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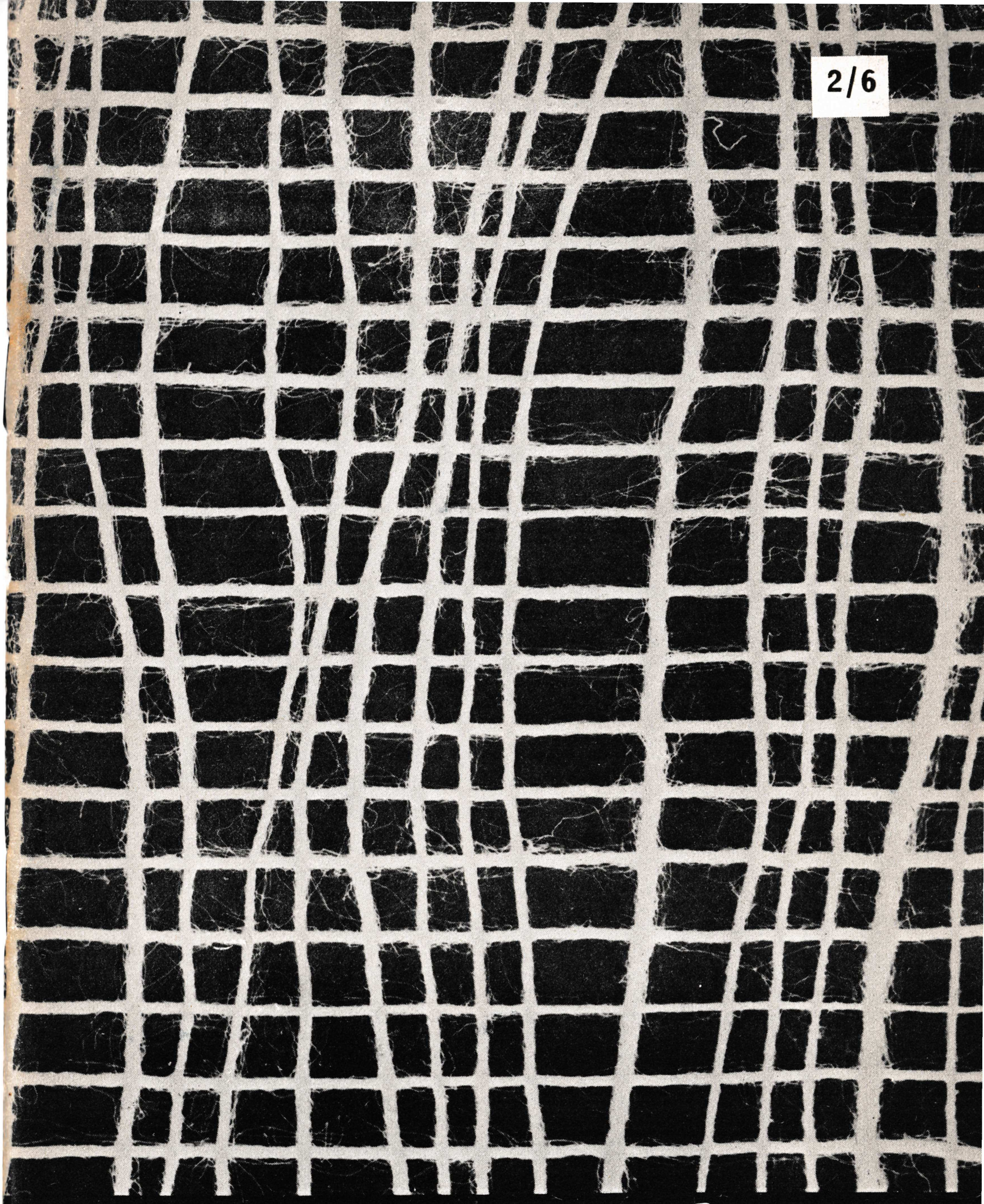

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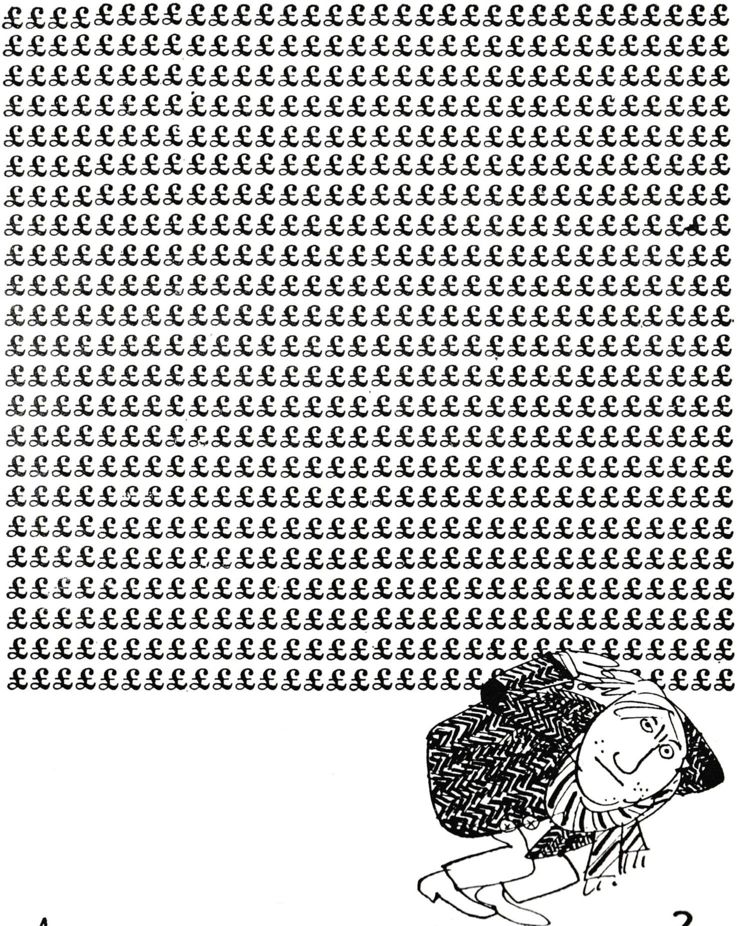

OCTOBER 1966

C O N T E N T S

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THE ARCHIMEDEANS' JOURNAL

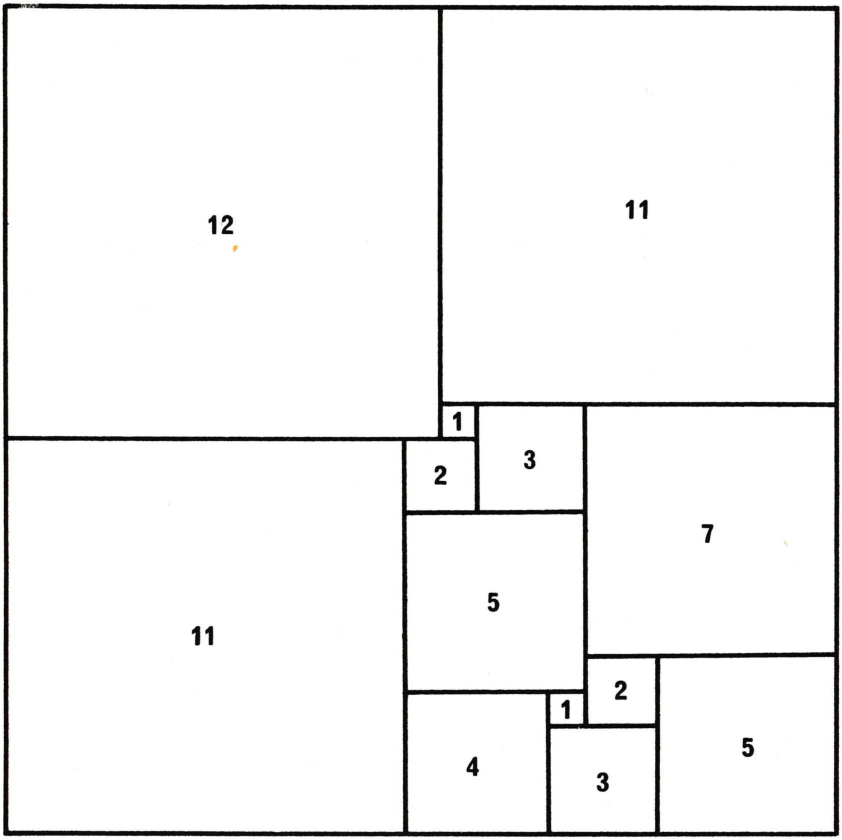
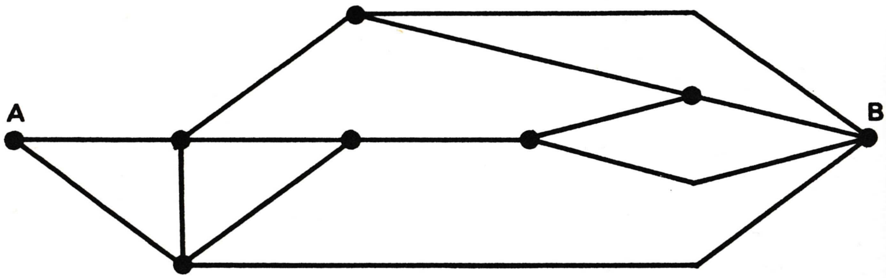


Figure 1, above, shows the perfect dissection of a square of side 23 into smaller squares. Figure 2, below, shows one of the corresponding electrical resistance networks. If each wire has resistance 1 unit, then the resistance between A and B is also 1 unit. (See the review of Mesichowski's book on page 35)



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EDITORIAL

The history of science over the last few decades has been remarkable. Many nations have vastly increased their support of scientific research in order to quicken their technological progress; the name 'scientific revolution' has been given to the effects of the resulting discoveries on everyday life. There has been less discussion, however, of the rapid increase in complexity of the mathematical techniques used; within the scientific revolution has occurred a mathematical revolution.

At the beginning of the 1930's work in most fields of science required only elementary mathematics, but now the situation is quite different. Sophisticated mathematical methods are becoming indispensable in any scientific investigation. Physics, economics, and much of chemistry are now almost entirely branches of applied mathematics. Modern engineering demands a high degree of mathematical aptitude. The application of mathematical techniques to industrial processes has resulted in the new disciplines of cybernetics and operational research. Even in some of the 'humanities' advanced statistical analysis is almost commonplace.

There are two principal reasons for these developments. The first is the introduction of new theories to explain more refined experimental observations. The second is the advent of the electronic computer, with its ability to process quickly enormous quantities of data. Both of these allow the study of more complicated systems.

It is difficult to avoid the conclusion that the applied mathematician will be the nearest approach to a 'universal man' that the future will be able to offer. It will be fairly easy for him to learn enough economics, anthropology, molecular biology, or any other subject to apply his techniques, whilst it will be very difficult for specialists in these subjects to work on others. The mathematician will be able to reverse the fragmentation of scientific effort, and to obtain results on the shadowy boundaries between disciplines. He may break down the barriers between the 'Two Cultures', and between pure science and technology.

How should all this affect the education of mathematicians? Obviously, emphasis should be laid on the development of powerful techniques rather than on the solution of particular problems. Only the Russians have grasped this point; the work of Sobolev and others has given them the lead in the study of partial differential equations, variational methods, and control theory. They avoid any rigid division into 'pure' and 'applied' mathematics; such a division seems to impede progress in this field.

Should there not also be a wider range of courses on applications, at present mainly confined to mathematical physics? There is scope for economics, theoretical engineering, operational research, and much else. How can repetition of elementary electrostatics and the extensive study of gyrostats occupy so much time in many university courses, when in his 'Lectures on Physics' Feynman has shown how quantum mechanical analogues of many dynamical problems can be considered without advanced mathematical techniques. Should not courses be more flexible, so that students might cover the ground at varying speeds? The need for some wider study is pressing; might not the Mathematical Tripos examinations include a general paper, as do those for the Natural Sciences Tripos?

We are living in an age in which the dependence of the human race on its mathematicians for scientific and technical progress is rapidly increasing. Mathematicians must be ready for this responsibility.

The last three editions of EUREKA were produced at a financial loss, mainly because of rising letterpress printing costs. After examining several tenders, we decided that to continue using letterpress would necessitate a large price increase. UNEOPRINT, a process based on offset-lithography developed by Messrs. Unwin Brothers Limited, The Gresham Press, Old Woking, Surrey, is about 25% cheaper than letterpress. The master copy is prepared on an electric typewriter, using special devices to type mathematical symbols. This method has great advantages for the reproduction of illustrations. We have seized the opportunity to modernize the cover design and to adopt an International Standard paper size.

To finance a larger number of pages we have increased the price to postal subscribers. Our rates are now

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Cheques and postal orders should be made payable to 'The Business Manager, EUREKA', whose address is The Arts School, Bene't St., Cambridge, England.

It is well-known that some members of the Department of Applied Mathematics and Theoretical Physics tremble at the suggestion that problems of convergence should be treated rigorously. This trembling is known as DAMTP oscillation.

—a supporter of the Deification, Purification, and Mummification of Mathematics Society.

