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No. 28

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THE JOURNAL OF THE ARCHIMEDEANS
The Cambridge University Mathematical Society; Junior Branch of the Mathematical Association

Contents

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Editorial

WHEN I started out to prepare this issue of *Eureka* for publishing, I wondered how to make it just that little bit different from its predecessors. A colour supplement, perhaps? Free autographed photo of Titan?

For reasons which I cannot go into, these and several other schemes failed to come to fruition. Then the answer struck me. I noticed that all previous issues had been called "*Eureka* no. n" where $n \leqslant 27$. To be different, I decided to call this issue number 28. This gives it the correct (and customary) degree of uniqueness.

Which reminds me of a number of interesting and totally irrelevant facts concerning the number 28. It is, of course, a perfect number (equal to the sum of its divisors), for $1 + 2 + 4 + 7 + 14 = 28$. It is the sum of the first 7 integers. It is also the sum of the first 7 digits in the decimal part of π. The only other numbers which are simultaneously the sum of the first n digits of the decimal part of π and also the sum of the first n integers are 1 and 6 (I have an excellent proof of this, but the margarine is too small to contain it, so I will have to work out a butter proof). And these are also perfect numbers! Does this hide some deep property of π?

The Archimedeans

THE society has again enjoyed a successful year with large attendances at most of its meetings. We are fortunate to be addressed by such well-known mathematicians and are grateful that they can spare us their time. Particularly notable during this past year were an excellent talk by Professor Zeeman on "Lens Spaces," and a "Careers Meeting" organised by Mr. J. N. Coope from the Appointments Board, at which we heard four mathematicians from various walks of life talking about their own work and branches of mathematics. Our tea meetings too were successful, and we were pleased that a number of "Invariants" from Oxford took part in the Problems Drive. The Computer Group, with a growing membership, meets fortnightly, and from the number of programmes written appears to keep Titan fairly busy. Both the Music Group and the Bridge Group are flourishing, but the Chamber Music Group has been suffering from small attendances. The proposed dinner was not held, possibly owing to lack of cash in undergraduate pockets, but it was disappointing that the punt party also had to be cancelled through lack of support.

This year's meetings begin with a talk from Professor L. Fox, the director of Oxford's Computing Laboratory, on "The Mathematics of Digital Numbers." This will be followed by a talk from Dr. H. M. Cundy, joint author of "Mathematical Models," on "Tiling and Patterns." Also in the Michaelmas term we have a talk from Professor S. Vajda on the Theory of Games, and tea meetings include a talk by Dr. J. V. Narlikar on "Observations and Theories of the Universe." The Lent term opens with a talk from Professor M. F. Atiyah, and also includes a talk postponed from last year by Professor W. K. Hayman of Imperial College, London on "Symmetrisation." In February we will be holding another Problems Drive, and a careers meeting, again arranged by Mr. Coope.

It is hoped that the programme contains items to interest all members of the society, but any suggestions of possible improvements will be welcomed by the secretary, either personally or through the book kept near our notice-board in the Arts School.

R. H. J. Harris, *Secretary.*

Postal Subscriptions and Back Numbers

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Some copies of *Eureka* Nos. 24, 25, 26, 27 (2*s.* 6*d.* each) are still available (postage 2½*d.* extra per copy). We would be glad to buy back any old copies of Nos. 1 to 14 which are no longer required.

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Transcendental Numbers

By C. J. Myerscough

An *algebraic number* is a complex number x which satisfies an algebraic equation of finite degree n, i.e.

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_n = 0 \quad \ldots \quad \ldots \quad (1)$$

where $a_0, a_1, \ldots a_n$ are integers, and $a_0 \neq 0$.

If x satisfies an equation of degree n, but none of lower degree, we say that x is of *degree n*.

Evidently all rational numbers are algebraic; other examples are $\sqrt{2}$, $3\sqrt[3]{7} + 4\sqrt[5]{10}$, i, and quantities such as the smallest real root of $x^5 - x + 1 = 0$, which cannot be expressed as a finite combination of radicals.

It can be proved that any root of a polynomial with algebraic coefficients is also algebraic.

