure in the product appears when and only when the multiplicand has reached $3.\dot{3}$. This number is called the Turnover Point. It is absolutely certain that up to that point there will be no digit in the second place, even if the multiplicand extends to infinity, e.g. 3.332999999 etc to infinity $\times$ 3 = 9.998999999 etc to infinity. (Note also that $.3\dot{3}$ is equivalent to $\frac{1}{3}$ or number 1 in the radix of 3.)

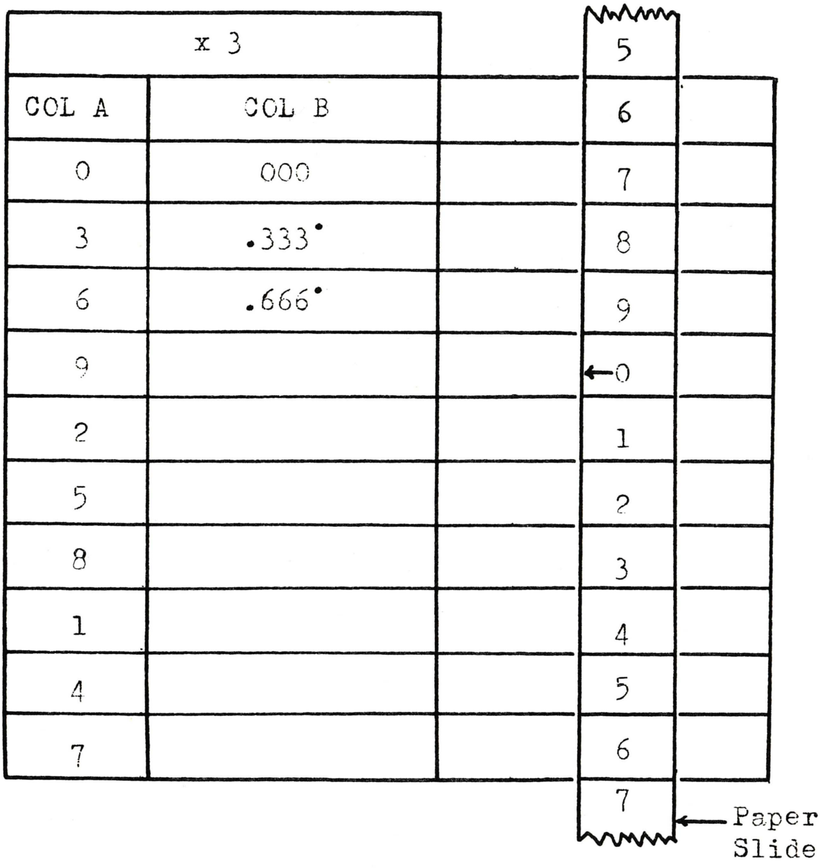
The second Turnover Point for the multiplier $\times 3$ is $.6\dot{6}$. Thus:

$$3 \times 6.65 = 19.95$$
$$3 \times 6.\dot{6} = 20$$

($.\dot{6}$ is of course the same as $\frac{2}{3}$ or 2 in radix 3.)

The Turnover Points can be determined very simply by dividing numbers lower than the multiplier by the multiplier. Thus, in the case of $\times 3$, we divide 1 and 2 by 3, getting $.\dot{3}$ and $.\dot{6}$.

Now we will see how these Turnover Points are applied to a mechanical Calculator. (See diagram)



Col. A represents the figure in the multiplicand which is to be multiplied by 3. Col. B represents the Turnover Points for the multiplier 3.

Let us take an example: 3 × 967341887999

We can start anywhere in the multiplicand. Let us take the left-hand figure 9. Set the arrow on the Slide to 9 in Col. A. Opposite the 0 in Col. A is 7 on the Slide. This is the 7 in 27 which is the product of 3 with 9 (9 with nothing after it, or 900000000 etc). But here the first 9 is followed by 673 etc. This is higher than .666˙. So we read off on the Slide, opposite to .666˙ in Col. B, the figure 9. This will be the 100% correct figure in the product falling below the 9. Similarly, without re-setting the slide, we obtain

```
0967341887999
            3
_____________
09........997
```

Now the product is bound to be longer than the multiplicand, so we have put an 0 before the latter. Set the arrow to 0. It is followed by 967 which is higher than .666˙. In Col. B, opposite this 666..., we find on the Slide the figure 2, giving 29........997.

Clearly no more than 10 separate settings of the Slide are required for any one multiplicand, however long it may be.

Each multiplier has its own proper List of Turnover Numbers. In some cases the Turnover Numbers form sequences. There may be two or more sequences.

The Turnover Numbers for the multipliers from 4 to 9 are as follows:

The Slides are always identical in all cases.

×4

Col. A	Col. B
0 : 5	000
2 : 7	250
4 : 9	500
1 : 6	750
3 : 8	

×5

Col. A	Col. B
2, 4, 6, 8, 0	000
	200
	400
	600
	800
1, 3, 5, 7, 9	

×6

Col. A	Col. B
0 : 5	000
	1.66
3 : 8	.333
	500
1 : 6	.666
	8.33
4 : 9	

×7

Col. A	Col. B
0	000
7	.142..
4	.285..
1	.428..
8	.571..
5	.714..
2	.857..
9	
6	
3	

×9

Col. A	Col. B
0	000
1	.11..
2	.22..
3	.33..
4	.44..
5	.55..
6	.66..
7	.77..
8	.88..
9	.99..

":" denotes "or".

The Turnover Numbers for ×7 represent a recurring sequence of .142857˙... This is regenerative i.e. it has the remarkable property that whatever number it is multiplied by, it will always return to the same sequence. There are other and longer such sequences. In the first 100 numbers, regenerative sequences are formed by the Prime Multipliers 7, 17, 19, 23, 29, 47, 59, 61 and 97. These correspond to the Atomic Numbers of Nitrogen, Chlorine, Potassium, Vanadium, Copper, Silver, two Lanthanides and Actinium. Are these elements in any way linked?

So far tables of Turnover Numbers have been worked out for some three hundred multipliers, with some constants and Metrical Equivalents.

A Shuffling Problem

by P. T. Johnstone

We consider the following problem: given a pack of n cards (where n is an even number), we cut it exactly in half and give it a perfect shuffle (i.e. we interleave the two halves exactly). There are two ways of doing this, depending on which half we choose for the first card; we make our choice so that the original top card remains on the top, and the bottom card on the bottom. However, the remaining cards will all be permuted; what are the conditions on n for this permutation to be an $(n-2)$-cycle?

Before investigating this question, we remark that the alternative choice of definition does not yield a new problem, since it is equivalent to adding an extra card on the top and bottom of the pack, performing the shuffle with our original choice, and removing the extra cards. We now number the cards from 1 to n; it is readily seen that the effect of the shuffle is to take the rth card to position $2r-1$ or $2r-n$, whichever lies within the range 1 to n.

Let d be a divisor of $n-1$; then if $r \equiv 1 \pmod d$ so are both $2r-1$ and $2r-n$, i.e. the shuffle permutes the set of numbers congruent to 1 amongst itself. If $1 < d < n-1$, this set is a nonempty proper subset of $\{2, 3, \ldots, n-1\}$ and so the shuffle cannot be an $(n-2)$-cycle. Thus we require that $n-1$ shall have no divisors in this range, i.e. it is a prime. We write $n-1 = p$ when convenient. It is now instructive to examine the cycle type of the permutation for the first few primes, by actual computation. We find that it is indeed an $(n-2)$-cycle for $n = 4, 6, 12, 14, 20, 30, 38, 54, 60, 62, 68, 84, \ldots$ (class (i)), and that it fails to be so for $n = 8, 18, 24, 32, 42, 44, 48, 72, 74, 80, 90, 98, \ldots$ (class (ii)).

In all the latter cases the $n-2$ elements split into q disjoint cycles of equal size, where q is a power of 2 except for $n = 32$ $(q = 6)$ and $n = 44$ $(q = 3)$. The case $n = 44$ also stands out as being the only number in the second sequence not congruent to 0 or 2 (mod 8), whereas all the numbers in class (i) are congruent to 4 or 6. We will now attempt to explain these results by examining the effects of the permutation more closely.

The permutation can clearly be written as $r \to 2r-1-\alpha p$, where $\alpha = 0$ or 1 as appropriate. Suppose that after k shuffles the rth card returns to its original place; then

$$2^k r - 2^{k-1} - 2^{k-2} - \ldots - 1 - Np = r$$

where N can take any integer value between 0 and $2^k - 1$, depending on the value of α at each stage. This reduces to

$$(2^k - 1)(r-1) = Np \equiv 0 \pmod p.$$

Since p is a prime, the ring of integers mod p has no zero divisors; thus if we exclude the cases $r = 1$ and $r = n$, we must have $2^k - 1 \equiv 0 \pmod p$, and this will ensure that the earlier equation holds independently of r, i.e. that the kth power of the permutation is equal to the identity.

This explains why the cycles of the permutation all have the same length when $n-1$ is a prime, although they do not when it is composite. Now suppose 2 is a qth power residue mod p, where q divides $p-1$; then it is a simple consequence of Fermat's Theorem that $2^{(p-1)/q} \equiv 1 \pmod p$. Hence the length of the cycles divides $(p-1)/q$, and their number must be a multiple of q. Conversely, if 2 is not a qth power residue for any q dividing $p-1$, then 2 is a primitive root mod p, and $2^k \not\equiv 1$ for $0 < k < p-1$. In this case the permutation can have only one cycle, since its length is $p-1$.