Double Doppler

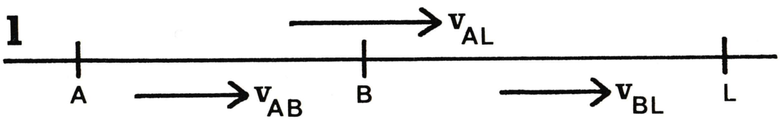
by Prof. W.H. McCREA, F.R.S.

Director, Research Group in Theoretical Astronomy,

University of Sussex

This note deals with the composition of optical Doppler effects, mainly in special relativity. The law of composition is simple, but is probably not familiar; there is some interest in approaching it, as here, from a postulated 'Doppler principle'. An application with certain interesting features is to reflection at a moving mirror. A particular velocity-transformation puts all the results into a specially simple form.

The Doppler principle. Let l be a straight line fixed in an inertial frame, and consider cases of free motion along l only. Relative velocities of pairs of objects moving in this way are supposed measured in some well-specified manner, and we suppose the free motion to be such that these are uniform in time. Also for any pair A, B we assume that the velocity of A relative to B is equal to the velocity of B relative to A. We count this velocity positive in the sense of mutual recession; this sign convention should be carefully noted. We write $v_{AB} = v_{BA}$ for this velocity.



Let A, B, L be objects moving in the manner contemplated. Suppose B is between A and L for all relevant events, in the sense that in any case of interest, a light-signal from L to A, or A to L, may be supposed to encounter B on the way. In particular, if A, L are at relative rest, we can agree that according to any acceptable definition of relative velocity the velocities of B relative to A and L are equal and opposite.

Now suppose L to be a source of monochromatic light of intrinsic frequency ν_L. Let A, B be observers who observe this light as having frequencies ν_A, ν_B respectively.

We postulate that the relative velocity of a light source and an observer being given, the observed frequency is proportional to the emitted frequency, the factor of proportionality depending only on the velocity. We call this the Doppler principle.

In symbols the principle asserts that a function $Z(v_{AL})$ exists such that $\nu_A = \nu_L/Z(v_{AL})$ (1) for all admissible values of v_{AL}, where the function Z depends only upon the way in which relative velocity is defined. We call Z the Doppler factor.

Properties of the Doppler factor. Two properties follow trivially. Taking the particular case A = L, we see that $Z(0) = 1$ (2) Also, from the stated properties of relative velocity, the principle shows that the Doppler factor for light emitted by B and observed by A is the same as that for light emitted by A and observed by B.

Now observer A can regard the light he sees as being emitted by L with frequency ν_L, or by B with frequency ν_B. Applying the postulate to A, we have therefore $\nu_A = \nu_L/Z(v_{AL}) = \nu_B/Z(v_{AB})$. Applying it to B, we have $\nu_B = \nu_L/Z(v_{BL})$. Combining these formulae, we obtain $Z(v_{AL}) = Z(v_{AB})Z(v_{BL})$ (3) for all possible values of v_{AB}, etc. This is the law of composition of Doppler factors.

Taking the particular case $v_{AL} = 0$ for which, as we have seen, $v_{AB} = -v_{BL} = v$, say, from (2) and (3), $Z(v)Z(-v) = 1$ (4) for all admissible values of v.

Classical physics. In classical physics, the law of composition of velocities gives in our case $v_{AL} = v_{AB} + v_{BL}$. The Doppler factor might be quoted as $Z(v) = 1 + v/c$, where c is the speed of light. Clearly these two relations are not compatible with (3) (although it is worth noticing that there is agreement apart from terms in $1/c^2$). The reason for the disagreement is that if c is the speed of light relative to one observer, according to classical kinematics it is not the speed of light relative to a differently-moving observer. It is easy to write down the classical counterpart of (3), but it involves different functions Z for different observers. Thus the postulate embodied in (3) requires something other than classical physics.

Special relativity. In special relativity, for the present case of motion in a single line and with the sign-convention adopted, the law of composition of velocities is

$$v_{AL} = \frac{v_{AB} + v_{BL}}{1 + v_{AB}v_{BL}/c^2} \tag{5}$$

The Doppler factor is

$$Z(v) = \left[\frac{c + v}{c - v}\right]^{1/2} \tag{6}$$

where the positive square root is taken; c is the universal speed of light. We see at once that (6) agrees with (2) and (4).

Using (6), the law (3) becomes

$$\frac{c + v_{AL}}{c - v_{AL}} = \frac{c + v_{AB}}{c - v_{AB}} \cdot \frac{c + v_{BL}}{c - v_{BL}} \tag{7}$$

It is easily seen that (5) and (7) are equivalent, so the law (3) in this case follows from the law (5), and conversely. Thus the postulate embodied in (3) is entirely compatible with special relativity. The composition of Doppler factors expressed by (3) is indeed a very simple feature of special relativity, but I have not seen it mentioned in any account of the subject.

Reflection. Here we take reflection of light to mean reflection by a mirror orthogonal to ℓ and moving freely along ℓ. We assume that radiation falling upon such a mirror is reversed in direction, and that its frequency is unchanged as seen by an observer moving with the mirror.

We infer that if A emits light of frequency ν which is reflected at L then B observes it as light of frequency $Z(v_{AL})Z(v_{BL})\nu$, (8) the same result holding good if the light is emitted by B and observed by A.

Image in a moving mirror. We again use special relativity. Putting $v_{AL} = v_{BL} = v$, say, in (8) and using (6), we conclude that if an observer sees himself reflected in a mirror receding from him with speed v, he sees himself to be 'red-shifted' by a factor

$$[Z(v)]^2 \equiv \frac{c + v}{c - v} \tag{9}$$

Setting this equal to $\left[\frac{c + V}{c - V}\right]^{1/2}$, we get $V = \frac{2v}{1 + v^2/c^2}$ (10)

Thus A would infer that his image is receding from himself with speed V given by (10). (Were A to suppose his image to be an equal distance on the other side of the mirror, he would infer a speed of recession 2v, and this would be in error again in the term in $1/c^2$.)

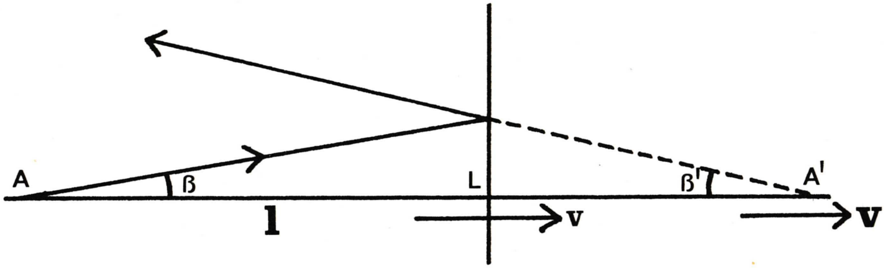
It is instructive to re-derive (9) in two other ways. First, let t be time reckoned by A in such a way that L moved past A at $t = 0$. Then if A emits a photon at $t = t_A$, it reaches L at $t = t_L$ where $c(t_L - t_A) = vt_L$ The photon, after reflection at L, returns to A at $t = t_A^*$, say, where $t_A^* - t_A = 2(t_L - t_A)$, and so $(c - v)t_A^* = (c + v)t_A$ (11)

The change of frequency is clearly such that

$$\frac{\text{frequency of emitted light}}{\text{frequency of reflected light}} = \frac{dt_A^*}{dt_A}$$

Using (11), this does agree with (9)

Secondly, as judged by A, let a photon travelling from A at angle β with ℓ be reflected at L and so become a photon travelling at angle β' with ℓ, as shown



Then according to a well-known formula, to the first order in β, β'

$$\frac{\beta}{c - v} = \frac{\beta'}{c + v} \tag{12}$$

It is sufficient to work to this order because we are finally interested in the limit when $\beta \to 0$, and so also $\beta' \to 0$. As before, let the photon return to the vicinity of A at $t = t_A^*$, then in figure 2 the point A' is the image of A as seen by A at t_A^*. Noting that $AL = vt_L$ and using (12) we have

$$AA' = \frac{2vt_A^*}{(1 + v/c)^2}$$

Now if an object leaves A at $t = 0$ and recedes at speed V, then A will observe it at time t_A^* to be at distance X, say, given by

$$X = \frac{Vt_A^*}{1 + V/c}$$

We then have $X = AA'$ if $V = 2v/(1 + v^2/c^2)$, again in agreement with (10).

Thus we have verified that the Doppler effect in a moving mirror got by a simple application of the composition law agrees with that obtained by direct calculation of the Doppler effect using (11), and agrees with the speed of recession of the image inferred from its apparent position.

Velocity transformation. The foregoing treatment may make the physics as simple as possible, but the mathematics becomes slight, in a perhaps elegant way, after a transformation. We write $v = c \tanh \gamma$, inserting suffices as required. Then (5), (6) become

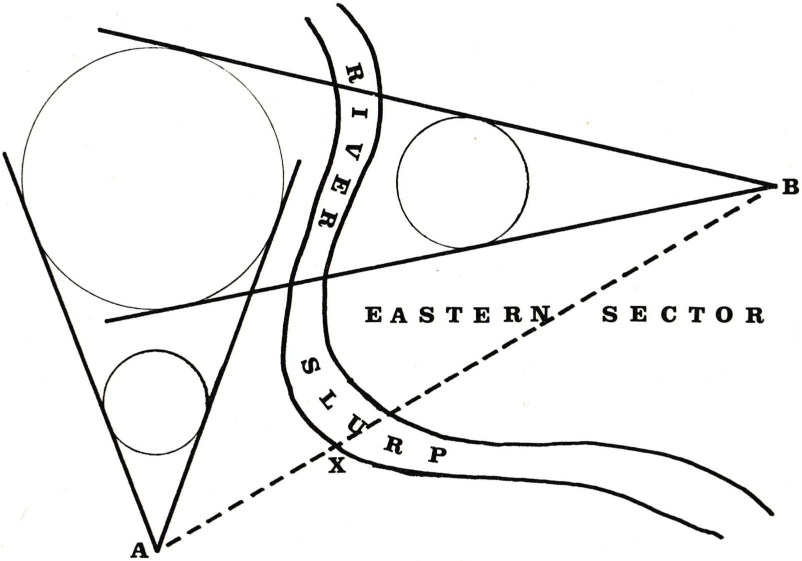
$$\gamma_{AL} = \gamma_{AB} + \gamma_{BL}, \quad Z(v) = \exp \gamma$$

Thus the law of composition of Doppler factors becomes obvious.

From (10) we see that if γ is the 'velocity' of the mirror and Γ the 'velocity' of the image, then we have the satisfactory property $\Gamma = 2\gamma$, and the Doppler factor (9) for the image becomes $\exp 2\gamma$. The formulae for t_A^* and β' take corresponding forms.

Hochsplodjzgass!!

by Dr. H.T.CROFT , Peterhouse.



In troubled times in the Central European city of Splödjz, the Burgomeister, on the point of fleeing, hides his family heirlooms in a crevice in the west bank of the river Slurp, on which the city stands. He had noted in childhood that this point was such that it lay on the line AB, where A and B were two points such that the sides of two of the city's three gas-holders—Splödjz's finest mediaeval monuments—coincided, as shown.

With the catacalysm over, our hero, by now a citizen of the United States of America, Homer. X. Shufflebottom, returns for his diamonds. The historic gas-holders had of course been respected by the cultured soldiery, but the Eastern sector, to the right of the river in the diagram, was held by the Tartar Horde, and no Westerner dared cross the river to construct the point B.

Advise Homer X!

TITAN

by **DR.D.W.BARRON**, Mathematical Laboratory

The TITAN computer at the Mathematical Laboratory is the University's largest computer. Although several departments have their own small computers, and Control Engineering will shortly have a large one, Titan is the only one to provide a computing service to all the University Departments and Faculties (or almost all—Theology is one of the few Faculties so far to escape the computing urge). Despite its being one of the largest and most powerful installations of any British University, Titan is fully occupied by work for the University, and is run round the clock, except at weekends when it is switched off from 10 p.m. on Saturday to 8 a.m. on Monday. There are about 500 registered users of Titan in the University, and in the course of a day the machine gets through 300 to 400 jobs, ranging from very small calculations taking only a few seconds to really substantial pieces of computation that take perhaps half an hour. (On a machine that can do a multiplication in 5 microseconds, half an hour's computation is really substantial.) In addition to providing a computing service, the machine is used for Computer Science research—the development of new ways of using computers. Work in this field is supported by a special grant from the Science Research Council.

Vital Statistics. Average speed is 5 microseconds per operation. Arithmetic is carried out on numbers with magnitude in the range 10^{115} to 10^{-115}, held to a precision of 12 decimal places. The main memory is a core store of 65, 536 words, which is backed up by a magnetic disc store of 8 million words capacity and six magnetic tape transports. A magnetic tape has a capacity of about $2\frac{1}{2}$ million words. Primary input is via punched paper tape or punched cards, output is either in printed form or on paper tape. At present, output is printed at 600 lines per minute: a new printer, now on order, will print at 1000 lines per minute.