A number which is not algebraic is called *transcendental*. By considering the set of all algebraic equations of degree n, it is clear that the set of all algebraic numbers is countable, and so has plane measure zero, and the set of all real algebraic numbers has linear measure zero. Thus almost all (real or complex) numbers are transcendental.

However, the task of determining whether any given number is transcendental or not is generally difficult, and only limited progress has been made. In 1873 Hermite proved that e is transcendental, and in 1881 Lindemann proved that π is transcendental. More recently, in 1934, Gelfond and Schneider proved that α^β is transcendental if α and β are algebraic, α is not 0 or 1, and β is irrational. It follows from this result that e^π, which is one value of $(-1)^{-i}$, is transcendental. The Bessel functions $J_0(x)$, $Y_0(x)$ are transcendental for algebraic non-zero x. However, it is not known whether $e + \pi$ or $e\pi$ are transcendental, and it is not even known whether 2^e, 2^π, π^e, or Euler's constant γ are irrational.

It is fairly easy to produce examples of transcendental numbers. This was first done by Liouville in 1844, using the theory of the approximation of irrational numbers by rationals. His work will now be outlined, together with that of Mahler (1932) which gives a classification of transcendental numbers into 3 different types.

A real number x is *approximable to order n* if there exists a quantity K depending only on x for which

$$|p/q - x| < K/q^n$$

has an infinity of solutions in coprime integers p, q.

A rational number is approximable to order 1, since the equation $bp - aq = 1$ has an infinity of solutions in coprime integers p, q; and so we have

$$|p/q - a/b| = 1/qb$$

By means of the theory of continued fractions it may be shown that any irrational is approximable to order 2. LIOUVILLE proved as follows that a real algebraic number of degree n is not approximable to any order greater than n.

Let x satisfy (1). Now there exists a quantity $M(x)$ such that $|f'(y)| < M$ for $x - 1 < y < x + 1$.

Suppose that $p/q \neq x$ is an approximation to x. Without loss of generality we may assume that $x - 1 < p/q < x + 1$, and p/q is closer to x than any other root of $f(z) = 0$, so that $f(p/q) \neq 0$.

Then $|f(p/q)| = (|a_0p^n + a_1p^{n-1}q + \ldots + a_nq^n|)/q^n \geqslant 1/q^n$.

By the Mean Value Theorem,

$$f(p/q) = f(p/q) - f(x) = (p/q - x)f'(\xi)$$

where ξ lies between p/q and x. So

$$|p/q - x| = (|f(p/q)|)/(|f'(\xi)|) > 1/Mq^n.$$

Since M depends only on x it follows that for any $K > 0$ there can exist only a finite number of p,q such that

$$|p/q - x| < K/q^r \text{ for } r > n,$$

and so x is not approximable to any order greater than n. This result has been further refined; in 1955 ROTH proved that no irrational algebraic number is approximable to any order greater than 2. Thus any more rapidly approximable number must be transcendental.

Now consider $x = \sum_{k=1}^{\infty} (-1)^k a^{-b_k}$,

where $a > 1$ is a fixed integer, and $b_1, b_2, \ldots$ is an increasing sequence of positive integers such that

$$\limsup_{n \to \infty} (b_{n+1}/b_n) = \infty.$$

Given $w > 0$ there exists an n depending only on w for which $b_{n+1}/b_n > w + 1$. If we put

$$q = a^{b_n} \text{ and } p = q \sum_{k=1}^{n} (-1)^k a^{b_k}$$

then p and q are integers, and

$$0 < |qx - p| < q/a^{b_{n+1}} = q^{1-(b_{n+1}/b_n)} < 1/q^w.$$

Thus x is approximable to any degree, and is transcendental. A number approximable to any degree is called a *Liouville number.*

Now it would appear possible to obtain an idea of how near a transcendental number is to being algebraic by considering how nearly a polynomial function of the number of given degree can be made to approximate to zero by suitably choosing the coefficients, subject to some restriction on their size. This leads to the following definition:

A function $\phi(n,t)$ is called a *transcendence measure* for the transcendental number z if for each n there exists a constant c_n such that, for every $X \geqslant 1$, all sets of numbers $x_0, \ldots, x_n$ such that

$$X = \max(|x_0|, |x_1|, \ldots, |x_n|)$$

have the property that

$$|x_0 + x_1 z + \ldots + x_n z^n| > c_n \phi(n,X).$$

It may be proved that $\phi(n,t) \leqslant t^{(1-n)/2}$; this maximum is attained for $z = e$.