The answer to our original question is thus 'n must be one greater than a prime with respect to which 2 is primitive'. Unfortunately, this is not the end of the problem, for there is no standard rule for telling when 2 is primitive with respect to a given prime p. However, we can deal with the quadratic residue cases easily enough since it is a standard result of Number Theory that 2 is a quadratic residue mod p if and only if $p \equiv \pm 1 \pmod 8$. This explains why all the values of n congruent to 0 of 2 (mod 8) appear in class (ii), and have even values of q. However the exceptional case $n = 44$ shows us that odd values of q (other than 1) can occur, and we have no way of predicting when these will appear. The exceptional cases up to about 300 are $n = 44, 110, 158, 230, 278, 284, 308$ $(q = 3)$ and $n = 252$ $(q = 5)$.

If $p_1 = \frac{1}{2}(p-1)$ is also prime, then 2 cannot be a p_1th power residue, since this would imply $2^2 = 4 \equiv 1 \pmod p$. If additionally $p_1 \equiv 1 \pmod 4$, then $p \equiv 3 \pmod 8$ and 2 is not a quadratic residue. Hence 2 is a primitive root in this case. Again, if $p_2 = \frac{1}{4}(p-1)$ is prime, 2 cannot be a p_2th power residue, and since this implies $p \equiv 5 \pmod 8$, 2 is primitive. These two cases yield the sequences $n = 12, 60, 84, 108, 180, 228, \ldots$ and $n = 14, 30, 38, 54, 150, 174, 270, 294, \ldots$ of numbers satisfying the original problem. Unfortunately, it is unknown whether or not these sequences terminate, and it is quite evident that they leave out a substantial number of solutions (e.g. $n = 20, 62, 68, 102, 132, 140, 164, 182, \ldots$).

Are there infinitely many solutions althogether? Almost certainly; E. Artin has conjectured that if a is not a power of any smaller integer b, the proportion of primes less than N for which a is primitive tends to a constant A as $N \to \infty$. Probabilistic arguments suggest that A is Artin's constant, the product over all primes p of $(1 - 1/p(p-1))$. This constant has been computed to 40 decimal places by J. W. Wrench; its value is about 0.3740. A. J. C. Cunningham's tables of primes less than 10^5 for which various residues are primitive, which date from 1913, also render this conjecture quite plausible; although the ratios he obtains are consistently greater than A, they do appear to tend towards it as N increases.

Finally, we consider a generalisation of the problem, which actually admits of a more complete solution than the original one in some cases. If we start with a multiple of k cards $(k \geqslant 2)$, we can cut the pack into k equal sections and interleave them exactly in a 'generalised shuffle'. This shuffle is defined by the permutation $\theta_k : r \to kr - (k-1) - \alpha(n-1)$ where α is chosen suitably in the range $0 \leqslant \alpha \leqslant k-1$. Clearly, θ_k can also be defined thus when k does not divide n (although it can no longer be readily performed as a shuffle), provided k is prime to $n - 1$. The reader may prove that $\theta_k = \theta_{k+n-1}$ and that $\theta_k\theta_j = \theta_{kj} = \theta_j\theta_k$, so the subgroup of the symmetric group Σ_{n-2} generated by the θ_k is Abelian, and indeed isomorphic to the group of reduced residue classes modulo $(n-1)$.

We now ask the same question as before; for what values of k and n is θ_k an $(n-2)$-cycle? By precisely the same arguments as before, we find that $n - 1$ must be a prime having k as a primitive root; but if we now restrict our attention to the actual shuffling problem (i.e. the case where k divides n), we find that, for some values of k, the question may be completely solved by consideration of quadratic character. Thus if $k = 4^m j$, where $m \geqslant 0$ and $j \equiv 1 \pmod 4$, the Law of Quadratic Reciprocity tells us that

$$\left(\frac{k}{n-1}\right) = \left(\frac{j}{n-1}\right) = \left(\frac{n-1}{j}\right) = \left(\frac{-1}{j}\right) = +1,$$

i.e. k is always a quadratic residue mod $(n-1)$. Thus no solutions of the problem exist. Though this is only a very partial answer, it is nevertheless consoling to pull <u>some</u> solid result out of the morass of conjecture.

Uniform Curves and Surfaces

by D. G. Hayes

I shall use the phrase 'uniform curve' (abbreviated to UC) to mean a curve, embedded in space of any number of dimensions, having the property that any arc of it is congruent to any other arc of the same length.

In space of two dimensions, the only uniform curves are the circle and straight line. In three dimensions, the most general UC is the helix. In space of one dimension, the only UC, and indeed the only curve, is the straight line.

The equation of a UC is most easily written in parametric form.

In one dimension the equation is $x = \theta$

In two dimensions, we have $x = a\cos\theta, y = a\sin\theta$

In three dimensions, we have $x = a\cos\theta, y = a\sin\theta, z = \lambda\theta$

In four dimensions, the equation of a UC is

$$x = a\cos\lambda\theta, y = a\sin\lambda\theta, z = b\cos\mu\theta, t = b\sin\mu\theta \tag{1}$$

Thus a four dimensional UC is bounded, and closed if and only if λ/μ is rational.

The above results can be generalized to space of n dimensions. If n is even, the most general UC is bounded and has equation

$$x_{2i-1} = a_i\cos\lambda_i\theta, x_{2i} = a_i\sin\lambda_i\theta \tag{2}$$

If n is odd, say $n = 2m + 1$, the most general UC is not bounded. Its equation is

$$x_{2i-1} = a_i\cos\lambda_i\theta, x_{2i} = a_i\sin\lambda_i\theta\ (1\leqslant i\leqslant m), x_n = \mu\theta \tag{3}$$

Figure 1 is a picture of the six dimensional UC

$$x = \cos 45\theta, y = \sin 45\theta, z = 4\cos 18\theta, t = 4\sin 18\theta, u = 10\cos 4\theta, v = 10\sin 4\theta.$$

If λ/μ is rational, then (1) is the equation of a closed curve in four dimensional space. If λ/μ is not rational, then the curve is not closed, and traces out the surface

$$x = a\cos\theta, y = a\sin\theta, z = b\cos\phi, t = b\sin\phi \tag{4}$$

or, in non-parametric form,

$$x^2 + y^2 = a^2, z^2 + t^2 = b^2$$

More precisely, given any point P on the surface, one can find a point on the curve arbitrarily close to P, although the curve may not actually pass through P.

We shall see later that (4) is the equation of what I shall call a uniform surface.

Consider the six dimensional UC

$$x = a\cos\lambda\theta, y = a\sin\lambda\theta, z = b\cos\mu\theta, t = b\sin\mu\theta, u = \cos\nu\theta, v = c\sin\nu\theta$$

If λ/μ and λ/ν are both rational, then the curve is closed.

If λ/μ and λ/ν are not both rational, but λ, μ, ν satisfy a relation of the form $p\lambda + q\mu + r\nu = 0$, with p, q, r integers, then the curve traces out the uniform surface

$$\left.\begin{array}{l} x = a\cos r\theta, y = a\sin r\theta, z = b\cos r\phi, t = b\sin r\phi, \\ u = c\cos(p\theta + q\phi), v = c\sin(p\theta + q\phi) \end{array}\right\} \tag{5}$$

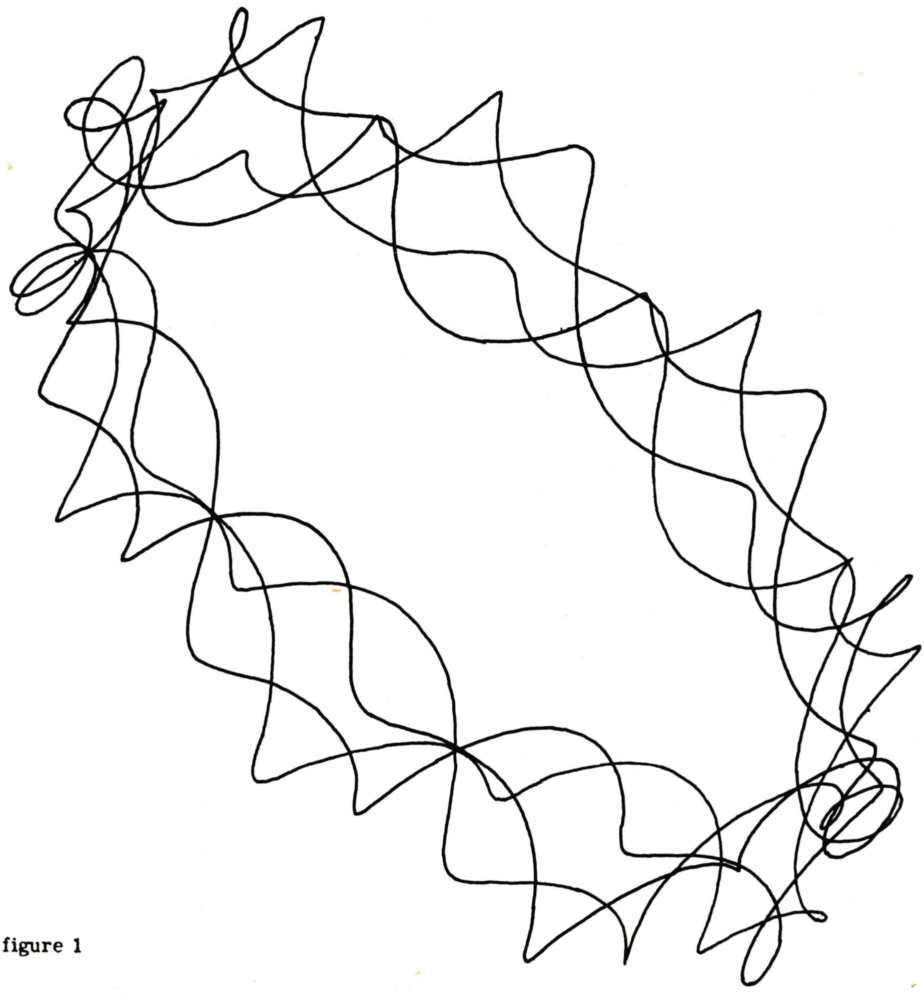


figure 1

If no such relation is satisfied, then the curve traces out the uniform three dimensional manifold