Titan is a prototype of the ICT Atlas 2 computer. Work started about four years ago, as a joint project between the Mathematical Laboratory and Ferranti Ltd to design a smaller and simpler version of the Atlas 1 computer. Whilst the project was in progress, the computer department of Ferranti Ltd was taken over by ICT. Only one Atlas 2 was ever sold—it is installed at the Atomic Weapons Research Establishment at Aldermaston.

Like the Atlas 1, Titan is a time-sharing computer. One of the major difficulties encountered in designing a computer is the disparity in speeds between the electronic central processor, and the mechanical input-output devices. A common way of getting round this is to employ a small satellite computer, which transcribes input material onto magnetic tape, and prints information from magnetic tape. In this way, the main computer can be isolated from slow mechanical devices, using magnetic tape for input and output. In Titan a different approach is employed. Whenever a peripheral device requires attention the computer is interrupted, and breaks off for a short time from what it is doing to deal with the device. Thus it appears to do three things at once—copy information from input media to the magnetic disc store, execute jobs which take their input from the disc store, and send their results to it, and transfer information from the disc store to the printer and tape punches. In fact, of course, at any instant in time it is only doing one of these things, but the quickness of the hand deceives the eye, and they appear to progress in parallel.

Another aspect of time sharing that is currently receiving much attention is multi-access working. In this scheme several teletypewriters (not necessarily in the same building) are connected 'on-line' to the computer, and by devoting a little time to each teletype on a 'round-robin' basis, the computer gives each user the impression that he has the computer to himself. In this way one gets the advantages of a 'personal' computer without the attendant inefficiency, since whenever one user is thinking, the computer's time which would otherwise be wasted is absorbed by the other users. Systems like this have been operating in America for some years now, and work is in an advanced state for such a system on Titan, which we hope will be the first British on-line system to be operational. The Titan on-line system will incorporate a graphical display and light-pen input, achieved by using a small American PDP7 computer as a satellite. This will enable us to investigate graphical input and output techniques, and a joint group from the Mathematical Laboratory and the Engineering Laboratory is working on the development of these techniques for Computer-Aided Design, a field which is already being explored by the American motor and aircraft manufacturers, among others.

The Archimedeans Computer Group are among the many users of Titan. One of their more spectacular achievements is the Tower of Hanoi program, which uses the visual output to good effect. They are currently helping us to test the compiler for a new programming language, CPL, and we hope that they will help us to try out our other systems as they come into service.

THE FIBONACCI SEQUENCE

by J.W. PORTER, Churchill College

As is well-known, the Fibonacci sequence is the sequence of numbers u generated by the recurrence relation

$$u_n = u_{n-1} + u_{n-2}; \quad u_1 = 1, u_2 = 2. \qquad (1)$$

This recurrence relation may be solved by the usual methods to give

$$u_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^{n+1}$$

We shall, however, find it simplest to use (1) as much and (2) as little as possible.

Among the most interesting properties of the Fibonacci sequence are its 'properties of reproduction'. For example, we take the Fibonacci sequence 1, 2, 3, 5, 8, 13, 21, 34, ...,

form the square of each term: 1, 4, 9, 25, 64, 169, ...,

and add consecutive terms, obtaining 5, 13, 34, 89, ...,

the sequence of Fibonacci numbers with even index. Similarly the sequence of Fibonacci numbers of odd index may be generated by adding consecutive terms of the series whose terms are the products of consecutive Fibonacci numbers, for the identities $u_{2n} = u_n^2 + u_{n-1}^2$ and $u_{2n+1} = u_n(u_{n-1} + u_{n+1})$ are easily proved using (2).

(1) gives expressions for the sums of series of Fibonacci numbers.

(a) $$\sum_{r=1}^{n} u_r = \sum_{r=2}^{n} [u_{r+1} - u_{r-1}] + u_1 = u_{n+1} + u_n - u_2 - u_1 + u_1 = $$
$$= u_{n+2} - 2.$$

(b) $$\sum_{r=1}^{n} u_{2r} = \sum_{r=2}^{n} [u_{2r+1} - u_{2r-1}] + u_1 = u_{2n+1} - 1$$

(c) $$\sum_{r=1}^{n} u_{2r-1} = \sum_{r=2}^{n} [u_{2r} - u_{2r-2}] + u_1 = u_{2n} - u_2 + u_1 = u_{2n} - 1.$$

(d) $$\sum_{r=1}^{2n} (-1)^r u_r = \sum_{r=1}^{n} u_{2r} - \sum_{r=1}^{n} u_{2r-1} = u_{2n+1} - u_{2n} = u_{2n-1}$$

and similarly $\sum_{r=1}^{2n+1} (-1)^r u_r = - u_{2n}$; thus $\left| \sum_{r=1}^{n} (-1)^r u_r \right| = u_{n-1}$.

The results (a), (b) and (c) can clearly be used as alternative methods of generating the sequence.

Another class of properties concerns the divisibility of the Fibonacci numbers. It is easy to see that any two consecutive Fibonacci numbers must be coprime, for by (1) any common factor m of u_n and u_{n-1} also divides u_{n-2}, whence it divides u_{n-3}, and so on down to $u_1 = 1$. So m cannot exceed 1.

We may also prove that a necessary and sufficient condition for u_m to divide u_n is that $m + 1$ should divide $n + 1$, unless $m = 1$. This result tells us which Fibonacci numbers are divisible by any given Fibonacci number. For any integer m, it is found that there always exist Fibonacci numbers divisible by that integer, and that there is at least one such Fibonacci number in the first m^2 members of the sequence. Furthermore, if m should be a prime, we may confine our search for a Fibonacci number divisible by m to the first m members.

An example of a problem which is as yet unsolved is the following: is the number of primes in the Fibonacci sequence finite or not?

GROUPS OF EXPONENT 2

by I.N.STEWART , Churchill College

The exponent of a group is the l.c.m. of the orders of its elements. Thus a group of exponent 2 is one in which every element other than the identity is of order 2. We will call such a group an E2-group.

It is well-known [1] that every finite E2-group is isomorphic with a direct product of cyclic groups of order 2.

There is, however, a generalization of this theorem to any (finite or infinite) E2-group. The proof is an example of the common mathematical trick of endowing the object under consideration with more structure than it at first sight possesses. In this case we 'embed' the group in a vector space— but no additional elements are needed! But first, a few preliminaries [2]: Let $\{X_\alpha\}$ $\alpha \in A$ by any set of groups indexed by A.

The direct product ΠX_α is the collection of all sets $\{x_\alpha\}$ $\alpha \in A$, such that $x_\alpha \in X_\alpha$. The multiplication on ΠX_α is defined by $\{x_\alpha\}\{y_\alpha\} = \{x_\alpha y_\alpha\}$. The identity is then $\{z_\alpha\}$, where $z_\alpha = 1$, the identity of X_α. And $\{x_\alpha\}^{-1} = \{x_\alpha{}^{-1}\}$. The external direct sum ΣX_α is that subset of ΠX_α consisting of elements $\{x_\alpha\}$, where $x_\alpha = 1$ for all but a finite number of values of α. It is clear that both ΠX_α and ΣX_α are groups.

Lemma: Any E2-group is Abelian.

Let G by any E2-group, and x, y any elements of G. Then $x^2 = y^2 = 1 = (xy)^2$, $xyxy = 1 = x^2y^2$, so $yxy = xy^2$, hence $yx = xy$.

THEOREM: A group G is an E2-group if and only if it is isomorphic with an external direct sum of cyclic groups of order 2.

PROOF: An external direct sum of cyclic groups of order 2 is clearly an E2-group.

To prove the converse, we give G the structure of a vector space over the field GF(2) with 2 elements, 0 and 1, where:

$$00 = 01 = 10 = 0, \ 11 = 1,$$

$$0 + 0 = 1 + 1 = 0, \ 0 + 1 = 1 + 0 = 1.$$

Write G additively (i.e. use + for the multiplication, and 0 for the identity. No confusion between 0's need arise.) For $g \in G$, define $0g = 0, 1g = g$, where the 0 and 1 on the left are elements of GF(2), and the 0 on the right is the identity of G.

Then this makes G a vector space over GF(2). The only slight difficulty in showing this is in proving that $(1 + 1)g = g + g$. But since $1 + 1 = 0$ and g is an E2-group, both sides are 0. By the Axiom of Choice, G has a basis A. Any $g \in G$ can therefore be written $g = \lambda_1\alpha_1 + \ldots + \lambda_n\alpha_n$, where $\lambda_i \in GF(2)$, $\lambda_i \neq 0$, and $\alpha_j \in A$. Hence g is uniquely expressible as $g = \alpha_1 + \ldots + \alpha_n$, $\alpha_i \in A$ (for the λ_j must be 1).

Each $\alpha \in A$ generates a subgroup X_α of G, and X_α is cyclic of order 2 since $\alpha + \alpha = 0$. We now claim that G is isomorphic with the external direct sum of the X_α. (In fact it is the internal direct sum.)

We can define a mapping $f\colon G \to \Sigma X_\alpha$, as follows: If $g = \alpha_1 + \ldots + \alpha_n$, then $f(g) = \{x_\alpha\}$ where $x_{\alpha_n} = \alpha_n$, all other $x_\alpha = 0$ (X_α is written additively).

This map is clearly an isomorphism, and is onto by the finiteness condition in the definition of the external direct sum. Hence the result is proved.

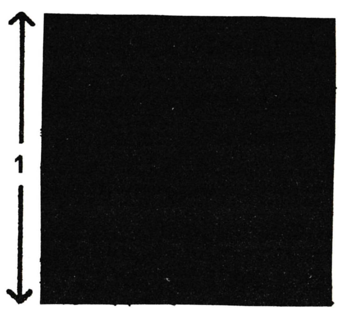
Extensions: We may by similar means prove a corresponding result for Abelian groups in which every element has order p, p a fixed prime > 2. We give the group a vector space structure over GF(p), and proceed as before. We can only deal with Abelian groups, however, since we cannot now prove they must be Abelian, as in the lemma.

References:

[1] Ledermann, The theory of finite groups, p. 47.

[2] Scott, Group theory, p. 14.

A SPACE-FILLING ARTICLE



Peano's space-filling curve in the unit square.

ART BY COMPUTER

by J.L.DAWSON, King's College

Take a real function $f(x, y)$ and a positive integer r. Choose real functions $x_j = x(j)$ and $y_i = y(i)$, for $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n-1$. Let A_{ij} be the m by n matrix defined as follows: a_{ij} = parity of the r'th digit after the decimal point in the number $f(x_i, y_j)$. (i.e. $a_{ij} = 1$ if this number is odd, 0 if it is even.) Write out the matrix, replacing 1 by a symbol such as *, and 0 by a space. This results in a pattern, which, given suitable choices of $f(x, y)$, x_j, y_i, and r, can be very beautiful.

So far I have only experimented with $x(j)$ and $y(i)$ linear functions of j and i respectively. I have reason to believe, though, that simple trigonometric functions of j and i would give interesting results. The best patterns produced have been those where f is either a combination of simple trigonometric functions, or of exponentials of such functions. The reason for this is that f is then periodic, giving the pattern a certain amount of symmetry. It is found that, with m and n around 100, and f a function taking values between 2 and -2, that the first and second decimal places ($r = 1$ or 2) give the best results. Taking $r = 3$, the third decimal place is changing too rapidly, and a rather uninteresting random pattern emerges.

It has been suggested that patterns with circular symmetry be attempted. This has not yet been done, but the results may be promising.

The patterns 1, 2, and 3 on the following pages have the descriptions:

	$f(x,y)$	m	n	r	$x(j)$	$y(i)$
(1)	$\sin x \cos y + \sin y$	100	100	2	$\pi j/98$	$\pi i/98$
(2)	$\sin x \sin y$	111	111	2	$\pi j/110$	$\pi i/110$
(3)	$\exp(\cos x + \cos y)$	100	100	2	$\pi j/198$	$\pi i/198$

I am making a 6 ft. by 4 ft. pattern of number (3), which will be mounted and framed. I hope to publish a photograph of it in the next issue of EUREKA.

My thanks go to D. M. Stanford of King's, who first thought of the idea, and to the University Mathematical Laboratory for the use of the Titan computer.