MAHLER has extended this theory by letting

$\alpha_n(X) = \min(|\sum_{k=0}^{n} x_k z^k|)$, where the minimum is taken over all those sets of integers $x_0 \ldots, x_n$ such that

$$\max(|x_0|, \ldots, |x_n|) \leqslant X \quad \text{and} \quad \sum_{k=0}^{n} x_k z^k \neq 0.$$

Evidently, $\alpha_n(X) \leqslant 1$ and is a non-increasing function of both X and n. Put

$$\begin{aligned} \rho_n(X) &= -\{\log \alpha_n(X)\}/\log X, \\ \omega_n &= \limsup_{X \to \infty} \rho_n(X), \\ \omega(z) &= \limsup_{n \to \infty} (\omega_n/n), \end{aligned}$$

so that ω_n and ω are either $+\infty$ or a non-negative number. If ω_n is infinite and $m > n$, ω_m is also infinite; hence there exists a quantity $\mu(z)$ which is either finite or infinite such that ω_n is finite for any $n < \mu$ and infinite for $n \geqslant \mu$. Now μ being finite implies that there exists a finite N such that $\omega_N = \infty$, and so $\omega = \infty$.

We call z

an *A-number* if $\omega = 0$, $\quad \mu = \infty$

an *S-number* if $0 < \omega < \infty$, $\quad \mu = \infty$

a *T-number* if $\omega = \infty$, $\quad \mu = \infty$

a *U-number* if $\omega = \infty$, $\quad \mu < \infty$.

The importance of this rather artificial-looking classification is that if there exists a polynomial $F(x,y)$ with integer coefficients such that $F(u,v) = 0$, then u and v belong to the same class.

The A-numbers are found to be precisely the algebraic numbers.

If μ is finite, there exists an n such that for every $\sigma > 0$ there exist integers $x_0, \ldots, x_n$ such that

$$|x_0 + x_1 z + \ldots + x_n z^n| < X^{-\sigma}$$

For $n = 1$, this is equivalent to the definition of the Liouville numbers. Thus the U-numbers may be regarded as higher degree analogues. LE VEQUE showed in 1953 that there exist U-numbers of every degree.

e is an example of an S-number. It is still not known whether T-numbers exist.

BIBLIOGRAPHY

This article is based mainly on chapter 5 of *Le Veque—Topics in Number Theory Vol. II—Addison-Wesley.* Proofs of theorems omitted here can be found there. *Hardy and Wright—Introduction to the Theory of Numbers—Oxford* also contains much useful material.

Problems Drive

The Editor recently received a letter from a reader in Canada suggesting that we publish all the past Problems Drives in book form. We would be pleased to hear readers' views on the matter. In the meantime, here follows a selection from past Problems drives (date in brackets).

1. (1947). Prove that the determinant a_{ij} such that $a_{ij} = (i+j)^{i+j}$ is composite or zero.

2. (1949). Since the building of the 200-in. telescope several towns have been observed on Mars. They have been provisionally named by Astronomers A, B, C, D, E, F, G, H, and J. They are connected by a series of canals which are used by the Martians as a means of transport, since each night they have been seen punting their massive gondolas along these waterways. According to the latest reports the canals run as follows:

   A to B, D, E, F, H, and J.

   B to C, D, and G.

   C to D, G, and H.

   D to H.

   E to G and J.

   F to G and H

   G to H and J.

   Moreover no two canals cross except at a town. Into how many regions is the surface of the planet divided by these canals?

3. (1951). Find unequal rational numbers p,q (other than 2,4) such that $p^q = q^p$.

4. (1951). A large town lies on a main road and has an unrestricted semicircular bypass; in the town the speed limit is 30 m.p.h., and the traffic lights are spaced on an average 220 yards apart. The red and green each last half a minute (amber may be neglected). Assuming your car can stop and start instantaneously, what must be its maximum speed to make it worth while taking the bypass?