$$x^2 + y^2 = a^2, z^2 + t^2 = b^2, u^2 + v^2 = c^2$$

Similar results apply in space of any even number of dimensions, but not in space of an odd numbers of dimensions. In space of 2m dimensions, it is possible for a UC to trace out a uniform p-dimensional manifold, where p may be any integer satisfying $1 \leqslant p \leqslant m$.

We can define a uniform surface (henceforth abbreviated to US) in either of two ways. In one sense an infinite circular cylinder is a US. Suppose we have such a cylinder, and a rigid lamina whose every point touches the cylinder. Then we can slide the lamina over any part of the cylindrical surface, without any point of the lamina losing contact with the surface.

A sphere is a US in the above sense, and it also has the property that the lamina can be rotated about any of its own points without any point of the lamina losing contact with the sphere. We can either define a US to be uniform in the sense that a cylinder is uniform or in the stronger sense that a sphere is uniform

I have tried without success to find other surfaces, in space of more than three dimensions, uniform in the way that a sphere is uniform. I have also tried without success to prove that there are none. Henceforth I shall use the US to mean a surface uniform in the way that a circular cylinder is uniform.

As examples of US's, I have already mentioned the sphere, the cylinder and the surfaces given by (4) and (5).

It is possible to find US's in space of n dimensions for any $n \geqslant 2$. If n is even, say $n = 2m$, then

$$x_{2i-1} = a_i \cos(\lambda_i \theta + \mu_i \phi), x_{2i} = a_i \sin(\lambda_i \theta + \mu_i \phi) \text{ for } 1 \leqslant i \leqslant m \qquad (6)$$

gives a US.

If n is odd, say $n = 2m + 1$, then the equations (6) together with

$$x_n = \lambda_{m+1}\theta + \mu_{m+1}\phi$$

give a US. (7)

Any geodesic on such a US is a UC. All such US's have zero total curvature; i.e. if a triangle, whose sides are geodesics, is drawn on such a surface, then its angles add up to π.

In space of an even number of dimensions, a US can trace out a uniform p-dimensional manifold, in the same way as a UC can.

Figure 2 is a picture of the US in four dimensional space

$$x^2 + y^2 = 1, z^2 + t^2 = 3.$$

The picture is made to appear three dimensional. It shows a family of curves x = constant, y = constant on the surface, and a family z = constant, t = constant.

The equations (2), (3), (6) and (7) can be generalized to give the equation of a uniform p-dimensional manifold in n-dimensional space, provided that $n \geqslant 2p - 1$. The equation of such a manifold is

$$\left.\begin{array}{l} x_{2j-1} = a_j \cos(\lambda_{ij}\theta_i), x_{2j} = a_j \sin(\lambda_{ij}\theta_i) \text{ (summation convention)} \\ \text{together with } x_n = \mu_i \theta_i \text{ if n is odd.} \end{array}\right\} \qquad (8)$$

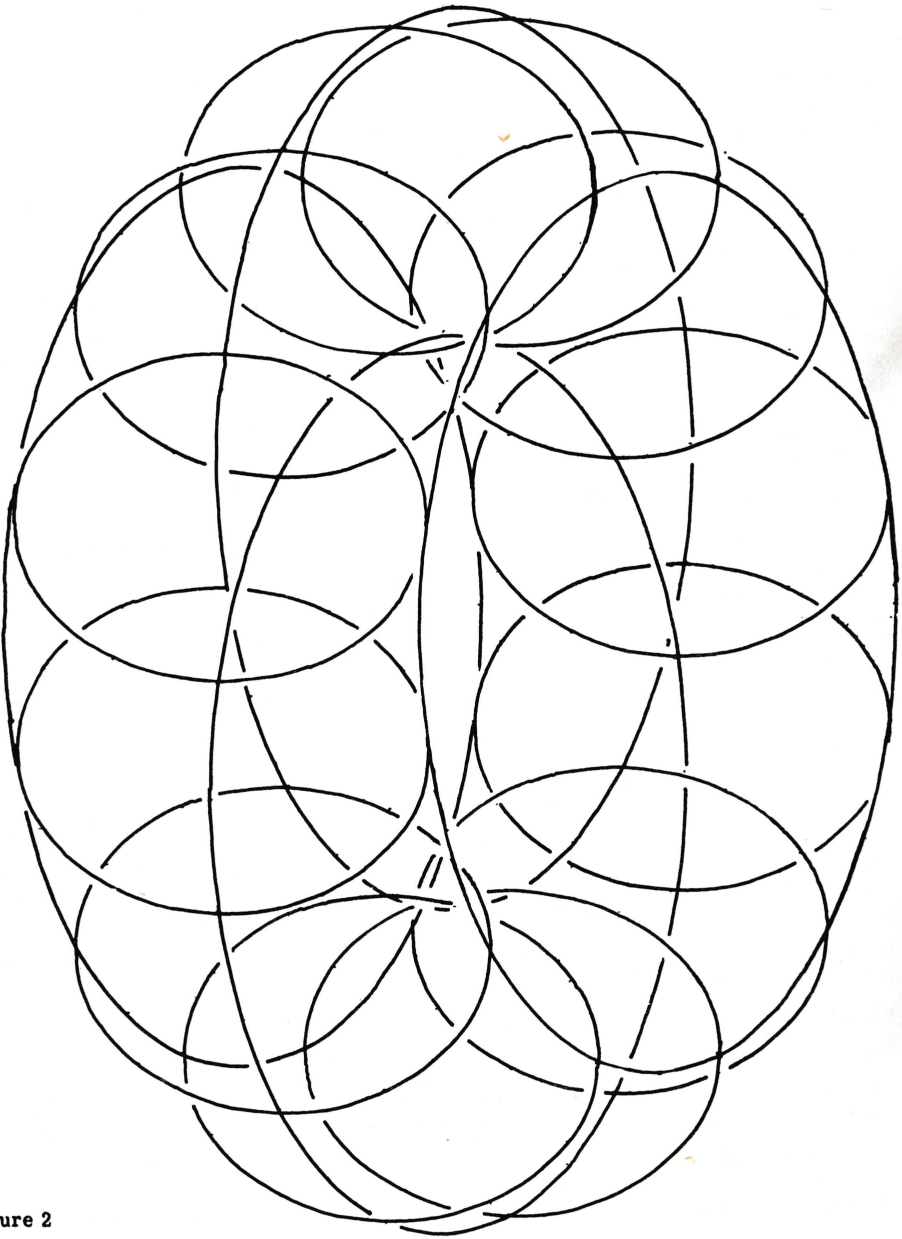


figure 2

As in the case of a US, any such manifold has zero total curvature. If $n < 2p - 1$, then the equation of a uniform p-dimensional manifold is, in non-parametric form,

$$\sum_{i=m_j+1}^{m_{j+1}} x_i^2 = a_j^2 \qquad 0 \leqslant j \leqslant n-p$$

where $m_0 = 1$ and $m_{n-p} \leqslant n$.

Such a manifold need not have zero total curvature.

If $n = 2p$, equation (8) coincides with equation (9). If $n = 2p - 1$, equation (8) is one possible case of equation (9).

Examples of manifolds given by (9) are the sphere $x^2 + y^2 + z^2 = a^2$, the cylinder $x^2 + y^2 = a^2$, z unrestricted, the 'spherical cylinder' in four dimensional space $x^2 + y^2 + z^2 = a^2$, t unrestricted, and the three dimensional manifold in five dimensional space given by $x^2 + y^2 + z^2 = a^2, t^2 + u^2 = b^2$.

A Further Examination of the 'Surprise Examination Paradox'

by C. Hudson, A. Solomon, R. Walker

The paradox is the following:

'A school-teacher tells his class that there will be a surprise examination on one day during the term in the sense that they will not know in advance that it will be held on any specific day. The children reason inductively that it cannot happen on the last day (as it would not be a surprise) and it cannot happen on the penultimate day as it cannot happen on the last and so would not be a surprise.......
By induction it cannot happen on any day!'

It has been suggested that the backward inductive argument is invalid for 'entropy considerations' but we find this solution unsatisfactory as we see no connection between a statistical phenomenon and a logical argument.

Mr. Woodall (Eureka No. 30) deduces, correctly, that the two statements:

(1) Teacher says that there will be a surprise examination this term.

and (2) Children are absolutely certain that teacher is telling the truth.

are mutually inconsistent.