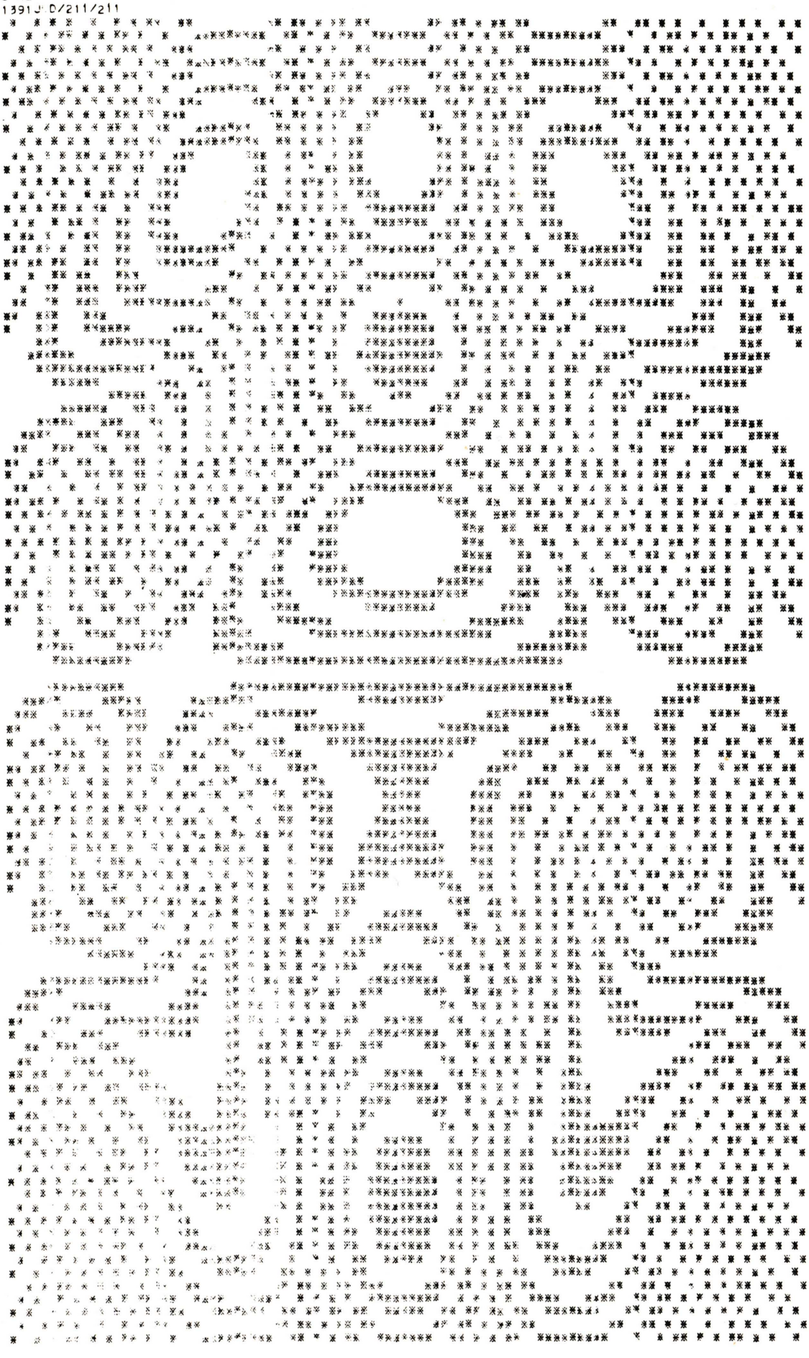
Then as Now

(EUREKA, 1959)

It so happens that the present editorial staff cannot read German and have always been puzzled by the remarkable variety of German Bs and Gs which appear in some mathematics apparently just to confuse the reader. Hence it was with delight that we happened to discover a complete German alphabet whilst preparing this issue, and with some surprise that we find it contains actually only one B and one G. Perhaps others have been puzzled likewise?

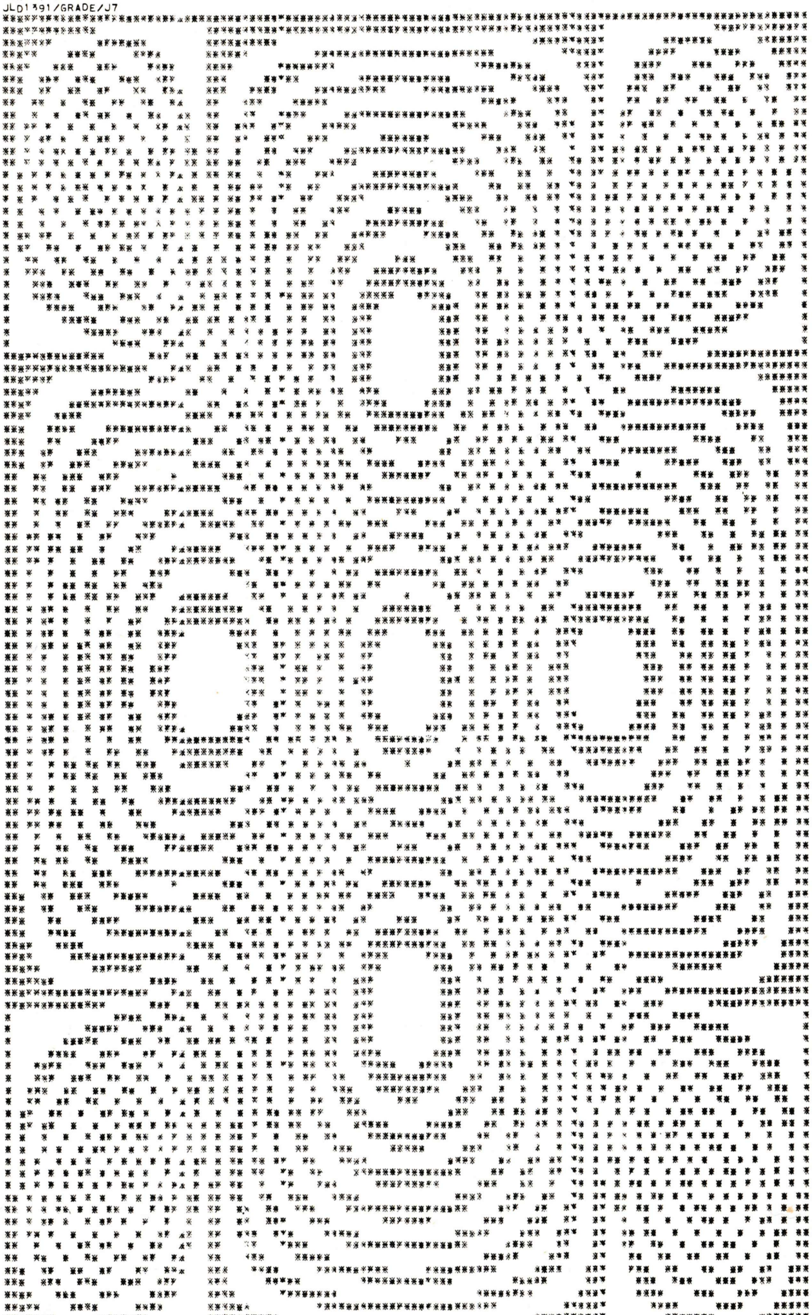
$\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{D}\ \mathfrak{E}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H}\ \mathfrak{I}\ \mathfrak{J}\ \mathfrak{K}\ \mathfrak{L}\ \mathfrak{M}\ \mathfrak{N}\ \mathfrak{O}\ \mathfrak{P}\ \mathfrak{Q}\ \mathfrak{R}\ \mathfrak{S}\ \mathfrak{T}\ \mathfrak{U}\ \mathfrak{V}\ \mathfrak{W}\ \mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}$

STREA' 1
1391J D/211/211



PATTERN 1

STREAM 1
JLD1391/GRADE/J7



TIME 21 CHARS 24d98

PATTERN 2

STREAM 1
1391JLD/211/211

TIME 34 CHARS 40419

PATTERN 3

PROBLEMS DRIVE 1966

set by **D. MOLLISON**, Trinity College

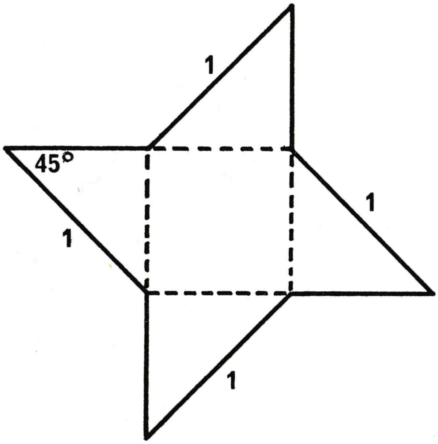


figure 1

1					
1	2				
1	$\sqrt{2}$	3			
2	1	$\sqrt{5}$	4		
1	$\sqrt{2}$	$\sqrt{2}$	?	5	
$\sqrt{2}$	1	?	$\sqrt{2}$	?	6

figure 2

1) S is the surface obtained by rotating a circle of radius r (>0) about a line in its plane, distant d from its centre ($d > r$). Show that any two points of S can be joined by a line in S of length <u>less</u> than $\pi(d + r)$.

2) A circular island of circumference 1 has an inpenetrable interior, a narrow path round its edge, and two first-aid posts (FAP's) at <u>random</u> points on the path. A woman is washed up on the island at night at a random point of the path; how far may she expect to walk before she reaches a FAP?

3) 21 men of equal weights are hanging onto an overhanging cliff. The feet of the n'th man are 10n feet from the ground, for $n = 1, 2, \ldots, 21$. The top man lets go; as he passes the next man, the latter also lets go, and falls clinging to the former in such a way that energy is conserved (this is wrong, but most competitors <u>rightly</u> assumed it, (?-Ed.)), and so on: as n men pass the n + 1'st he lets go and clings to them. Neglecting air resistance and their energy of rotation, find with what velocity they hit the ground, taking $g = 8$ ft/sec/sec. (this is not a misprint-Ed.)

4) (a) How many points can be placed on figure 1, so that no two are less than 1 unit apart?

(b) How many configurations are there for the maximal number of points (assuming the points to be indistinguishable)?

5) (a) Complete the table of 'distances' given in figure 2.

(b) Would the 'distance' between 2/3 and 3/4 be meaningful?

(c) If so, what is it? If not, what is the 'distance' between 17 and 29?

6) A railway and a road run together for seven miles from P to Q. Two miles from P there is a level crossing, which is closed one minute before, and opened one minute after, a train passes.

A train passes a stationary car at P and travels on to Q at 60 m.p.h., and, forgetting to slow down, crashes at Q; the car passes the train as it crashes. Assuming that stopping for an instant from full speed loses the car one minute, of what speed must it be capable?

7) Find four distinct digits A, I, N, and T between 1 and 9 (inclusive) such that $A^m + (AN)^m = AINT$ in the scale of 10, for m = 1, 2, or 3.

8) Let Q be a string of 0's and 1's. Q^1 is the string of 0's and 1's obtained by taking the (positive) differences of successive terms of Q; $Q^2 = (Q^1)^1$; etc..

For example, Q = 11100101010011

Q^1 = 0010111111010

Q^2 = 011100000111

An infinite periodic string......mmmm... is written as (m)- e.g. ...101101101... is (101), (011) or (010). For which of the following is $Q^n = (0)$ for large enough n:

(101) (1110) (11010) (1111000111010) (1110000101010111)

9) An early warning system consists of sets of three buzzers, (a_0, b_0, c_0), (a_1, b_1, c_1), ..., (a_n, b_n, c_n). Each a_m can be fired by just b_{m-1} or c_{m-1}, and similarly for b_m and c_m; it is certain to fire if both its inputs fire, is equally likely to fire if only one of them fires; and will not fire if neither does. a_0 and b_0 fire, but not c_0. What is the probability of:

a) c_2, but neither a_2 nor b_2, firing.

b) a_2, b_2 and c_2 all firing.

10) Write down the next three terms in each of the following series:

a) 3, 9, 17, 21, 27, ...

b) 1714, 1727, 1760, ...

c) 1001, 1100, 0011, 1110, ...

d) 1, 18, 3, 8, ...

e) 1, 2, 3, 4, ...

SPUDSAC I

by **RUNNY DUSTCARTES**, University of Outer Mongolia

The following is a description of an automatic food-processing computer developed by the Cambridge Mathematical Laboratory Cleaning-Staff. The main components of the machine include:

Storage: 1) The main store. This consists of 2^{15} wire-wound apple-cores, on which the basic substances required are stored. Each word of storage consists of 32 chips, each of which may be in one of two states—soggy or burnt.

2) There is also a high-speed low temperature random-access store, for perishable goods.

Registers: 1) The Accumulator Register, which holds the address of the current operand.

2) The Operation Register, which holds the address of the next operation to be obeyed.

3) The Cash Register, which holds the address of the local grocer's shop.

Input/Output: Input and output are on five-channel punched macaroni, with a direct feed mechanism for large output. Peripherals include an on-line potato-peeler in the Domestic Science Lab, and a number of bins in case of overflow.

Programming: Most programming is done in Spudsac Tomatocode, which is similar to ordinary language. An example is given below:

```
TITLE: PREPARATION OF VEG TABLES AT INTERVALS OF FIVE
       DEGREES
0) REGULO:= INPUT
PAN:= PAN + CARROTS
REPEAT till CARROTS > 67
for HEINZ = 1: 1: 57
BOIL until done
RETURN
X:= INNER PRODUCT (PAN)
JUMP if X IS HOT
STOP!
```

A number of subroutines are included, e.g. 'SPUD' (Suet Pudding UnDump). A large range of compilers is available as soggyware, particularly for Fortran, Saucepan, Frypan, Algol, Soupbol, and Coalhol.

Optional extras include facilities for input on baked beans instead of macaroni.