5. (1953). A gambling game is played as follows:—

A player throws the same dice twice, and the scores are added up.

If the score adds up to an *odd* number, the player wins.

If it adds up to an *even* number, he loses.

In the unlikely event of a house allowing a dishonest player to provide his own dice, how would he bias the dice, in order to give himself an improved chance of winning?

6. (1954).

$$\begin{array}{r} \text{O N E} \\ \text{T W O} \\ \text{F O U R} \\ \hline \text{S E V E N} \end{array}$$

S is not zero, and no digit is repeated.

7. (1954). "What's wrong with these dice?" I said suddenly. "I've just rearranged them a bit," said Sam. "The chances of a 7 are better by a third, and I've changed as few other chances as possible."

What were the numbers on the two dice?

8. (1957). A pile of counters, coloured red, orange, yellow, green, and blue is given to 4 men, 3 of whom are colour-blind.

One confuses yellow and orange.

Another confuses green and blue.

The other confuses red and orange, also green and yellow.

Each writes down how many he thinks there are of each colour. You are given the guesses, but don't know who made which set of guesses. Give a way of deducing how many counters there are of each colour. Can you always find out who made which set of guesses?

9. (1959). In the year 1960, question 3.14159265. . . . of the Archimedeans' Problems Drive was: "Find the next term in each of the following series:

(i) 5, 7, 9, 13, 17, 19, 21, 25, . . .

(ii) 0, 1, 1, 3, 11, 43, 225, 1393, . . .

(iii) 1, 3, 7, 13, 19, 31, 43, . . ."

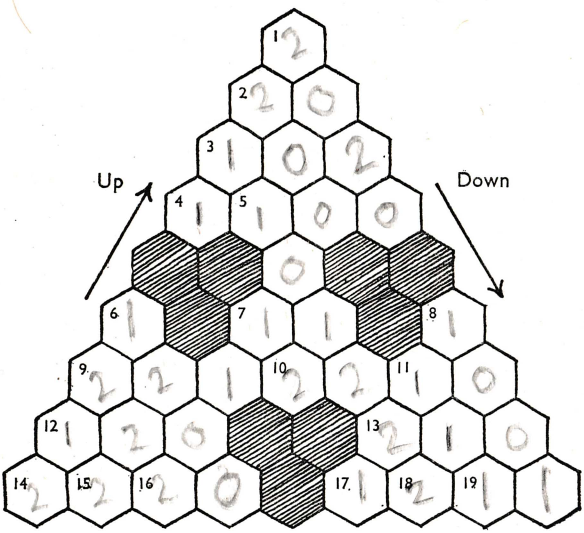
Unfortunately, the printer had altered one number in each of the three series. Find his mistakes.

10. (1961). In how many distinct ways can the faces of a cube be coloured using every colour at least once, if the number of colours available is:

(*a*) 2, (*b*) 5?

Three

BY PEREGRINE



Clues and solution are in the scale of 3. $a, b, c, d, e, f, g, h, i, j$, are the first ten triangular numbers in some order.

UP

4. $e - 1$
5. $f + h$
10. $c + h$
14. $j + d$
15. $i - 2$
16. $2i^2$
17. $g + a + hf$
18. g
19. $(d - a)/j$

ACROSS

2. h
3. $g - a$
4. b
7. $(b - i)/2$
9. a^2g
12. $b - g$
13. $h + j$
14. $2b + h$
17. $g + i$

DOWN

1. $b + e - g$
2. $i - a$
3. $e^2 - a^3$
6. e
7. $d/(a + c)$
8. i
9 $2(a + f)$
11. $(b + d)/(a - f)$
12. j/f

Parallel Curves

By D. R. Woodall

The concept of parallelism is one generally restricted to straight lines. My object here is to extend the definition to curves. My starting idea was that parallel curves should be something like railway lines, having everywhere a common normal and being everywhere the same distance apart measured along the normal; but other definitions are possible. In particular, two curves are often said to be parallel if they can be made to coincide by a simple translation, and I shall consider this case later. The only other intrinsically different property of parallel straight lines seems to be that used by Euclid as a definition, namely that they do not meet however far they are produced; but this does not lead to a generalisation of any interest.