We shall assume the validity of (2) and shall examine (1). The definition of surprise was taken to be 'the inability to forecast with certainty that the examination will occur today'. We shall call this absolute surprise. We now define partial surprise to be 'the probable inability to.... '. We see that there is no paradox with partial surprise as the last day is no longer excluded.

Considering the time factor, we see that the argument requires a 'last day' i.e. a known upper bound upon the time period during which the examination can be set.

We now observe there is no paradox with either:
(a) absolute surprise and no known time bound, or
(b) partial surprise and no known (or known) time bound.
We note also that if there may be more than one examination, or no examination, then again there is no inductive argument and hence no paradox.

We thus conclude that the paradox is resolved in the inconsistency of a definition of absolute surprise with one examination within a known time bound.

John Adams

(A Gothick Necromance)

by A. G. Smith & P. E. Smith

Young John Adams was at Cambridge reading Alchemy and Astrology in the Supernatural Sciences Tripos. He had been called before his Tutor, who told him that his Director of Studies reported that the horoscopes he had cast for supervision had been consistently wrong all term, and that the Junior Bursar had complained that he had transmuted all the college gold into lead. When he returned to his rooms he fell to wondering why his predictions failed. Suddenly the thought crossed his mind that the theories might be at error. What Influence could they be neglecting? 'Suppose', he thought, 'there were another zodiacal sign; for are not there 13 months to the lunar year?' Excitedly he took up his quill to explore the consequences of this conjecture; there would have to be 390 degrees to the circle, and the world's population would have to be $^{13}/_{12}$ of the accepted figure, but he was not interested in such mundane matters; he was interested in Impure Research, and the astrological consequences of his theory. He seized his copy of Ghoulson's 'Ætheric Waves' and worked all night on the Mathlab abacus to come up with a prediction. On both his theory and the conventional one his college would defeat the TMS cricket team next day, but the margin of victory would serve to distinguish the two theories.

By tea-time these prophecies had, of course, failed, and John Adams was wondering why. He had checked his calculations, only to find that the prediction had indeed accurately embodied his theory. For a week he tried to find some other possible Hidden Influence, but at length he decided he needed help. He went to see his Chaplain, and told him that he wished to sell his soul to the Devil. 'Have a drink, old man,' said the Chaplain, 'and do take a seat. I always feel that selling one's soul is such an intensely Personal matter, don't you? But don't let me force my own opinions on you, will you? Won't you have another glass?'

The Chaplain having been of no help, John Adams went to the Occult Periodicals Library, on the top floor of the Black Arts School, to try to find out how one set about selling one's soul to the Devil. Having found the reference at last, he took down Sir Harold and Lady Margaret Beautclub's standard work 'Methods of Mathematical Psychics' (6gns, Pandemonium Press; or 2nd-hand from the Archimedeans' Bookshop). At the head of chapter XIX he found the quote, 'Then fear not, Faustus, but be resolute, and try the uttermost Magic can perform', and a footnote on the selling of souls, saying, 'too complex to discuss here'.

He went to the Porters' Lodge and asked for his Diabolical Coll. Rep.. 'You'll be meaning the Anti-Chaplain, sir; I expect he'll be in the Anti-Chapel, sir, preparing for tonight's Black Mass. 'Tis Walpurgisnacht tonight, Mr. Adams, sir, and many of the non-resident fellows have flown up specially. Aren't you going to the Feast of Commemoration of Malefactors, sir? Oh, I do beg your pardon, sir, I was thinking you were one of them scholars.'

After the Anti-Chaplain had conducted the Black Mass, John went to call on him. He explained that at Syngeon's College the matter is dealt with through the Amalg. Clubs, and that all he had to do was sign 'one of these cards' in his blood, and return it to his tutor as soon as possible.

That night the Devil appeared to John Adams in a dream, and suggested that he try postulating another planet. Adams protested that that would be ridiculous, but the Devil vanished into a Great Red Omnibus, and left him.

The next day, John Adams began collecting information on all the unpredicted disasters that had befallen the human race down the ages; working out in what sign of the zodiac the new planet must have been, and what the magnitude of its Influence must be. After several months of calculation, he had determined to his own satisfaction the position and strength of Influence of the new planet, and announced his results by reading a paper at a tea-meeting of the Cambridge University Astrological Society. Urged by his tutor, John Adams wrote to the Astrologer-Royal, predicting that Princess Charlotte would die in childbirth of a still-born son. The Astrologer-Royal, who had predicted to the Regent that Princess Charlotte and Prince Leopold would present him with many grandsons, and reign long and gloriously after his own reign, understandably ignored John Adams' claims.

Meanwhile in France, a young man named Le Verrier had been investigating the orbital eccentricities of Uranus, using Newtonian Mechanics. As soon as he had published his findings, the Prussian Astronomer-Royal pointed his telescope at the spot marked, and found a new planet, whose name he perceived to be Neptune.

That night an Angel of Light appeared to John Adams in a dream, and told him that as the Devil had fallen down on his part of the compact, it was void, and John could be saved. The next morning, the card was returned to him by college post, with 'Refer to Drawer' written across it in blood.

The Archimedeans

The Archimedeans have again had a very successful year, and attendance at the Evening Meetings has been good. The talk by Sir Richard van der Woolley, the Astronomer Royal, on 'Box Orbits of Stars in the Galaxy', and Professor J. F. C. Kingman's address on 'Change and Decay — variations on a theme of Croft (Eureka 1957)' were especially notable, as was also Professor P. H. Fowler's talk 'Very Heavy Elements in the Cosmic Radiation', which was accompanied by coloured slides of the launching of a balloon. Amongst the Tea Meetings, particular mention should be made of the talk '... To Preserve its Symmetrical Shape' given by Dr J. H. Conway, and Mr A. G. Smith's address on 'Looking for Stars by Computer', the stars of the title being polyhedra.

Members of the Invariants, the Oxford Mathematical Society, visited Cambridge in February to take part in the Problems Drive, which was again a success. The Computer Group has had a very active year, meeting most weeks to listen to talks or for discussion. The Puzzles and Games Ring has continued to thrive, and the Music and Bridge Groups have met regularly. The Tiddlywinks Match against the Dampers was again won by the Archimedeans, and although the Punty Party was held in traditional cloudy weather, it was nevertheless a success. There were visits in the Michaelmas Term to Oxford and Pye Telecommunications Ltd, and to 'Iolanthe' in the Lent Term. The Bookshop has continued to provide an excellent service to undergraduate mathematicians.

This year's Evening Meetings will include an address from Professor A. Fröhlich, who was prevented by illness from speaking to the Society in Feburary. His talk 'The Quadratic Reciprocity Law, and After' will take place on 10th October, and will be the first meeting of the academic year. There will be a Careers Meeting, similar to those held in previous years, and a visit to Oxford in the Michaelmas Term; the Invariants will again be challenged to take part in the Problems Drive to be held in February. A Tiddlywinks Match against the Dampers and a Punt Party to Grantchester will be arranged as usual.

It is hoped that all members of the Society will find something in the coming year's programme, but all suggestions as to possible changes in future years will be welcomed. These should be made either directly to the Secretary, or through the suggestions book kept in the Arts School.

S. J. TURNER, Secretary

Movement without Speed

by K. J. Evans

"Since you ask, it's been a wretched morning. What obscure lectures! Which is worse algebra or analysis? — and then like a fool I had to go and call on Big Jack Bill. He was racing his pig to market: 'Why this emphasis on reality?' he was saying, 'real vector spaces, real that, real this. Really it's a conspiracy to hide freedom: freedom from the bonds of plus and minus signs. There's a demo. tomorrow — proctors to be stopped by banners saying 'Equality of plus and minus signs! Equal wages for plus and minus signs'. Coming?'

I only wanted to humour him when I pointed out that an analyst would object without a metric.

'That's easy' he said. 'Let F be the field of characteristic 2 consisting of $\{0, 1\}$ and define d, a distance, as

$$d(0, 0) = d(1, 1) = 0, d(0, 1) = d(1, 0) = 1.'$$

'Very interesting' I said sarcastically.
'If you want to be like that, let $F[t]$ be the ring of polynomials over the transcendental t and define for all x, y, z belonging to $F[t]$

$$d(x, y) = |x + y|, |z| = \sum_{r=0}^{N} a_r \phi^r \text{ if } z = \sum_{r=0}^{N} a_r t^r \text{ where } \phi = (\sqrt{5} - 1)/2; \text{the } a_r\text{'s are}$$

either 0 or 1.
d forms a metric* such that for all x, y, x_i in $F[t]$,

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* Proofs omitted throughout.

lemma 1 $|x + y| \leqslant |x| + |y|$

lemma 2 $|xy| \leqslant |x|.|y|$

lemma 3 $x_i \to x$ as $i \to \infty$ implies $|x_i| \to |x|$ as $i \to \infty$

lemma 4 $|x| \leqslant 1/(1-\phi)$

'Having got F[t] we might as well make it metrically complete and so form $\hat{F}[t]$.

Any series like $\sum_{r=0}^{\infty} a_r t^r$, a_r in F, will be in $\hat{F}[t]$ and this is the standard form for any element of $\hat{F}[t]$.

'Define $Lt(x_n) + Lt(y_n) = Lt(x_n + y_n)$ and $Lt(x_n).Lt(y_n) = Lt(x_n.y_n)$. This is well defined and $\hat{F}[t]$ is still a ring in which the lemmas hold.