PROBLEM COMPETITION

We have had many requests for more problems in EUREKA. Accordingly, here is a competition of a rather more difficult nature than our usual problems drive. Three prizes of £1 are offered jointly by the Archimedeans and EUREKA, one for the best set of solutions from a 1966 freshman, one for the best set from a resident junior member of the Archimedeans, and one for the best set from a postal subscriber. The closing date for solutions, which should be sent to the Editor, c/o the Arts School, Bene 't St., Cambridge, England, is January 31st. We hope to post solutions on the Arts School Notice Board on February 1st, and to publish some in next year's EUREKA. The Editor's decision is final. Don't get too disheartened if you cannot get very far — we consider it unlikely that anyone will do them all.

Our thanks must go to Dr. J. H. Conway, who provided us with nos. 4, 5, 6, 8, and 9, Dr. H. T. Croft (nos. 2, 7, 11, 12, and 14), Mr. Max Rumney (no. 13), T. W. Cusick (no. 15), D. Mollison (no. 10), and K. F. Wylie (no. 16). We would like to have our readers' comments on the competition, and suggestions for problems for next year.

1) Let P and Q be polygons, each with a finite number of sides, (these numbers are not necessarily equal), and with the same area. Prove that each of P, Q can be divided into triangles $P_1, P_2, P_3, \ldots, P_n$ and $Q_1, Q_2, Q_3, \ldots, Q_n$ respectively, such that P_i and Q_i are congruent, for $1 \leqslant i \leqslant n$.

2) Suppose that $ny_n = y_{n+1}^2 + ny_{n+1}$, $n = 1, 2, 3, \ldots\ldots$

and $y_n > 0$. Prove that $y_n \log n \to 1$ as $n \to \infty$.

3) Two polynomials P(z) and Q(z) of the complex variable z have the same zeros, though not necessarily with the same multiplicities. The same is true of the polynomials P(z) + 1, Q(z) + 1. Prove that these polynomials are identically equal.

4) Prove that for sufficiently large n there is one and only one sequence of positive integers $a_1, a_2, \ldots. a_n$ in which a_i is equal to the number of values of r such that $a_r = i$.

5) a, b, c, d, e are elements of a group and satisfy

$$ab = c,\ bc = d,\ cd = e,\ de = a,\ ea = b$$

Prove that $a^{11} = 1$, the unit element. Try to generalize the result to n elements $a_1, a_2, \ldots a_n$ satisfying $a_i a_{i+1} = a_{i+2}$ (mod n).

6) An overheard conversation:

A: 'My children's ages are positive integers whose sum is the number of people in this room and whose product is my own age.'

B: 'I need more information if I am to determine their ages. Perhaps you would tell me the number of your children and your own age.'

A: 'I'm afraid that even this information would not be sufficient.'

B: 'Then you needn't give it to me, since I can work it out for myself. You are. . . years old and have . . . children.'

Fill in the blanks. Since the conversation obviously took place between mathematicians, no 'realistic' assumptions may be made about the ages concerned.

7) Prove that any closed curve of diameter ≤ 1 encloses an area at most $\pi/4$.

8) For all positive integers p and q, $p > q$, find a way of linking p strings together so that when any $q + 1$ are broken the remaining strings separate completely, but that when any q are broken the system remains completely connected.

9) A: B and C are not both true.

B: The number of false statements in this system is 0.

C: If the numeral appearing at the end of statement B were increased by 1, the resulting system would have no solution.

Solve this system of statements. A solution means an assignment of values true and false to the statements such that the value assigned to any statement is the same as that deduced from its meaning.

10) In no. 8 of the problems drive, prove that $Q^n = (0)$ for large enough n if and only if Q has period a power of 2.

11) Is it possible to cover an infinite plane with convex heptagons?

12) Prove that an infinite set of points in a plane such that any pair are an integral distance apart must be collinear. Can one find an arbitrarily large finite number of points with this property, and no 3 collinear?

13) A Perfect Digital Invariant in scale s is a number, N such that its representation in scale s has n digits $a_1, a_2, \ldots, a_n$ where $N = a_1^n + a_2^n + \ldots + a_n^n$. For example in scale 10, $153 = 1^3 + 5^3 + 3^3$, and $1634 = 1^4 + 6^4 + 3^4 + 4^4$; in scale 13, $491 = 4^3 + 9^3 + 1^3$. Prove that any Perfect Digital Invariant is composite.

14) A 6 by 6 'chess-board' of 36 one-inch squares is covered by 18 2" by 1" dominoes in the obvious way, each domino covering two squares. Can a line always be drawn either across or down the board to divide the dominoes into two non-zero portions, without cutting any dominoes in half?

15) Let $f(n) = 2n + 2$ for each non-negative integer n. We can divide the non-negative integers into two sequences S_1 and S_2 such that

1) Every non-negative integer is in one and only one of S_1, S_2.

2) n is in S_2 if and only if $n = f(m)$ for some m in S_1.

For we put 0 in S_1 and put any $n > 0$ in S_2 if and only if $n = f(m)$ for some m in S_1, this being decidable by induction since we necessarily have $m < n$.

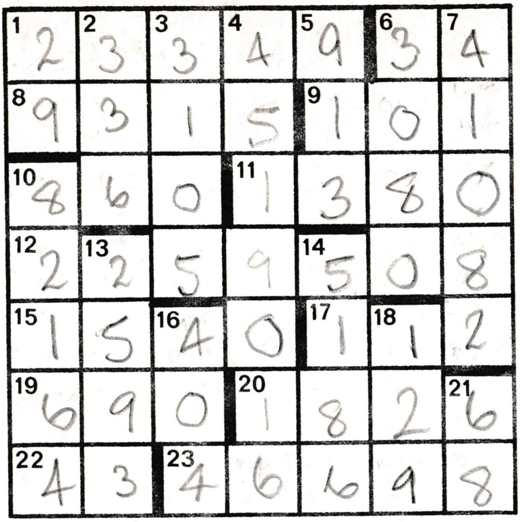


figure 1

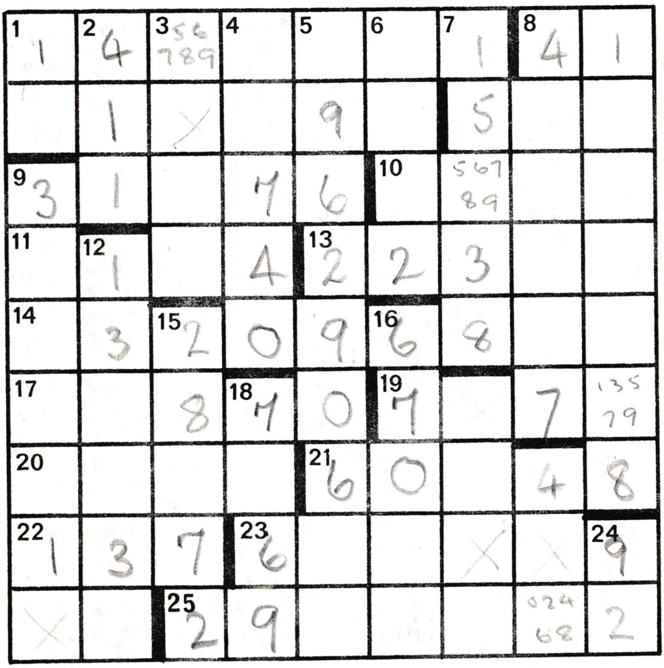


figure 2

Thus we get

$$S_1 = (0, 1, 3, 5, 6, 7, 9, 10, 11, 13, 15, \ldots)$$

$$S_2 = (2, 4, 8, 12, 14, 16, 20, 22, 24, 28, \ldots)$$

Find a method by which the difference sequences of S_1 and S_2 - i. e. the sequences formed of the differences of consecutive terms — may be written out as far as one liked without writing out S_1 and S_2.

16) Solve the following two crossnumber puzzles.

a) Figure 1:

ACROSS

1: See 23 across
6: See 21 down
8: 3 × (3 down)
9: See 16 down
10: No clue
11: 2 × (19 across)
12: See 4 down
15: See 6 down
17: See 2 down
19: See 11 across
20: 2 × (5 down)
22: See 18 down
23: 2 × (1 across)

DOWN

1: No clue
2: 3 × (17 across)
3: See 8 across
4: 2 × (12 across)
5: See 20 across
6: 2 × (15 across)
7: See 10 down
10: 2 × (7 down)
13: See 14 down
14: 2 × (13 down)
16: 4 × (9 across)
18: 3 × (22 across)
21: 2 × (6 across)

b) Figure 2:

ACROSS

1: 3 × (8 down); see 25 across
9: 2 × (7 down); see 12 down, 21 across
10: 3 × (20 across)
11: See 6 down
13: See 12 down, 16 down
14: See 5 down
17: See 4 down
19: The sum of four other answers
20: See 10 across
21: 9 across + 15 down
22: See 2 down
23: Half of some other answer
25: 2 × (1 across)

DOWN

2: 3 × (23 across)
3: Some other answer
4: 2 × (17 across)
5: 3 × (14 across)
6: 3 × (11 across)
7: 2 × (18 down); see 9 across
8: See 1 across
12: 2 × (7 down + 9 across + 13 across)
15: See 21 across
16: 3 × (13 across)
18: See 7 down
24: The sum of three answers

Topological equivalents of the axiom of choice

by P.G.DIXON, Churchill College

In general topology there are one or two theorems which are nearly always discussed with reference to the axiom of choice. These are generally the theorems which depend on the axiom of choice in its strongest form. However, what is less often mentioned is that many more of the theorems of elementary general topology depend on weaker forms of the axiom; for instance, on forms allowing only a countable number of choices.

It is the purpose of this article first to state some of the results concerning fairly strong forms of the axiom, then to examine some of the theorems depending on weaker forms; in particular, to specify this dependence precisely by proving certain theorems equivalent to purely set-theoretic forms of the axiom. Finally, we shall look at some theorems which seem at first sight to use infinitely many arbitrary choices, but which, in fact, do not need any further axioms beyond the basic ones of Zermelo-Fraenkel set theory, for their proof.

We start by listing some of the weaker and stronger forms of the axiom of choice to which it will be necessary to refer later on:

I The Axiom of Choice: If $\{X_\alpha\}_{\alpha\in A}$ is a family of non-empty sets, there is a function $f: A \to \bigcup_{\alpha\in A} X_\alpha$ such that for every $\alpha\in A$, $f(a)\in X_\alpha$.

II The Axiom of Dependent Choices: If X is a non-empty set, with a binary relation R defined on it, such that the range of R is included in the domain of R, then there is a sequence $x_1, x_2, \ldots$ such that, for all i, $x_i \in X$ and $x_i R x_{i+1}$.

III The Axiom of Countable Choice: If $X_1, X_2, \ldots$ is a sequence of non-empty sets, there is a sequence $x_1, x_2, \ldots$ such that $x_i \in X_i$, for all i.

IV The Axiom of Countable Choice from Sets of Reals: If $X_1, X_2, \ldots$ is a sequence of non-empty sets of real numbers, there is a sequence $x_1, x_2, \ldots$ such that $x_i \in X_i$, for all i.

V The Boolean Prime Ideal Theorem: In every Boolean algebra, every proper ideal is included in some prime ideal.

(The following logical relations between these are known: I $\Rightarrow$ II $\Rightarrow$ III $\Rightarrow$ IV; I $\Rightarrow$ V; III $\not\Rightarrow$ II; V and IV are independent of the basic axioms of Z-F set theory, assuming those to be consistent.)

Theorems depending on strong forms of the Axiom of Choice (I, II and V)

The classic example under this heading is, of course, the Tychonoff Product Theorem. The proof of this from the axiom of choice is well known (ref. 3 p. 143). The proof that this theorem implies the axiom of choice is due to Kelley (ref. 1), but the proof given there has a slight error.

In that paper, Kelley raised the question whether Alexander's subbase theorem is equivalent to the axiom of choice. It has since been shown that it is equivalent to the Boolean Prime Ideal Theorem. It has also been shown that Tychonoff's theorem for products of Hausdorff spaces and the Stone-Čech compactification theorem are both equivalent to V (ref. 2).

Another equivalence which may be stated in this section is that Baire's category theorem, stating that a complete metric space is second category, is equivalent to II, the axiom of dependent choices.

Theorems depending on weak forms of the Axiom of Choice (III and IV)

(i) The Lindelöf Covering Theorem is equivalent to IV.
The Lindelöf Covering Theorem states that if (X,τ) is a second-countable space, and if $\{O_\alpha\}_{\alpha \in A}$ is a collection of open sets covering X, then there is a countable subcollection $(O_{\alpha_i})_{i=1,2,\ldots}$, $\alpha_i \in A$, also covering X.

We first prove IV implies Lindelöf's theorem.

(X, τ) is second-countable, so we can choose a countable base (B_j) for the topology. (This is a single act of arbitrary choice, which is admissable.) For each $\alpha \in A$, let

$$J_\alpha = \{j \mid B_j \subseteq O_\alpha\}.$$

Then, since (B_j) is a base, $O_\alpha = \bigcup_{j \in J_\alpha} B_j$.