Consider, then, two simple, well-behaved segments of curve AB and CD (Fig. 1). We say that two segments are *parallel* if any normal to one is also a normal to the other. It can readily be shown that in this case the two curves have the same centre of curvature along any such normal, and that two curves are parallel if and only if one of them is the locus of points which lie a fixed distance from the other measured along the normals to it: i.e. the curves are the same distance apart along all the normals. This is the only bit of real mathematics in the whole article, so I follow the well-established precedent of leaving it as an exercise for the reader!

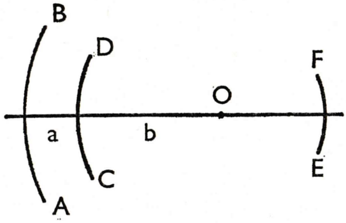


Fig. 1

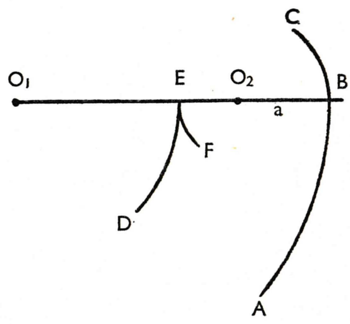


Fig. 2

As Fig. 1 shows, it will sometimes be found (though not on the railways) that the centre of curvature lies between the curves. It is convenient to distinguish this case and to call it *negative parallelism*, non-negative parallelism being called *positive parallelism*. It is also convenient to think of a curve as being described in a specific direction, which automatically fixes the direction of any parallel curve, and to define the distance of a parallel curve from it to be *positive* if it lies on the left and *negative* if it lies on the right. For example, in Fig. 1, the curve AB is distant $+a$ from curve CD, and the curve BA is distant $-a$ from curve DC. The sign of the parallelism and the sign of the distance are independent, one being an intrinsic property of the curves, the other depending on the direction of description of the curves and on which is being considered as parallel to which. If the direction of description of the curves is fixed, the parallelism has the opposite sign to that of the product of the distances of the curves from each other. Thus, in Fig. 1, curves CD and FE are negatively parallel, distant $+b$ from each other. Clearly in general, if curve MN is positively parallel, distant x, to curve PQ and PQ is positively parallel, distant y, to RS, then MN is positively parallel, distant $x+y$, to RS; and if MN is parallel (positively or negatively) to PQ and PQ is parallel to RS, then MN is parallel to RS. Thus parallelism and positive parallelism are equivalence relations.

We are now in a position to consider two situations which may arise when such segments are joined to form a continuous and differentiable curve. The first is illustrated in Fig. 2. On passing through B, the radius of curvature of the curve ABC becomes less than the distance of DEF from it, with the result that the sign of the parallelism changes, producing a cusp at E. This cusp gives rise to the following phenomenon: as the curve DEF is described, the change of direction at E causes a sudden change in the sign of the distance of ABC from it. For AB is distant $-a$ from DE, but BC is distant $+a$ from EF. Thus at a point where the sign of the parallelism changes, there is a corresponding change in the sign of the distance of one curve from the other. (The sign of the distance has no meaning, of course, at points such as E where the direction of the curve is not defined, and the sign of the parallelism no meaning at points such as B and E where the radius of curvature is not defined.) We say that two curves are *properly parallel* if the distance of each from the other, in the sense defined here, remains constant as each of the curves is described. This implies that the sign of the parallelism does not change, and that there are no cusps.

The second situation is illustrated in Fig. 3. Here segment AB is parallel to EF, and BC to DE, so there is clearly a sense in which ABC is parallel to DEF, but there is a pretty violent discontinuity

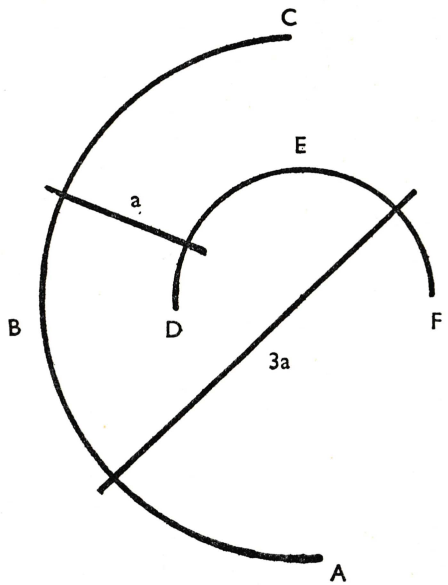


FIG. 3

at B and E. We call such parallelism *discontinuous*. Two curves are said to be *continuously parallel* if to any continuous segment of either there is parallel a continuous segment of the other. Note that proper parallelism is continuous, but that the converse need not be true (Fig. 2 providing a counter-example).