'It seems a shame not to be able to divide, so let's form the field of fractions of $\hat{F}[t]$ and call it $f\hat{F}[t]$. As the algorithm will show, the subring of $f\hat{F}[t]$ generated by $1/t$ and $\hat{F}[t]$ is in fact $f\hat{F}[t]$ itself.

Algorithm. It will be convenient to define A as a subset of $\hat{F}[t]$ whose standard forms begin with a 1 i.e. x is in A if and only if $x = 1 + \sum_{r=1}^{\infty} a_r t^r$. If y is in A, then $1/y$ is also in A and $1/y = P = Lt(P_r)$ where P_r is defined inductively as follows: $P_1 = 1$; if P_r is defined, then $yP_r - 1$ is a polynomial series whose lowest term is t^{s_r} say; define $P_{r+1} = P_r + t^{s_r}$.

Proof. We note that $s_{r+1} \geqslant s_r + 1$, so $s_r \to \infty$ as $r \to \infty$. Hence P is in standard form $1 + \sum_{r=1}^{\infty} t^{s_r}$ so that LtP_r exists.

Also $|yP_r - 1| \leqslant |\sum_{p=s_r}^{\infty} t^p| \leqslant \phi^{s_r}/(1 - \phi)$.

Therefore $Lt|yP_r - 1| = 0$ and $yP = 1$ as required.

'Perhaps an example will clarify this:-

If $y = 1 + t + t^2, P_1 = 1, P_2 = 1 + t, yP_2 - 1 = t^3, P_3 = 1 + t + t^3, yP_3 - 1 = t^4 + t^5$,

$P_4 = 1 + t + t^3 + t^4, yP_4 - 1 = t^6, P_5 = 1 + t + t^3 + t^4 + t^6, yP_5 - 1 = t^7 + t^8$,

$P_6 = 1 + t + t^3 + t^4 + t^6 + t^7$ etc.

Hence every element of $f\hat{F}[t]$, e.g. z, has the standard form $z = t^r y$ for some integer r and some y in A. The metric can be restored by defining $|z| = \phi^r |y|$ which is consistent, and the first three lemmas still hold.

Theorem. For elements of $f\hat{F}[t]$, if $x_i \to x, y_i \to y, z_i \to z$ ($z_i \neq 0, z \neq 0$) as $i \to \infty$, then $x_i + y_i \to x + y, x_i y_i \to xy, 1/z_i \to 1/z$ as $i \to \infty$.

Proof of (iii) $\left|\frac{1}{z_i} + \frac{1}{z}\right| = \left|\frac{z + z_i}{z_i z}\right| \leqslant \left|\frac{1}{z}\right| |z + z_i| \left|\frac{1}{z_i}\right|$

Now $z_i \to z$ implies that sooner or later the factor t^r which puts $t^r z_i$ in A is always the same (proof by reductio ad absurdum). For that value of r, for sufficiently large i,

$$\left|\frac{1}{z_i}\right| = \left|\frac{t^r}{t^r z_i}\right| \leqslant |t^r| \left|\frac{1}{t^r z_i}\right| \leqslant |t^r| \frac{1}{1 - \phi} \text{ since } \frac{1}{t^r z_i} \text{ belongs to A.}$$

Therefore $\left|\frac{1}{z_i} + \frac{1}{z}\right| \leqslant \left|\frac{1}{z}\right| \cdot |tr| \frac{1}{1-\phi} |z + z_i|$, which tends to zero as $i \to 0$ as required.

'Our analyst should be happy with that: $f\hat{F}[t]$ is a field with a metric happily blended with the field properties.

'We can start differentiating:

Non-Theorem: If a function f is such that $f'(x) = 0$ for all x if $f\hat{F}[t]$ (f maps $f\hat{F}[t]$ into itself) then $f(x)$ is constant.

Un-Proof: If $f(x) = x^2$, then $f'(x) = 0$. In general,

$$\frac{d}{dx} x^n = \begin{cases} 0 \text{ if n is an even integer} \\ x^{n-1} \text{ if n is an odd integer.'} \end{cases}$$

'And for integration?'† I asked.

Big Jack didn't hear. 'It's the algebraist that's going to kick up; try solving $x^2 + x + 1 = 0$,' he said. 'But ignoring the temptation to complete the square, we can sometimes solve

$$x^2 + bx + c = 0 \tag{1}$$

by $x = \frac{c}{b} + \frac{x^2}{b}$ (2)

Therefore $x^2 = \frac{c^2}{b^2} + \frac{x^4}{b^2}$ (3)

So, substituting from (1), $x = \frac{c}{b} + \frac{c^2}{b^3} + \frac{x^4}{b^3}$

Therefore, $x^2 = \frac{c^2}{b^2} + \frac{c^4}{b^6} + \frac{x^8}{b^6}$, and $x = \frac{c}{b} + \frac{c^2}{b^3} + \frac{c^4}{b^7} + \frac{x^8}{b^7}$ from (1) etc.

Hence, $x = \frac{c}{b} + \frac{c^2}{b^3} + \frac{c^4}{b^7} + \ldots$ if the remainder term tends to zero.'

I was glad to get away before he tried getting tied in knots trying to make up a 'Taylor's Theorem'. Would you like some coffee; I think I need it. But what does Taylor's Theorem look like in a field of characteristic two?"

† The difficulty is in getting a decent interval: for any distinct A and B in $f\hat{F}[t]$, there exists a positive ϵ such that if $x_o, x_1, \ldots\ldots, x_n$ is any sequence of points in $f\hat{F}[t]$ with $x_o = A, x_n = B$, then an r can be found such that $d(x_r, x_{r+1}) > \epsilon$.

Mathematical Association

22 Bloomsbury Square, London, W.C.1

President: Lady Jeffreys, M. A., Ph.D.

The Mathematical Association, which was founded in 1871 as the Association for the Improvement of Geometrical Teaching, aims not only at the promotion of its original object but at bringing within its purview all branches of elementary mathematics.

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The Mathematical Gazette is the journal of the Association. Published four times a year, it deals with mathematical topics of general interest. The present Editor is Dr. E. A. Maxwell.

Polyhedra

by A. G. Smith

The reference for this article is the paper 'Uniform Polyhedra', by Coxeter, Longuet-Higgins and Miller (all Trinity men, I believe) [1].

We shall use G(P) to denote the group of automorphisms of P; that is, the group of all orthogonal transformations of R^3 which leave fixed the set of points of P.

A polyhedron P is said to be uniform if

(i) two faces meet at each edge

(ii) each face is a regular polygon (not necessarily convex)

(iii) G(P) is transitive on the vertices of P.

Clearly condition (iii) implies that the vertices of P are congruent, but it is in fact stronger; for an example, due to Miller of a polyhedron satisfying (i), (ii) and having congruent vertices but not satisfying (iii), see [2], page 137. A uniform polyhedron is either connected or a compound. We observe that if Q is a compound uniform polyhedron, consisting of n copies of the uniform polyhedron P, G(Q) need not contain a subgroup G(P); indeed the subgroup of symmetries of P need not even be transitive on the vertices.

Examples: (i) There is a compound of 12 decagonal prisms, which has total symmetry group G (dodecahedron), but in which the subgroup of symmetries of a single prism is C_{10}, which cannot be transitive on a 20-element set.

(ii) The symmetry group of an individual cube in the wellknown compound of five cubes is a 24-element group called T_h, which is the direct product of C_2, generated by re-

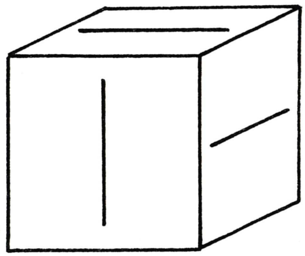


Fig. 1

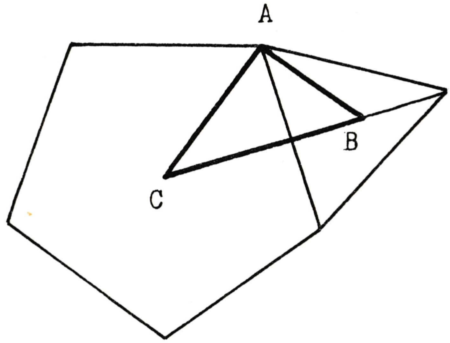


Fig. 2

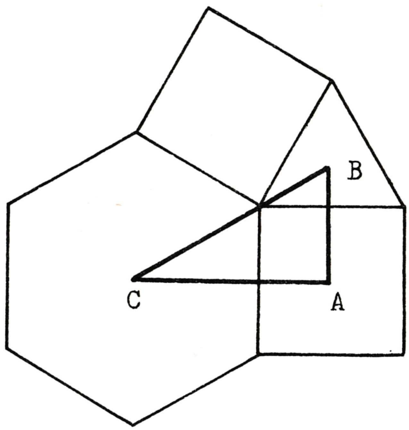


Fig. 3

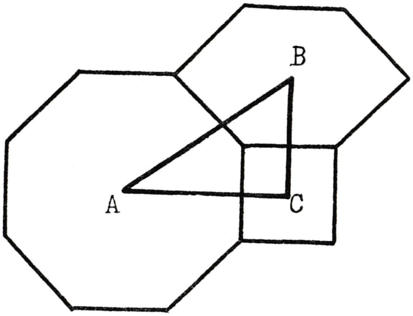


Fig. 4

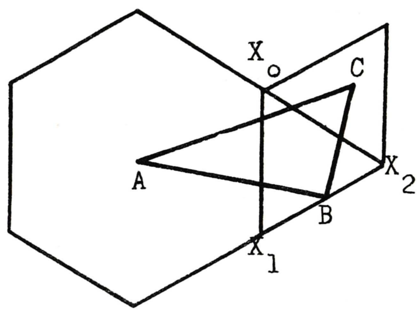


Fig. 5

flexion in the centre, with G (tetrahedron). This is in fact the group of symmetries of a cube marked as in fig. 1.