Thus we have made the sets O_α correspond to sets J_α of integers.

For each i, let $X_i = \{J_\alpha \mid B_i \subseteq O_\alpha\}$.

Now the X_i are sets of sets of integers, but since there is a simple one-to-one correspondence between reals and sets of integers, we may apply IV to them and obtain a sequence (J_{α_i}) such that, for each i, $B_i \subseteq O_{\alpha_i}$.

The sequence (O_{α_i}) thus has a union which includes every set B_i, and so it covers X.

To prove Lindelöf's theorem implies IV, we again consider IV as applying to sets X_j whose members are sets of positive integers. We then take as our space, X, the set of all pairs of positive integers, with the discrete topology. The space is countable; the collection of singletons forms a base; and so the space is second-countable. For the covering sets O_α, we take all the singletons $\{\langle i, j\rangle\}$ where $i \geqslant 2$, together with the sets

$$\{\langle 1, j\rangle\} \cup \{\langle k+1, j\rangle \mid k \in x\}$$

for each $x \in X_j$ and each $j \geqslant 1$.

Since each of the sets X_j is non-empty, we have singletons containing each $\langle i, j\rangle$ for $i \geqslant 2$ and at least one set containing each $\langle 1, j\rangle$ for every j; so the sets O_α do cover X. Applying Lindelöf's theorem, we obtain a countable subcovering (O_{α_i}). Now this must contain some set covering $\langle 1, j\rangle$ for each j; in particular, since the subcovering is a countable sequence, there is a first set in it containing $\langle 1, j\rangle$ for each j. This set defines a member x_j of X_j, and so we have our required choice sequence.

(ii) 'A second-countable space is separable' is equivalent to III

In this case, the proof of the theorem from III is straightforward; we simply choose the sequence of points to form the dense subset from the given sequence of base sets (ref. 3 p. 49). The proof that the theorem implies III proceeds as follows.

We are given a sequence $X_1, X_2, \ldots$ of non-empty sets which we can take to be disjoint. We introduce a topology on $\bigcup_i X_i$ by taking the sets X_i and unions of these to be open, i.e. by taking as base the collection of sets $\{X_i\}_{i=1,2,\ldots}$ This topology is therefore second-countable, so the theorem gives us a sequence of points $y_1, y_2, \ldots$ which is dense in $\bigcup_i X_i$. Now there must be a point of this sequence in every open set of the topology. In particular, we have $\forall i, \exists n, y_n \in X_i$. Now take x_i to be the y_n for least n such that $y_n \in X_i$. Then the sequence $x_1, x_2, \ldots$ is the required choice sequence.

(iii) Consider the theorem, (hereafter referred to (α)):

'If X, Y are topological spaces, X is first-countable, and f is a function from X into Y, and if $x_n \to x$ always implies $f(x_n) \to f(x)$; then f is continuous.'

An examination of the proof of this theorem shows that it follows from III. To attempt to deduce a form of arbitrary choice from the theorem, we proceed as follows.

Given a sequence of pairwise disjoint sets, $(X_n)_{n=1,2,\ldots}$, we define a decreasing sequence of sets $(Y_n)_{n=1,2,\ldots}$ by:

$$Y_n = \bigcup_{i \geqslant n} X_i$$

Let y be some element not in any of the X_i. Define $Y = Y_1 \cup \{y\}$, and make Y a topological space by defining the basic open sets to be all the subsets of Y_1, together with the sets $Y_n \cup \{y\}$. It is easy to check that this defines a first-countable topology on Y---- a base at y is the collection of sets $Y_n \cup \{y\}$, and a base at any other point is simply the singleton of that point.

We define the topological space Z as the set Y with the discrete topology, and the function $f: Y \to Z$ as the identity on the set Y. f is continuous everywhere except at y, where we have $\{y\}$ open in Z, but $f^{-1}\{y\} = \{y\}$ not open in Y. Theorem (α) now gives a sequence $x_n \to x$ such that $f(x_n) \nrightarrow f(x)$. But since f is continuous at all pounts other than y, if $x \neq y$ and $x_n \to x$, then it can be shown that $f(x_n) \to f(x)$. Thus we must have a sequence $x_n \to y$ with $f(x_n) \not\to f(y)$.

Now produce a sequence $x'_n \to y$ with $f(x'_n) \not\to f(y)$ and $x'_n \neq y$ by removing all occurrences of y from the sequence (x_n). This must leave a sequence with the required properties, unless the original sequence had x_n eventually equal to y, in which case, $f(x_n)$ eventually equals $f(y)$, and so $f(x_n) \to f(y)$; a contradiction.

Now take a subsequence of (x'_n) by taking, for each X_i containing some x_j, the first x_j lying in X_i, and denoting this by y_i. i.e. we have defined a sequence (n_i) by:

n_1 = minimum i such that there is some $x'_n \in X_i$

n_{j+1} = minimum $i > n_j$ such that there is some $x'_n \in X_i$

and y_{n_i} = the x_j with least j such that $x_j \in X_{n_i}$.

Thus we have shown that, given a sequence of (disjoint) sets X_i, we can find a sequence of numbers $n_1 < n_2 < \ldots . < n_i < \ldots .$ and a sequence $(y_{n_i})_{i=1,2,\ldots}$ such that for all i, $y_{n_i} \in X_{n_i}$.

We shall call this the axiom of Sequence Choice. Clearly, Sequence Choice is a weaker form than III. However, it is sufficient to prove theorem (α), as can be seen by examining carefully the usual proof of that theorem (ref. 5 p. 78). It can also be shown equivalent to other theorems involving sequences, such as: 'If X is a Lindelöf space and every sequence in X has a cluster point, then X is compact.' (ref. 3 p. 137).

One might suppose Sequence Choice to be a new form, strictly weaker than III, but it turns out that we can prove that it implies III, as follows.

Assume Sequence Choice. Given a sequence of non-empty sets $X_1, X_2, \ldots,$ let F_n be the set of all finite sequences $\langle x_1, x_2, \ldots, x_n \rangle$ with $x_i \in X_i$ for all $i \leq n$. Each of these sets F_n is non-empty by the admissable rule allowing a finite number of arbitrary choices. Apply Sequence Choice to the sequence of sets (F_n) and obtain a series of numbers $n_1 < n_2 < \ldots . < n_i < \ldots .$ and a sequence $f_{n_1}, f_{n_2}, \ldots$ such that $f_{n_i} \in F_{n_i}$ for all i. Now for each i, there is a minimum j for which $n_j \geq i$; define $x_i = \left(f_{n_j}\right)_i$, using this j. x_i thus defined is in X_i and so we have our required choice sequence.

By identifying finite sequences of real numbers with real numbers, we may obtain as a corollary that Sequence Choice from sets of reals is equivalent to Countable Choice from sets of reals, IV.

(iv) To conclude this section, we list some other equivalents of IV.

(a) A second-countable metric space is separable.

(b) A subspace of a separable metric space is separable.

(c) A subspace of the Hilbert cube is separable and metrisable.

(ref. 3 p. 125).

Theorems not depending on the Axiom of Choice

There are, of course, many theorems which are usually proved in a way involving arbitrary choice, but which are not essential applications of the axiom of choice. We shall consider here two examples, each characteristic of a collection of such theorems.

(i) 'In a Hausdorff space every compact set is closed.'

In the usual proof, we consider a compact set C in a Hausdorff space X, and a point $y \notin C$; and, for every point $x \in C$, we choose disjoint neighbourhoods U_x, V_x of x, y respectively. We take a finite

$$\{x_1, x_2, \ldots, x_n\} \subseteq C \text{ such that } C \subseteq \bigcup_i U_{x_i}, \text{ (by the compactness of C).}$$

Then $\bigcap_i V_{x_i}$ is a neighbourhood of y not intersecting C.

Apparently, we have used arbitrary choice in selecting, for each $x \in C$, a specific pair of neighbourhoods U_x, V_x out of the set of possible pairs; a set which the Hausdorff axiom guarantees to be non-empty for each x. To circumvent this, we proceed as follows.

Let $\{\langle U_{x,\alpha}, V_{x,\alpha}\rangle\}_{\alpha \in A_x}$ be the family of suitable pairs of neighbourhoods of x, y.

Then $\{U_{x,\alpha}\}_{x \in C,\, \alpha \in A_x}$ is an open cover of C.

Take a finite subcover $\{U_{x_1,\alpha_1}, U_{x_2,\alpha_2}, \ldots\ldots, U_{x_n,\alpha_n}\}$.

Then $V_{x_1,\alpha_1} \cap V_{x_2,\alpha_2} \cap \ldots\ldots \cap V_{x_n,\alpha_n}$ is a neighbourhood of y not intersecting C.

(ii) 'If X is a topological space, $A \subseteq X$ and $x \in \overline{A}$, (the closure of A), then there is a net $\{x_d \mid d \in D\}$ of points of A, with $x_d \rightarrow x$.'

To prove this, we note that $\mathcal{N}_x$, the collection of neighbourhoods of x, preordered by inclusion, is a directed set. (A preordering is a transitive and reflexive binary relation, and a preordered set is directed if for any pair of elements there is a third element which follows both in the preordering, c.f. ref. 3 p. 65). For each neighbourhood, N, $N \cap A \neq \emptyset$, so we choose points $x_N \in N \cap A$, and $\{x_N \mid N \in \mathcal{N}_x\}$ is the required net.

To avoid the choice of an x_N for each N, we form our directed set, instead, as the collection, S, of ordered pairs $\langle x, N\rangle$, where $x \in N \cap A$ and $N \in \mathcal{N}_x$; preordered by:

$$\langle x, N\rangle \geq \langle y, M\rangle \quad \text{iff} \quad N \subseteq M.$$

(It is important here that we do not require an antisymmetric preordering). We then define our net as $\{x_s \mid s \in S\}$ where $x_{\langle x, N\rangle} = x$.

Continued on page 40

The Archimedeans

The society has once again enjoyed a very successful year with the evening meetings, on the whole, very well attended. Particularly notable were talks by Dr. H. M. Cundy on 'Tiling and Patterns', and by Professor R. Rado on 'Arithmetic ad Infinitum', and also a 'Careers Meeting' organised by Mr. J. N. Coope from the Appointments Board, at which we heard three mathematicians from various walks of life talking about their own work and branches of mathematics. The tea meetings also were very successful especially notable were Professor J. E. Littlewood who gave an intriguing talk entitled 'Some Surprises in Differential Equations' and Mr. R. W. Shephard of Defence Operational Analysis Establishment who gave a very enlightening talk on 'Operational Research and Defence'.

As usual the Problems Drive was well attended and we were pleased that a large number of 'Invariants' from Oxford could take part. The Computer Group has thrived this year with large attendances at its weekly meetings, and is now keeping Titan even more busy. The Bridge Group is also flourishing, however attendances at both the Music Group and the Chamber Music Group have dwindled, and these Groups may have to enter the defunct category. To counteract this it is hoped to revive the Puzzles and Games Ring next year. Both the Tiddleywinks Match with the Dampers and the Punt Party were well supported and thoroughly enjoyed. Visits tended to be poorly subscribed and some had to be cancelled, although the visit to the Atomic Energy Establishment at Harwell was fairly popular and very rewarding despite being unable to visit Atlas. The Bookshop has been fairly busy and is providing a valuable service to undergraduate mathematicians.

This year's meetings begin with Professor L. Rosenhead talking on 'Mathematicians and Social Purpose' which should provoke a lively discussion, and also the Careers Meeting, again arranged by Mr. Coope. Dr. D. R. Taunt opens the tea meetings with a talk entitled 'Straight and Crooked Thinking'. It is also hoped to arrange a visit to Oxford during the Michaelmas Term. The Lent Term is opened by Professor D. B. Sawyer of Waikato University, New Zealand who will be visiting Cambridge next year, he will be speaking on 'Polyhedra'. This is followed by Professor R. S. Scorer, and then Professor G. F. J. Temple giving a talk entitled 'Newtonian Aerodynamics', and finally Professor K. A. Hirsch will give a talk on 'Braids and Groups'. The Problems Drive will again be held in February, and we look forward to challenging the 'Invariants' again. There will be the usual visits next year, and it is hoped to include some London theatres.

As usual it is intended to arrange a Tiddleywinks Match with the Dampers for the Michaelmas Term, and a Punt Party to Grantchester in the Easter Term.

The programme has been designed to cater for the needs of all members of the Society, but the Secretary would greatly appreciate any suggestions as to possible improvements or alterations, which may be made either directly or through the book kept for this purpose in the Arts School.

R. S. A. TUFF,
Secretary.

Mathematical Association

President: F. W. Kellaway, B.Sc., F.I.M.A.

The Mathematical Association, which was founded in 1871 as the Association of for the Improvement of Geometrical Teaching, aims not only at the promotion of its original object but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 30s. per annum; to encourage students and those who have recently completed their training the rules of the Association provide for junior membership at an annual subscription of 10s. 6d. Full particulars can be had from the Mathematical Association, 29, Gordon Square, London, W.C.1.

The Mathematical Gazette is the journal of the Association. It is published four times a year, and deals with mathematical topics of general interest.