There are essentially 3 ways in which pairs of parallel segments may be put together to form "parallel" curves. Firstly (Fig. 4) the resultant "curves" may consist of a collection of such pairs arranged haphazardly without any particular continuity, and having end-points. This case is interesting only in that one has to go to great lengths to specify just what each curve consists of, especially at the end-points. Thus in Fig. 4, for example, is the

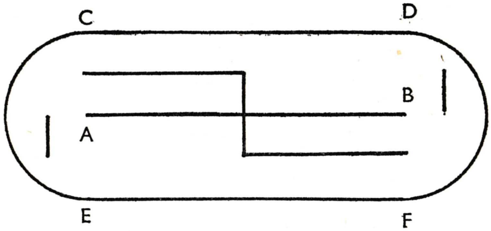


FIG. 4

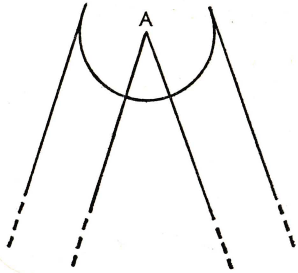


FIG. 5

line AB parallel to the line pair CD, EF? Or to the curve $CDFE$? I would prefer to say that the line AB described twice was (continuously) parallel to the line pair CD, EF, and that the curve formed by the line AB, the point semicircle at B described clockwise from direction BD to BF, the line BA, and the point semicircle at A described clockwise from AE to AC was (properly) parallel to the curve $CDFE$. But perhaps this is being pedantic.

Secondly (Fig. 5) there may be no end-points, but there may be points (such as A) where the curve is not differentiable. Similar peregrinations are necessary here.

Thirdly, the resultant curves may be everywhere continuous and differentiable. There may be cusps, but there are no end-points or points where there is a change of slope. The parallelism may be discontinuous and improper (Fig. 6), continuous and improper (Fig. 7), or continuous and proper. Two concentric circles, the