Compounds are not very interesting unless vertices coincide, for otherwise there are large families of essentially identical compounds; for example the 5 cubes give rise to 5 of anything with T_h as a transitive symmetry group, eg 5 octahedra, 5 stellae octangulae (alias 10 tetrahedra), 5 cuboctahedra, rhombicuboctahedra, or truncated cubes or octahedra. Since T_h does not act nicely on the snub cube or great rhombicuboctahedron, these do not form compounds of five in this way.

We thus define a uniform compound Q of n P's to be a uniform polyhedron which is a compound of n copies of the uniform polyhedron P, such that each Q-vertex contains more than one P-vertex.
Examples: the 12 decagonal prisms above, the 5 cubes, 10 tetrahedra, and compounds of 20 triangular, 12 pentagonal or 12 star-pentagonal prisms or 20 hexagonal prisms. All these prism-compounds were discovered by Mike Guy, and are not to my knowledge yet published; I rediscovered three of them for myself last December. Most of the cubic analogues degenerate, and the survivors do not seem to have double vertices.

It can easily be shown that the only possible rotation groups are those of prisms, antiprisms and the convex regular solids; see [3] for a proof. When we throw in reflexions as well, we get a collection of boring sequences of groups of symmetries of prisms and antiprisms, T_h, and the symmetry groups of the convex regular polyhedra. The list is given in gory detail, together with examples from molecular chemistry in [4].

For any given group, the problem of determining what polyhedra exist with the group in question is a large finite one. A general point has $|G|$ images under G, and we can choose which lines are to be edges, constrain them to be equal, and then plate in faces between the edges in finitely many ways. We can then see if the group of the polyhedron is G or larger.

The numbers involved are too large to allow this approach to be carried through; John Conway and Mike Guy did it for the small groups on Edsac, and showed that any unknown polyhedron must have group G (dodecahedron), 120 vertices, and something improbable like snub faces with at least 23 sides. (A snub face is one which has no axis of rotational symmetry through its centre).

So we would like a method of constructing polyhedra, even if it carries no guarantee that it gives them all. In fact it is not known whether all uniform polyhedra are known. The following method is essentially that of [1].

Consider where the axes of symmetry operations of a given group cut the surface of the sphere (all but the identity and the pure reflexion will have an axis). This is a finite collection of points on the surface of the sphere. Take three elements of the group which generate it. Since none of these can be the identity or a pure reflexion, there are corresponding points on the surface of the sphere, A,B,C, say. Suppose further that the product of the three operations is the identity; then evenly many are reflexional rotations, and there is a theorem to the effect that if the spherical triangle has angles π/a, π/b, π/c, respectively, the rotations are through $2\pi/a$, $2\pi/b$, and $2\pi/c$. If the group is one of the groups G (regular polyhedron) the images under the group of the triangle cover the sphere finitely many layers deep; otherwise there are gaps of the same shape as the triangle only reversed. Now in 1873 Schwarz determined all possible such triangles; his list is quoted in (1). Having chosen a Schwarz triangle, there are three obvious ways of getting a polyhedron:

(i) The images of a vertex, say A, give the vertices of a polyhedron whose faces are b-gons and c-gons, a of each at a vertex; an edge is given by joining A to its reflexion in BC. See fig 2.

(ii) The images of a point on BC give a polyhedron with b-gonal, c-gonal and 2a-gonal faces, (b, 2a, c, 2a) at a vertex. To make the edges equal we need the point to be on the bisector of angle A. See Fig 3.

(iii) The images of a point in the centre of the triangle give a polyhedron with 2a-, 2b-, and 2c-gonal faces; one of each at a vertex. We need the point to be the incentre to make the edges equal. See Fig 4.

Less obviously,

(iv) (ii) and (iii) degenerate if we need a 2t-gon, where t has even denominator (eg 5/2). Usually the polyhedron splits up into ones we've had already, but in (iii), if just one c, say of a, b and c has even denominator, we can fudge it. If a=b, we get a polyhedron that we have already had, but if a≠b we have to surround an odd-sided face with 2a's and 2b's alternately. This means going round twice; and so we throw away the 2c's in order to leave two faces at each edge. Incredibly enough, this works.

All the solids constructed so far have reflexional symmetry as well as rotational. For some solids with only rotational symmetry we proceed as follows:

(v) Take a point X_0 and its images under the group, which form a-gons about A, b-gons about B and c-gons about C. The triangle (X_0, X_1, X_2), where X_1 =the image of X_0 under the rotation about A, and X_2 that of X_1 under the rotation about B, need not be equilateral. The condition that it be so may be worked out, and solved to give us the snub polyhedra. See Fig 5. (Note that the image of X_2 under the rotation about C is X_0, since our three rotations are supposed to have product 1.)

There is no reason why this should not be carried out using a group with reflexions in it, but in fact the polyhedra so obtained all reduce to ones already obtained.

(vi) Similarly we can take four rotations with product 1 which generate the group, and get a polyhedron whose faces are a-, b-, c-, and d-gons and quadrilaterals. Solving for these to be squares, we get one new polyhedron (due to Miller) and three of Guy's uniform compounds. For all these a=c and b=d; moreover for Miller's solid A=C and B=D, and for the compounds A and C are antipodal and so are B and D. Presumably our geometric insight is incomplete, or this would be predictable.

If evenly many of the operations are reflexional we could carry out a similar procedure; I suspect that the remainder of Guy's compounds can be obtained in this way.

One could look for polyhedra with higher types of snub faces, or with several types of snub face, but, like the frontal assault on the problem, this runs into problems of computer space.

All the known uniform connected polyhedra are illustrated at the end of [1], and there is a partial collection in the Mathematical Gallery at the Imperial Science Museum in South Kensington.

References:

[1] Phil. Trans. Roy. Soc. London, Vol 246, Series A, 1954

[2] 'Mathematical Recreations and Essays' by Rouse Ball, revised by Coxeter

[3] 'Regular Polytopes' by Coxeter

[4] 'Group Theory and its Applications to Physical Problems' by Hammermesh

On Writing Text-Books

by **W. L. Ferrar**

The difficulties and dangers of writing mathematical text-books are many and varied; one has to be clear on so many points before one can even begin to follow a pattern and, human nature being what it is, glaring mistakes, omissions and over-elaborations invariably intrude to spoil that pattern as you try to work it out. Equally, if you succeed too well in adhering to a set pattern, then liveliness and personal touch diminish and the interest of the reader is lost.

Let me set out a few general observations, guide-lines to my own work, but so often transgressed in moments of carelessness or of absorption in some particular facet of the subject. These guide-lines were not laid down before writing my first book; rather, they came into place as experience, success and failure—those hard teachers of us all—forced them upon me.

The Wood and the Trees. In any one part of mathematics a professional mathematician knows so many details, tricks, turns and devices, pit-falls and snares for the unwary, that a real effort of analysis is needed to sort out the central themes which will, in their development, present the subject as a whole and carry the reader with them. To this day there are large tracts of mathematics in which I am familiar with many of the trees, but would be hard put to it to describe the wood. The first task of the writer is to resolve that he will try to display the shape of the wood, even though most of the detail will, in fact, be concerned with particular groups of trees.

The Subject and the Audience. The subject and the audience must be appropriate, each to each. Analysis all but died in the schools in the 1930's because the minds of young people of 17 were rarely prepared to receive the full rigour of the details that had so enchanted their teachers when they themselves had been undergraduates, and aged 20 or more.

In my own experience choice of subject and of audience for that subject have been simultaneous. I wrote my first book Convergence, because as a college tutor (I think the Cambridge word is 'supervisor') I was convinced that most undergraduates needed something aimed especially at their level of attainment and range of study. The choice having been made, the writer's task is firmly to adhere to his choice of audience. It sounds easy, but it is not. For the task of a text-book is to teach; at every stage the presentation must be appropriate to the reader and one is constantly forced to steer between the Scylla of a too elementary, even condescending approach and the Charybdis of a too advanced, monographic display of the intricacies and niceties of the subject.

It is not enough to set down the truth; it must be set in words that can be seen to be the truth—and by the people to whom those words are addressed. Perhaps this problem of level of understanding and grasp is a little illustrated by a comment from Professor W. H. Young—a notable mathematician of his day. I have forgotten the exact words, but they ran something like this. 'Differential Calculus is a subject that I understood perfectly when I was a boy of 16; but now, as a man of 60, I still have doubts about whether I have taken all the points at issue.' The author must decide whether he is writing for the boy of 16 or a young man of 20, or whether he is writing for the mature mathematician.

Liability to Error. The most effective way that I have found to remove errors, and I am very liable to make them, is to read and re-read what I have written; and to read every chapter, once at least, with the set purpose of misunderstanding it if I can. And even so the errors survive.