Problems Drive Solutions

1) Let O be the nearest point of the circle to the straight line; join P and Q to the points nearest to them on the locus of O. This can be done by two arcs of length $\leqslant \pi r$. Now join these points by an arc of the locus of O, of length $\leqslant \pi(d-r)$. If P (say) is not on the locus of O, the resulting path from P to Q has a right angle in it, and S locally approximates a plane, so a short cut across the right angle will give a shorter path; if P and Q are on the locus of O, the path is of length $\leqslant \pi(d-r) < \pi(d+r)$ anyway.

2) The three objects (woman, FAP1, FAP2) are at undistinguished random points; consider each in turn as moving to its right (say) till it reaches one of the others. We see that the three distances apart are identically distributed random variables with sum 1; hence each has mean $1/3$.

3) $4\sqrt{110}$ ft/sec.

4) (a) 5
(b) Infinitely many.

5) (a) Reading table in usual way, the distances are $\sqrt{5}, 1, \sqrt{3}$.
(b) Yes
(c) 13

6) 75 m.p.h.

7) $A = 9, I = 6, N = 8, T = 5, m = 2$

OR $A = 1, I = 7, N = 2, T = 9, m = 3$

8) The second and fifth.

9) (a) 0
(b) $3/8$

10) (a) 29, 33, 39, . . . (100-primes)
(b) 1820, 1910, 1936 (accession of Georges)
(c) 0000, 0111, 1111, . . . (forget the commas; strings of 0's and 1's get longer)
(d) 9, 13, 5, . . . (Code for ARCHIMEDEANS)
(e) 5, 6, 7, . . . (!)

Average marks obtained out of 10 were as follows

No.	1	2	3	4	5	6	7	8	9	10
Marks	1.9	0.6	2.2	6.3	0.2	5.2	4.5	3.6	0.4	4.5

The winners were A. G. Smith and P. N. Toye, both of Trinity College, with 53% of possible marks (4 α-questions). 17 pairs competed, the average mark being 29.5%.

Book Reviews

SCHOOL MATHEMATICS PROJECT, Books 1, T, T4 and Teacher's Guides for Books 1 and T. Edited by A. G. Howson. (C.U.P.) 21/- each.

The Director of the Project, Professor Bryan Thwaites of Southampton University, in his introduction to the series says: 'the project was founded on the belief held by a group of practising school teachers, that there are serious shortcomings in traditional school mathematics syllabuses and that there is a need for experiment in schools with the aim of bringing these syllabuses into line with modern ideas and applications. . . . .'. To this end, the group concerned has succeeded admirably.

If the publishing schedule goes according to plan, pupils using Book 1 (intended for pupils at the beginning of their secondary school career) will be able to go on and use Books 2, 3, and 4 to cover the complete O-level course. Incidentally it is pleasing to note that all the examining Boards set joint G.C.E. examinations on syllabuses which follow the contents of these books. Eventually Books 6 and 7 will be available for the S.M.P. A-level course, while Book 5 will lead to the S.M.P. Additional Mathematics examination. Book T is intended for those pupils at the 13+

stage who for some reason will not have started the course at the 11+ stage and for those schools which wish to change over to S.M.P. without waiting for the full series Books 1 to 4 to be published. Book T4 is a continuation of Book T (and of course of the future Book 3).

Besides introducing topics which are new to an O-level syllabus, such as flow-charts, desk calculating machines and the idea of a set, Book 1 treats many older basic topics which have been retained, in an unfamiliar way. So for instance, chapter one deals with arithmetic by introducing the arithmetic operations in different number scales. Geometry is dealt with in more or less traditional fashion, but there is a whole chapter on symmetry and rotations. Throughout, the emphasis is on familiarising the pupil with mathematical concepts rather than making him adept at solving lengthy numerical problems—this is one of S.M.P.'s basic axioms, and a good one it is too.

There are included in Book T some topics which are in Book 1. The language and notation of sets are introduced from the beginning, and used to unify the various branches of mathematics. Euclidean space is studied by means of the geometrical transformations of rotation, reflection and translation. Trigonometry is given an entirely traditional treatment, but estimation of results and the use of the slide rule are encouraged. An important chapter on the displaying of data and interpretation of graphs forms the basis of a first study of statistics which will be taken a step further at the beginning of Book T4. Indeed, there are three chapters devoted to graphical work, the other two introducing functional ideas and co-ordinates. Shearing, as a transformation, is used as a starting point for studying the areas and volumes of simple figures.

Book T4, after the opening chapter on statistics, goes on to deal with transformations and matrices. Representing simple transformations as single letters, the algebra of translations, rotations, and reflections is built up. Matrices are introduced as means of storing information and are then neatly connected with transformations. Chapter 4, on more traditional lines, deals with trigonometry of the general angle, polar co-ordinates and the equation of the straight line, which leads to a chapter on linear programming, inequalities and problems of maximising and minimising. There follows more work on vectors (a vector base is introduced) and matrices. Then probability is introduced, employing set notation wherever possible. The following chapters deal with the geometry of the circle, proportion, areas under graphs using Simpson's Rule, loci and envelopes—allowing conics to be introduced—triangle geometry (where vectors are employed to advantage), solid geometry, projection and plans and elevations. Chapter 14 concerns practical, domestic arithmetical problems and the final chapter deals with algebraic structures ending with the definition of a group and checking various structures alongside the axioms.

An essential feature of S.M.P. is that not only the pupil is learning. The teacher is also being introduced to new ideas and to new methods for more established topics. It is for this reason that very comprehensive Teacher's Guides are available, providing not only answers to exercises (which sometimes include the whole working of the problem) but also a running commentary on the corresponding book, often treating topics from a different viewpoint to provide the teacher with an excellent supplementary course in themselves. Nevertheless, here are also to be found references for further reading.

All the books are pleasantly presented in colourful and very sturdy hard backs. The print is large and easy to read; the diagrams clear, explicit and plentiful. Amusing drawings and quotations start off each chapter in lively fashion and there is an adequate number of carefully chosen examples, including Revision Exercises. The undergraduates of tomorrow, coming up with this kind of mathematical training, will have a tremendous advantage over their 'traditionally' trained contemporaries. Yes, indeed, something very useful has been achieved here, and I have no misgivings whatever in wholeheartedly recommending this series. E. JENKINS

UNSOLVED AND UNSOLVABLE PROBLEMS IN GEOMETRY. By H. Mesichowski. (Oliver and Boyd) 55s.

This book deals with a variety of problems, both solved and unsolved, on which much work has been done, especially during this century. The author discusses packings and coverings in some detail, and gives the latest results on Lesbegue's 'tile' problem. The chapter on extremal problems includes a proof of Besicovitch's theorem. The impossibility of trisecting a general angle with ruler and compasses only is proved; constructions on the sphere and some problems of set theory are treated.

A topic of particular interest is the dissection of squares into smaller squares. Such a dissection is called perfect if all the squares are incongruent, and simple if no subset of the squares as arranged form a rectangle. We investigate the smallest number 1 (m, n) of squares used in a perfect (simple, simple and perfect) dissection of a square. It is known that m = 13; figure 1 on page 2 gives the corresponding dissection. l and n are unknown: $13<l\leq 24$ and $13<n\leq 38$.

Now consider a uniform conducting square plate with line electrodes on two parallel sides. Since the current flows normal to the electrodes throughout, we can cut the plate in this direction and introduce electrodes parallel to the original ones, to give a dissection of the plate into smaller square plates, without affecting the resistance of the system. If the original plate has unit resistance so do the smaller plates. Thus each dissection into squares corresponds to a system of unit resistances connected so as to have unit resistance. A perfect dissection corresponds to a network in which no simple parallel or series circuits occur, and so one of the two smallest such networks is given by figure 2 on page 2.

Thus even geometry has its uses! C. J. MYERSCOUGH

GEOMETRY, A CONTEMPORARY COURSE. By Dr. H. Lewis.

The introduction to this book tells us that it 'aims to bring the teaching of geometry more closely in line with current trends, without breaking the ties with the past.' This, together with its American origin, meant that I was prepared for many diagrams, and a fairly verbose text. However, I expected, too, a discussion of some of the newer ideas, such as simple transformations, reflections, and groups of symmetries. With what, in fact, does this book deal, and in what way is it 'more closely in line with current trends'?

The subject material is elementary Euclidian geometry, with some U mathematical works thrown in—'set' and the like—but the treatment differs greatly from that traditional. Instead of stating some definitions fairly quickly and developing the subject somewhat, then returning to consider the definitions more carefully, there is a laborious discussion of what is meant by 'equals', and an explanation of the proof of each result which would make the English staff go on strike, I hope!

The author has tried, albeit unsuccessfully, to produce a logical treatment for 13-year olds. On page 87 we are told 'if we were to deny the antecedent, this would not imply the truth of a denial of the consequent.' We state the 'Pons Asinorum' theorem on page 171 and culminate over 600 pages with a short section on 'the areas of polygons'.

This text is too long for the student to read himself (which seems to be the author's idea) and is too restrictive in style for teachers, whose interpretations tend to vary slightly.

M. A. LEWIS.

PUZZLES AND PARADOXES. By T. H. O'Beirne (Oxford) 30s.

Readers of the magazine 'New Scientist' will probably recognise the above title and author. This book is in fact a selection from a series of articles by T. H. O'Beirne which appeared in the magazine from January 1961 to February 1962.

But this is no jumble of miscellaneous and unrelated trivia—each problem is fully investigated in a chapter of its own, and usually more than one line of attack is used.

Problems treated include River-crossing, Coin-weighing, strange geometries, colour arrangements, games such as Nim, how to find Easter, the 'A affirms that B denies that C declares that D is a liar' problem, and the problem of the cows, pigs, and sheep.

The occasional analysis of results of competitions provides additional amusement—particularly one entrant who succeeded in crossing a river by boat in an <u>even</u> number of trips.

I. N. STEWART.

MATHEMATICAL INDUCTION. By B. K. Youse. (Prentice-Hall) 24s.

This book is not wholly bad; it is always perfectly clear, it is well set out and printed, and there are plenty of illustrative examples. Yet it succeeds in being very dull and not in the least stimulating in the course of its 55 pages. J. S. DENNIS.

RINGS AND RADICALS. By N. J. Divinsky (Allen and Unwin) 42s.

Ring Theory tends (undeservedly) to be a rather neglected subject. Vast quantities of texts on group theory and field theory appear every year, but the lowly ring seldom rises above the status of a sadly defective field.

This book is an attempt to plug the gap down which much potential seems to be disappearing, by providing a medium-level text.

Let P be a property that a ring may possess, and call a ring with this property a P-ring. Then P is a radical property

if 1) A homomorphic image of a P-ring is a P-ring,

2) Every ring contains a P-ideal S which contains every other P-ideal of the ring,

3) The factor ring R/S contains no non-zero P-ideals.

By means of such radical properties it is possible to prove a number of powerful structure theorems expressing rings with the given property in terms of matrix-rings over a division ring.

Eight different radicals are considered, and the final chapter gives a survey of recent results, particularly on the relations between the various radicals. I. N. STEWART.

INTRODUCTION TO FIELD THEORY. By I. T. Adamson (Oliver and Boyd) 12s. 6d.

This is a competent, though rather slow-moving, introduction to field theory, and Galois theory in particular. It is extremely explicit in treatment—perhaps too explicit, for example, the theorem that if F is a subfield of E and E is a subfield of K then $(K:F) = (K:E)(E:F)$ requires a page and a half of proof.

It ends, as might be expected, with the theorem that quintics cannot be solved by radicals. However, the actual theorem proved is that generic polynomials of degree > 4 are not solvable by radicals. This does not prove that there exists a quintic with rational coefficients which cannot be solved in terms of radicals (although there is no 'general formula' it might just happen that each equation can be solved by a different method). In fact there do exist such quintics, e.g. $x^5 + x + 1$, but a little more work is needed to prove this.

On the whole, however, this is a good buy for anyone interested in fields. I. N. STEWART.

THE THEORY OF FINITELY GENERATED COMMUTATIVE SEMIGROUPS. By L. Rédei. (Pergamon) 84s.

This very specialist book is an attempt to provide a structure theory for finitely generated commutative semigroups (hereafter called semimodules).

The method of attack used is to note that any semimodule is a homomorphic image of a free semimodule on n generators for some finite n, which can itself be embedded in the free Abelian group of rank n.

The problem thus reduces to finding all possible congruences on a free semimodule. This is done by embedding the free Abelian group in n-dimensional Euclidean space, to make the theory more intuitive.

The author then introduces the concept of a 'Kernel function' (which is a function whose values are ideals of the free semimodule) and shows that there is a natural one-to-one correspondence between kernel-functions and congruences.

This effectively reduces the problem to considerations of Kernel functions, the properties of which are developed in the remainder of the book. I. N. STEWART.