There are two sources of error that I have come to fear like the plague. The first is 'writing what I know' and failing to 'think the thing out' as I write; one acquires so many bits of knowledge that can be written down with complete sense to oneself but are valid only because of some vital thought that is so familiar that one omits to set it out explicitly. The reader either fails to follow you or is forced to take enormous pains to discover what thought you had in mind in arriving at your conclusion.

The second is 'last minute changes'. Twice at least in my career I have fallen into this trap. In the final reading of the manuscript I have suddenly decided 'that way of putting it is too severe and complicated for the present stage' and have altered it to make the matter more easily acceptable; only to find when the book has been finally and irrevocably printed that the join-up is faulty and some detail of definition or logic has disappeared with the discarded original text.

The Fascination of Book-writing. As a young don I eschewed all administration and concentrated on teaching and research. Later, by accident rather than design, I undertook a heavy administrative task and, having written one book, I soon found (a) that I itched to write another and (b) that research and demanding administration are poor bed-fellows. As the years went by I found my mathematical pleasure more and more in the writing of books and in wrestling with the problem of presentation of mathematics to some particular level of audience, be it sixth forms, undergraduates, scientists having some mathematics or, my last essay, economists having but little. It has not the thrill of research, but it has its own particular brand of challenge and satisfaction.

The Reading of Books. May I end with a word or warning and encouragement to young mathematicians. Do not accept everything you read. Be ready to quarrel with your author. Learning mathematics is not the meek acceptance of other people's doctrines and procedures, but the development of your own thinking and practice. Some things have to be known, and known unerringly without the need of references; but, for the rest, your mathematics should be your own thinking , based on your understanding of what you have acquired by first listening to other people; and then, perhaps, at the research stage, disregarding other people altogether.

Problems Drive 1969

Set by R. G. Newcombe & K. Loveys

(1) On a set of South Sea islands live three basic races of people, the Whites, Reds and Pinks. Whites always tell the truth, Reds always lie, whilst Pinks tell the truth and lie alternately.

On one of these islands live only Whites and Reds. I meet two of the natives, A and B, of whom only A speaks English. How can I find, in only one question,

(a) what colour A is

(b) what colour B is?

On another of the islands live three species of Pinks, the Roses, Corals and Salmons. To a pair of questions, a Rose will tell the truth to the first and lie to the second; a Coral will lie then tell the truth; and a Salmon will decide at random to which question he will lie. I meet one native.

(c) What two questions must I ask him to determine his tribe?

N.B. Most marks will be given to briefest simplest answers; all questions must call for yes/no answers, and must concern only tribes, so that such questions as 'Are you alive?' are inadmissible.

(2) Complete the following sequences by giving the next two terms:
(a) 19 14 1 5 4 5 13
(b) 15 20 20 6 6 19 19
(c) 2 3 7 1 1 4 8 1 2 5
(d) 2 3 5 11 17 37 67

(3) Complete the following sequences by giving the next two terms:
(a) MYOANDDSAE
(b) HHELIBEB
(c) BCEGAAACAGAI
(d) YYHLYEY

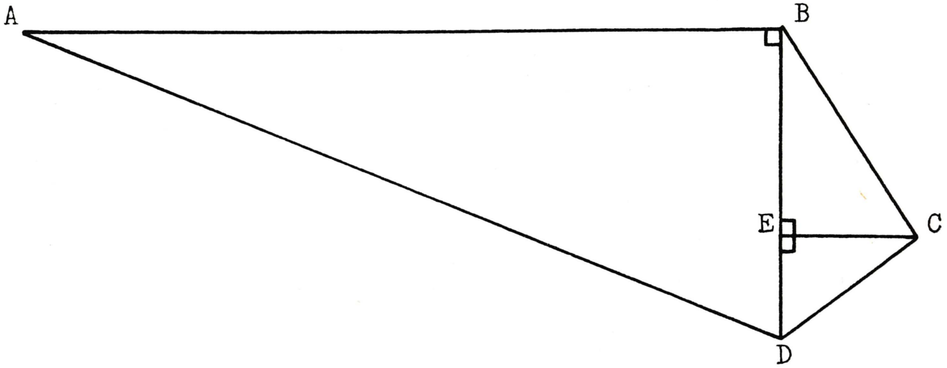
(4) In the game of Eleusis, one player invents a rule which determines, for each card in the pack, which cards are allowed as the next card to be played. E.g. 'Only black cards may be played on red, and only reds may be played on black'. This could produce the sequence:

3D 7S QD 10C 6H KS .....

Find a simple rule to produce the sequences:

(a) AS 8D 6D 2H 3H JH 7S 4C KC AC 5C 5D 10H JS QC 5H 7C AD
(b) AS 4H 9D JC 2S 7S 6D 8H AC 4S 5D 10C

(5) In the following figure, no pair of ABD, BEC, CED are congruent. All lines in the figure have integer length, and AD = 13BC. Find BA/CD



(6) A soldier faces an enemy cannon in battle. He may take refuge in any one of the five foxholes marked 1-5. The cannon may shoot at A,B,C, or D.

1 A 2 B 3 C 4 D 5

The soldier is killed if his foxhole is adjacent to the target chosen, e.g. if the shot hits C the soldier is killed if and only if he is in 3 or 4. How should the soldier choose his foxhole, and what is the probability that he will be killed by the first shot, assuming the gunner is a games theorist?

(7) 1 2 3 4 5 6 7 8 9 =100

Find as many ways as you can of inserting symbols in the gaps between the digits such that the resulting equation will be true. Further digits and unknowns must not be introduced, e.g. the manufacture of an extra 2 as $(x+x)/x$ is inadmissible. Consecutive digits may be used as the one number, e.g. 234.

(8) The sum of the ages of Mary and Ann is 44 years, and Mary is twice as old as Ann was when Mary was half as old as Ann will be when Ann is three times as old as Mary was when Mary was three times as old as Ann. How old is Mary?

(9) Four married couples went into town one day to do some shopping. They had to be economical, for among them they had only forty shilling coins. Ann spent 1/-, Mary spent 2/-, Jane spent 3/-, and Kate spent 4/-. Ned Smith spent as much as his wife, Tom Brown twice as much as his, Bill Jones three times as much as his and Jack Robinson four times as much as his. On their way home someone suggested they should divide the cash that remained equally among themselves; this was done. Give the full name of each woman.

Answers on page 40.

Book Reviews

THE THEORY OF GROUPS. By Ian D. Macdonald. Clarendon Press: Oxford University Press, 45s. hardback, 22s. 6d. paperback.

Why do you want a book on Group Theory? Ian Macdonald has written one which contains enough material in sufficient depth to keep the most enquiring mind busy for several months of spare-time study. However, to read what he has written is a serious undertaking. While ambiguities and omissions of logic are rare the book contains little redundancy and relies to a large extent on a single logical structure rather than being written in self-contained chapters. For this reason it is difficult to use it as a reference work without first reading every chapter, with a few exceptions, in the correct order.

For the serious reader there are many questions set on the text. Worked examples are rare except at the beginning of the book where most of the theorems proved are illustrated with reference to the symmetric group S_3. There is also a final chapter of remarks of a more specific nature than that of the theorems proved previously. However the book goes into much greater depth than that required in Maths. Pt. IA and I doubt if it is suitable as a text for any Algebra syllabus. It certainly does cover the whole of the Algebra I course from scratch, possibly in unfamiliar notation.

Roughly the topics covered are: finite and infinite groups and subgroups with specific chapters devoted to Normal Structure, Sylow Theorems, generators, Abelian groups, nilpotent groups and soluble groups. A. F. CHAMPERNOWNE.

SOLVING PROBLEMS IN VECTOR ALGEBRA. By E. M. Patterson. (Oliver & Boyd) 17s. 6d. (No. 2 in the series 'Solving Problems in Mathematics')

In keeping with the stated aim of this new series to present mathematics from a practical point of view, this book uses an intuitive geometrical approach to vector algebra.

Unfortunately this tends partially to obscure the algebraic structure of the subject, and I would say that the book would be primarily of use to the engineer or scientist rather than to the pure mathematician. All the same the approach is still quite systematic, and takes us through the concepts of bases, linear independence, scalar and vector products, and coordinate systems, and gives applications to solid geometry and mechanics. There are plenty of worked examples and questions with solutions. The diagrams, however, are poor and it is not easy to see the three-dimensionality. An index is a startling omission. Another criticism is that the book will not lie open flat very easily, whereas the other Oliver & Boyd paperbacks, which are cheaper, will. The layout is clear and the proof-reading nearly excellent. I would recommend the book to anyone who finds vector algebra difficult. G. LEVERSHA.

MATRIX THEORY. By J. N. Franklin. (Prentice-Hall). £5. 2s. 6d.

As Professor Franklin points out in his preface, matrix theory has applications in many fields—mathematical economics, quantum physics, geophysics, electrical network synthesis, crystallography and structural engineering, to name but a few. What is more he has produced a book that would be comprehensible to workers in any of these fields, but at the same time maintains a standard of rigour that would satisfy most pure mathematicians.

There are two approaches to matrices—the 'practical' one found in most books specifically on matrices, and the 'abstract' one to be found in the algebra courses of the Mathematical Tripos, which produces matrix results as by-products of more general vector-space results. Professor Franklin naturally chooses the former approach, but nevertheless manages to put his basic theory firmly in the context of vector spaces.

The book deals first with the basic properties of matrices and determinants, using theory of linear equations as motivation. It then goes on to discuss matrix analysis of differential equations, to lead to a discussion of eigenvalues and canonical forms, culminating in the proof of Jordan's theorem. There then follows an interesting and useful chapter on variational principles and perturbation theory. The final chapter deals with numerical methods for solving matrix problems.

This is a well written and well motivated book, although it perhaps delves rather deeper into fundamentals than some of its potential users might require. Nevertheless anyone who is faced with dealing with matrices on a digital computer would probably find it worth having just for the numerical methods chapter. J. J. BARRETT.

POLYOMINOES. By S. W. Golomb (George Allen & Unwin) 30s.

A polyominoe is a 'generalised' dominoe, i. e. a shape made up of a number of connected squares. For instance a pentominoe is made up of five squares. But while two squares can be juxta-posed in only one way to form a dominoe, five squares can be arranged in twelve ways, to form the twelve different pentominoes.

The book is concerned with the theory and application (to mathematical recreation) of polyominoes, confining itself mainly to pentominoes and tetrominoes, which manage to be not too trivial but not too complicated. It deals in less detail with higher dimensional polyominoes e. g. solid polyominoes, and with triangular and hexagonal animals.

A novel feature of the book is the free set of plastic pentominoes that comes with it. These provide a good deal of amusement on their own as well as illustrating the problem with which the book deals.

An amusing book, probably well worth its price for those with a bent for mathematical games. J. J. BARRETT.

THEORY OF OSCILLATORS. By A. A. Andronor, A. A. Vitt and S. E. Khaikin. (Pergamon). £10.

This is a new English edition of a work first published over 25 years ago. Although intended as a graduate reference amnual, it develops the subject completely from first principles—the first fifty pages deal with harmonic oscillators and elementary extensions, introducing the fundamental phase plane technique. This technique represents the behaviour of a system specified by two equations

$\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = g(x,y)$ by a plot of y against x; the system moves along curves satisfying $\frac{dy}{dx} = \frac{g}{f}$.

To the reviewer (a mathematician) the most interesting part of the book was the chapter discussing the topological phase—plane configurations possible. Even here, however, this book is unlike many Russian works in that it demands little mathematical knowledge on the part of the reader. The latter half of the book uses the general theorems previously set up to discuss a great variety of specific control systems; a selection at random includes the two-position automatic pilot with delay; approximately sinusoidal oscillations by Van der Pol's method, and several types of multivibrator circuit.

The reader should be warned that as the work first appeared so long ago, all the circuits involve valves rather than transistors.

The book is undoubtedly an extremely valuable reference work for those studying control engineering. However, it is unlikely that many will be able to afford the extremely high price demanded.

Such a work as this deserves to reach a far wider readership; if Pergamon published it as three or four volumes of their admirable new paperback series it would do so.

C. J. MYERSCOUGH.

TOPICS IN MATHEMATICAL SYSTEM THEORY. By R. E. Kalman, P. L. Falb and M. A. Arbib (McGraw-Hill). £7. 14s.

This book covers a wide and growing field which has not so far been covered in the Tripos. It may be helpful to give a simple example of the type of problem under consideration. Suppose that an oven has to be cooled by ten stages from an initial temperature x_0. We suppose that at each stage we can control the amount u_n by which its temperature falls so that the temperature x_{n+1} at time $n+1$ is given by $x_{n+1} = x_n - u_n$, but that large drops in temperature are costly, and yet on the other hand we wish to reach the desired temperature as soon as possible. It might then be that the cost of the operation could be represented by

$$C(u_0, \ldots, u_9) = \sum_{n=0}^{9} (au_n^2 + bx_n^2) + cx_{10}^2.$$

It is then of interest to find the optimal method of controlling the drop in temperature.

The authors of this book attempt to give a rigorous mathematical account of the general theory of problems of the above type. It would stand up well to a critical examination by a pure mathematician, but as is usual with books on applied mathematics, to say this is to pay a rather back-handed compliment. It is very short of motivation and almost devoid of explicit examples. On the other hand, the extent of the confusion existing in some other books means that it is useful to have this book available for reference, although few will read it from cover to cover.

It is divided into four sections; a compact treatment of elementary control theory, then an account of optimal control and filter theory within the framework of functional analysis, then a section on modern automata theory, and finally a fundamental treatment of linear system theory.

In conclusion, if you can think of the topics covered by the book as parts of pure mathematics, then you will probably like this book. P. M. LEE.

STUDIES IN GLOBAL GEOMETRY AND ANALYSIS edited by S. S. Chern (The Mathematical Association of America)

Six papers, each giving a stimulating introduction to a facet of this fascinating branch of mathematics.

What is Analysis in the Large? (M. Morse) Answered briefly by a few examples showing the interplay of 'geometric' topology and analysis, including variational problems and the qualitative study of differential equations. Of general interest.

Curves and Surfaces in Euclidean Space (S. S. Chern) A selection of intriguing geometric theorems. The notation occasionally obscures the presentation.

On Conjugate and Cut Loci (S. Kobayashi) A paper comparing the similar (but distinct) concepts of conjugate and cut points on geodesics in Riemannian manifolds. A geodesic is a path which locally minimises arc length. If G is a geodesic starting at P, the cut point is the first point Q on G such that G is no longer the unique shortest path between Q and P. The definition of conjugate points is more technical.

The remaining papers are condensed versions of the respective books with (almost) the same titles.

Differential Forms (H. Flanders) Mainly clear. An intuitive rather than rigorous approach. Very useful to Part II differential geometers.

Surface Area (L. Cesari) A study of the problem of assigning an 'area' to e. g. a (continuous) map of a subset of the plane into Euclidean 3-space.
Integral Geometry (L. A. Santalo) Consider a group G acting on a set X and suppose we are interested in a set Y of subsets of X, with G acting transitively on Y. Integral geometry looks at G-invariant measures on Y. This paper discusses the existence and uniqueness of such measures for subgroups of GL(n), with examples including X = the plane, Y = the set of all lines in X, G = Euclidean motions.

The book is mostly at Part II level. Each paper has a comprehensive bibliography, so the book makes good introductory reading in this subject. P. E. JUPP.

ELEMENTS OF THE THEORY OF FUNCTIONS. By R. S. Guter, L. D. Kudryavtsef & B. M. Levitan. (Pergamon). £2. 15s.

The title of this might suggest that it is a general introduction to the theory of functions. In fact, it is an attempt to bring to general attention some of the recent advances in particular branches of the subject. The necessary analytic tools are introduced in the first chapter; there follow two sections on approximations to functions and on almost periodic functions respectively. The treatment of these is both clear and extensive but mathematically leaves much to be desired. The book is notable for a total absence of proofs, which must surely be aggravating to the mathematician. It is unlikely that the nonmathematician will find it anything but incomprehensible. There are many gaps in the available mathematical literature but this book does little to alleviate this situation. B. R. GOALBY.

Solutions to Problems

An Alphametric (page 1)

The complete problem (noting that Eureka must not represent an even number) is solved uniquely by:

$\overline{750 + 8 +}\ 69185 + 69185 = 139128$

Problems Drive (page 34)

(1) (a) If I could ask B what he was, would he say White? (The answer 'yes' means that A is white, the answer 'no' implies A is red.)

(b) Are you and B in the same tribe? ('Yes' implies B white, 'no' implies B red.)

(c) (i) Are you a Rose or a Coral? (ii) Are you a Salmon? (Yes, yes' implies Rose, 'no, no' implies Coral, 'yes, no' or 'no, yes' implies Salmon.)

(2) (a) 9, 18. (Archimedeans backwards)
(b) 5, 14. (Initials of digits)
(c) 8, 9. (e and π interlaced)
(d) 131, 257. (Primes next above 2^n)

(3) (a) Y, U. (Monday, Tuesday)
(b) C, N. (Periodic Table)
(c) B, C. (Decimal Primes)
(d) T, R. (Last letters of names of months)

(4) (a) Club requires 1 (mod 4), diamond 2 (mod 4), heart 3 (mod 4), and spade 0 (mod 4).
(b) Low (2-7) on Black; high (8-A) on Red.

(5) 22. The complete diagram has AB = 220, BC = 17, CD = 10, CE = 8, BE = 15, DE = 6, DA = 221.

(6) Choose between 1, 3 and 5, each with equal probability, at random. Probability is $\frac{1}{3}$. (2 and 4 are useless.)

(7) $1+2+3+4+5+6+7+(8\times 9)$.
$-(1\times 2)-3-4-5+(6\times 7)+(8\times 9)$.
$1+(2\times 3)+(4\times 5)-6+7+(8\times 9)$.
$(1+2-3-4)\ (5-6-7-8-9)$.
$1+(2\times 3)+4+5+67+8+9$.
$(1\times 2)+34+56+7-8+9$.
$12+3-4+5+67+8+9$.
$123-4-5-6-7+8-9$.
$123+4-5+67-89$.
$123+45-67+8-9$.
$123-45-67+89$.

(8) Mary $27\frac{1}{2}$, Ann $16\frac{1}{2}$.

(9) Ann Jones, Mary Robinson, Jane Smith, Kate Brown.

The (joint) winning pairs were G. Foley (Clare) and A. Solomon (Fitzwilliam), and M. K. Harper and Davey (Invariants) who scored $33\frac{1}{3}$%.

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