LECTURES ON DIFFERENTIAL GEOMETRY. By S. Sternberg (Prentice-Hall)

This book competently discusses a great variety of topics in differential geometry. However, the exposition is unclear in many places and typographical errors are frequent. The author comments in his preface: 'I hope that there are no serious logical errors in the book. I do not have any similar aspirations on such matters as signs, factors of 2π, etc.' Indeed, I found no serious logical flaws, but the second sentence of the author's remark should definitely be kept in mind while reading his book. T. W. CUSICK

THE STRUCTURE OF THE REAL NUMBER SYSTEM. By L. W. Cohen and G. Ehrlich (Van Nostrand)

This book presents a detailed and rigourous construction of the real number system. An introductory chapter sets up the necessary set theory. The natural numbers are introduced by Peano's axioms, and the concepts of finiteness and infiniteness of sets established. The integers, rational numbers, and real numbers are each introduced as providing properties lacking in the previous set. It is finally shown that the reals are complete, whereas the rationals are not.

The necessary algebraic theory is set up at each stage, and the concept of order is stressed at each stage. Exercises are provided as an integral part of the text, many of them as important extensions of it. An adequate index and bibliography are provided. The book should be very useful to students of the foundations of analysis.

M. K. AYRES.

REAL NUMBERS. By Stefan Drobot. (Prentice-Hall) 28s.

This book claims to be an introduction to several aspects of the theory of real numbers–the axiomatic basis, the representation of real numbers, approximation to reals, cardinalities of sets of reals, and Lebesgue measure.

However, the book is, in fact, mainly devoted to the theory of continued fractions–a subject which has been treated better elsewhere. The section on axiomatics is fairly satisfactory, but the final chapter, on Lebesgue measure, is decidedly sketchy.

P. G. DIXON.

INTRODUCTION TO TOPOLOGY. By M. J. Mansfield. (Van Nostrand) 21s.

Whilst this book is good value at a guinea, it contains substantially less than the requirements of the Part II Analytic Topology course. But since the treatment is careful and thorough, the book can be strongly recommended as introductory reading for that course.

The basic ideas of continuity, topology, functions, continuous mappings, and homeomorphisms are first introduced. The reader is led to seek topological properties–i.e. those properties of a topological space which remain invariant under homeomorphisms. Such topological properties as connectedness and compactness are then examined. The discussion of separation properties includes a proof of Urysohn's lemma. Finally metric spaces are discussed. It is regrettable that the author devotes an excessive amount of time to the process of completion whilst not relating compactness to metric spaces.

The trouble with this book is that as soon as discussion of a topic becomes interesting, it stops. But perhaps this is a good reason for recommending it as introductory reading.

J. H. WEBB.

THE GENESIS OF POINT SET TOPOLOGY. By J. H. Manheim. (Pergamon) 25s

This little book consists of a history of those developments in mathematics from the time of Newton, which culminated in the foundation of the modern subject of point set topology. The book lays particular emphasis on the impact of the study of Fourier and other trigonometrical series on mathematics in the last century.

I found the book readable, and generally free from errors, but, unlike many mathematical historians, the author assumes considerable mathematical knowledge on the part of the reader. This is certainly not the book to give Aunt Edna for Christmas unless she happens to have a Ph.D. But I can strongly recommend it to those taking the Part II course in Analytic Topology, as background reading. There are plenty of quotations from original papers, dozens of references, and a very full bibliography.

J. S. DENNIS

THE THEORY OF SHEAVES. By Richard G. Swan (University of Chicago Press)

The theory of sheaves has important applications in topology, differential geometry, and similar branches of mathematics; its main use being to deduce global properties from local ones. This book concerns itself with the general theory of (abstract) sheaves. The reader should have some familiarity with the basic methods of homological algebra, which are used intensively throughout.

I. N. STEWART.

THEORY OF FUNCTIONS. By L. F. Toralballa. (Prentice-Hall) 94s

At first sight, this book looks like the answer to every impecunious undergraduate's prayer. In one volume, the author manages to cover three-quarters (at least) of a three-year university course in analysis. This is a very ambitious project, and it is scarcely surprising that it is not a complete success.

First the rationals, the reals, and the complex numbers are developed from the positive integers. After some elementary topology, in preparation for a general discussion of differentiation and integration over normed fields, come real functions of a real variable, measure and the Lebesgue integral, Fourier series, and functions of several real variables. The next (and longest) section is on complex variables. For a short, unadorned account of the theory, with clear and straightforward proofs of all the principal theorems, it would be difficult to beat. The concluding chapters deal, respectively with harmonic functions, a more rigorous survey of the ground covered, and functions of several complex variables.

Since the book tries to cover so much ground, it is inevitably rather superficial in its treatments of nearly all subjects. It is often pedestrian, there being only the briefest explanatory comments, and no digressions to encourage the reader to look further into the subjects covered. The order in which the material is covered is, whilst logical, unsuitable for a first year undergraduate. The book is very careless over details. Many of these faults are due to the author, but there are also very many misprints. I advise any prospective purchaser of this book to study it carefully before buying.

For the most part I have been critical of this book, but I ought to conclude with some praise. It is comprehensive, generally sound, and rarely obscure, but it is a pity that there are so many flaws.

J. S. DENNIS

A COURSE OF HIGHER MATHEMATICS. Vol. III Part II 'Complex Variables/Special Functions' By V. I. Smirnov (Pergamon) 110s

This book continues the translation of Professor Smirnov's course. Chapters I, II and III develop the theory of functions of a complex variable in the usual way. Cauchy's theorem is proved assuming continuous differentiability of the integrand, and a large number of applications to the theory of two-dimensional fields is included. Chapter IV contains material on functions of several complex variables and matrix functions, with applications to differential equations. Chapter V considers linear differential equations and their solutions about regular singularities; there is a detailed treatment of the hypergeometric equation and its special cases, Legendre's and Jacobi's equations. Chapter VI consists of 200 pages of information on special functions. There is a wealth of reference information in both these chapters.

The book is large, easy to read and well printed. It is rather classical in tone, and much of the material duplicates the contents of much cheaper and more rigorous books. It is well suited to the college library, but seems too expensive and not altogether indispensable to the undergraduate.

S. M. NEGRINE

FUNCTIONAL ANALYSIS IN NORMED SPACES. By L. V. Kantorovich and G. P. Akilov. (Pergamon) £7

This large volume is divided into two parts, the first provides a good general introduction to the theory of normed spaces---properties of normed spaces, (including Hilbert space); functionals; the Hahn-Banach theorem; conjugate spaces; convergence; adjoint operators; linear topological spaces---with, throughout, many applications of the theory to specific spaces. The second part is more specialised, being devoted to methods of solving functional equations---the method of steepest descent, the use of fixed point theorems, and Newton's method. Here, again, the general theory is illustrated by applications to specific integral and differential equations.

P. G. DIXON

A DISCOURSE ON FOURIER SERIES. By C. Lanczos. (Oliver and Boyd) 63s

This book deals with a wide variety of topics connected with the approximation of functions by Fourier series. The first few sections are rather confusing; historical notes on Fourier's discoveries alternate with the basic definitions of function and limit. The proof of Dirichlet's fundamental theorem for Riemann-integrable functions of bounded variation over the interval considered is a little difficult to follow.

The book then improves considerably; Fejer's method of summation, smoothing of experimental data, and the Laplace transform are among the topics discussed. The whole book is written in an informal 'Question and Answer' style which while pleasant to read lacks conciseness. Whilst the book fills the gap in standard of treatment between elementary and advanced texts, one cannot help feeling that without loss of subject matter the book could have been condensed to half the length and published at one fifth of the price in Oliver and Boyd's 'University Mathematical Texts' series.

C. J. MYERSCOUGH

A TREATISE ON TRIGONOMETRICAL SERIES. By N. K. Bary (Pergamon) Volume 1 : 84s. Volume 2 : 105s.

There are now two large works on the classical theory of trigonometric series, the present work which is a translation from Russian, and Zygmund's Trigonometric Series (2nd edition Cambridge 1959). Although the contents are to a great extent the same the two books are quite different in spirit; Zygmund is concise, high-powered and demands mathematical maturity, while Bary is unhurried and claims to cater for undergraduates who know Lebesgue integration. Miss Bary is as good as her word, for in her book trade secrets are passed on to the reader and the phrase 'easily seen' never calls for the usual half-hour with pencil and paper. The first chapter gives a very clear 160-page introduction to the basic concepts, including all the usual convergence tests and summation methods. Later chapters deal, among other things, with the convergence and divergence of a Fourier series in a set, conjugate series, absolute convergence, lacunary series, general trigonometric series and, one of the author's special interests, problems of uniqueness and multiplicity. Two chapters, those on adjustment of functions in a set of small measure, and on the representation of a function by a trigonometric series, contain material not to be found in Zygmund's book; otherwise Zygmund contains much more.

The book is beautifully produced and well-indexed. This English translation sometimes reads a little oddly, but the reviewer is not qualified to comment on its accuracy; R. P. Boas Jnr. has said that is plain wrong in places.

C. L. THOMPSON

PRINCIPLES OF RANDOM WALK. By F. Spitzer (Van Nostrand) £5

This is a fascinating and beautifully written account of one of the most highly developed branches of probability theory. The prerequisites are slight-anyone who has read the greater part of the first volume of William Feller's Introduction to Probability Theory and its Applications should be well equipped to embark on this intensive study of random walks although, as is inevitable in a book in which a large number of concrete problems are investigated in detail, there are occasional involvements with more sophisticated analytical techniques than Feller allowed himself. The potential theory of random walks is well represented, and anyone who wonders why capacity is of interest to probabilists will find the answer here. While the book is unreservedly recommended, the reader is reminded that a Markovian random motion need not be a 'random walk', and that not all random motions are Markovian. Thus this book should be supplemented by, for example, the second volume of Feller's work, if a bird's eye view of the whole subject is required.

D. G. KENDALL

VECTOR ANALYSIS FOR MATHEMATICIANS, SCIENTISTS, AND ENGINEERS. By S. Simons (Pergamon Press) 17s. 6d.

This book provides an easily understood course, reaching the divergence and Stokes' theorems, and curvilinear co-ordinates. Its principal defect is lack of rigour, for example in statements of the conditions under which theorems hold. The intention of the author is to omit 'finer mathematical points', but this is carried too far. The book lacks examples of the application of vector analysis in science and engineering. There is no index.

Not a book to be recommended, particularly as there are already so many on the subject.

M. K. AYRES

PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS. By S. L. Sobolev (Pergamon) 100s.

The author is a very distinguished Soviet mathematician who based this book on a course of lectures given by him in Moscow. The book is fairly self-contained, requiring only a knowledge of elementary analysis.

About half of the book consists of standard material on partial differential equations-the equations are derived from physical situations and classified into elliptic, hyperbolic and parabolic forms; for the hyperbolic equation the method of characteristics is explained and illustrated in superb lectures on D'Alembert's and Riemann's methods, and the wave equation for a retarded potential is also treated. The heat equation is discussed briefly, but there is much material on Laplace's and Poisson's equations, with harmonic and spherical functions. Particularly noteworthy is the treatment of Green's function for the Dirichlet and Neumann problems for the circle and plane.

The book also contains a treatment of multiple integrals and develops the theory of Lesbegue integration. There are also lectures on Green's function for ordinary differential equations and a full treatment of the Fredholm theory of integral equations, reaching the Hilbert-Schmidt expansion.

This book is indispensable to college libraries and is a very good buy for Part II undergraduates. However, it is rather ambitious in scope. It is a great tribute to Soviet mathematics that 'pure' topics such as Lesbegue integration and integral equations are unified with 'applied' partial differential equations, but it is a bad idea to have them all in one book.

S. M. NEGRINE

PROBLEMS OF MATHEMATICAL PHYSICS. By N. N. Lebedev, I. P. Skalskaya, and Y. Š. Uflyand. (Prentice-Hall). 70s.

Competently translated by R. A. Silverman, this is a further addition to Prentice-Hall's series of mathematical works originally published in Russian. The subject-matter is presented well, with plenty of diagrams and an extensive bibliography.

The work consists mainly of 566 problems, divided into chapters, at the beginning of each of which the mathematical techniques to be used are briefly explained. Answers to all the problems are given, with hints where necessary, and about eighty are solved in moderate detail in the second part of the book. The prospective buyer should note that the authors have confined themselves chiefly to mechanics, elasticity and electromagnetism, with little else. The variation in difficulty of the problems is very large however, ranging upwards from the fairly elementary. The general level seems to be just about right for the undergraduate. P. J. BUSSEY

Topological equivalents of the axiom of choice—continued from page 30

We conclude this article with some problems.

(i) What form of the axiom of choice is equivalent to Urysohn's Lemma?

Dependent choice is required in the usual proof of it, but it is not clear that the lemma implies even a simple form of countable choice.

(ii) Are the theorems:

'A totally bounded metric space is separable.' and: 'A totally bounded metric space is second-countable.' equivalent to IV? (A metric space is said to be totally bounded if for any $\delta > 0$ it is covered by a finite number of spherical neighbourhoods radius δ. This property is sometimes called precompactness).

(iii) Is the following theorem equivalent to some form of the axiom of choice?

'If a continuum is a Hausdorff space, then each two of its points lie in a subcontinuum irreducible between the two points.' (c.f. ref. 4 p. 44).

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