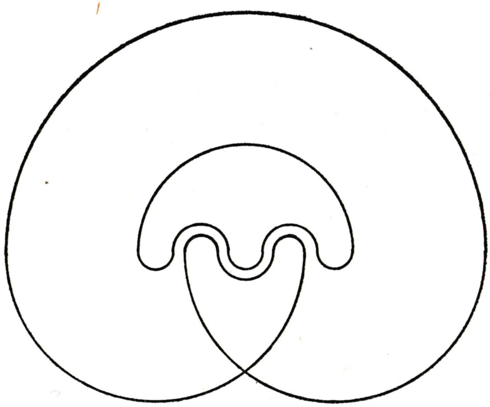


FIG. 6

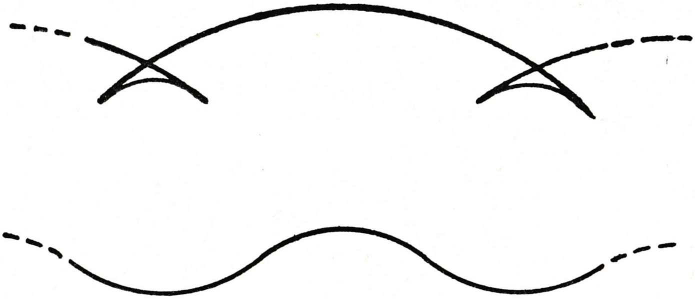


FIG. 7

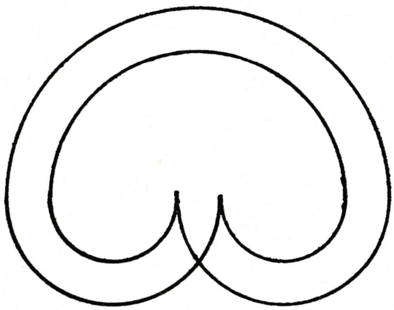


FIG. 8

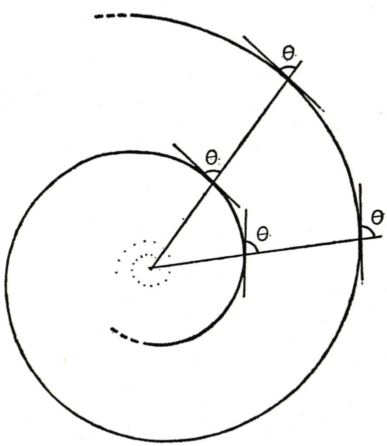


FIG. 9

outer of radius a, the inner of radius b, provide an extreme example of this case, for the circle radius a may be considered to be properly and continuously parallel

(1) positively, distant $\pm (a - b)$, to the circle radius b,

(2) negatively, distant $\pm (a + b)$, to the circle radius b,

(3) positively, distant $\pm a$, to the point circle, its centre,

(4) negatively, distant $\pm 2a$, to itself.

Circles and curves formed from sets of complete circles are, I believe, the only connected differentiable curves which are properly and non-trivially (i.e., at distance $\neq 0$) parallel to themselves; but there are other curves which are continuously parallel to themselves, the one in Fig. 8, for example.

I mentioned at the beginning that there were other possible definitions of parallelism. For example, one might say that two curves are *parallel with respect to a given set of straight lines* if each of them meets each of the lines at the same angle θ. Thus the successive coils of an equiangular (or logarithmic) spiral (Fig. 9) may be said to be parallel at angle θ to the set of straight lines through the origin. This might be called *radial* parallelism, since the set of straight lines taken is the set of "radii vectores" from the origin. If one takes instead the set of normals to the curves, this sort of parallelism reduces to that originally considered, and so this might reasonably be called *normal* parallelism. Two concentric circles are both normally and radially parallel. But in the case of non-normal parallelism, it is not possible to define the distance between the two curves in such a way that it remains constant as the curves are traced out. I also mentioned that two curves are often said to be parallel if they can be made to coincide by a simple translation, or, indeed, by translation and magnification, i.e., projection. Thus of the *pantograph*, an instrument for enlarging drawings, it is often said that as the pointer traces out the curve to be enlarged, the pencil traces out a parallel curve, in the sense that it move in the same direction. One could extend the definition to cover this by saying that two curves are *pseudo-parallel* or *semi-parallel* with respect to a given set of straight lines if any one line meets the two curves at the same angle, but different lines meet the curve at different angles. Thus translated figures are pseudo-parallel with respect to the set of straight lines parallel to the direction of translation, while projected figures are pseudo-parallel with respect to the straight lines through the fixed point of projection. Again, no invariant distance can be defined, except in the case of pure translation, when the distance moved, or shift distance, is invariant.

It must also be mentioned that there is no reason why one should not define a sort of *curvilinear* parallelism, by which two curves are parallel with respect to a given set of *curves* if they cross each of the curves at the same angle. Thus the statement that two concentric circles are parallel with respect to the set of radius vectors through their centre generalises to the statement that two confocal ellipses are parallel with respect to the orthogonal set of confocal hyperbolas. But this opens up a whole new field of geometry, and although it is a fascinating subject, I do not feel that the relation between these curves is sufficiently close to the intuitive idea of parallelism really to justify the name.

It is possible to go still further, and apply the basic definitions in 3 or more dimensions. Thus in 3 dimensions, for example, one would say that two *surfaces* were parallel if any normal to one were also a normal to the other, and that two *curves* were parallel if at any point of one there were a normal (i.e., a line in the normal plane) which was also a normal to the other. Much of the above theory will be seen to extend to these cases, although further complications arise.

An Uncommonly Pathological Recurrence Relation

BY ROGER HELMER

I recently came across the following problem whilst trying to formulate a mathematical model for an electrical system. The physical situation is this: Consider n electric terminals, and the possible way of joining them up in pairs using $n - 1$ wires. If the reader keeps this idea in mind, he will have no trouble in following the abstract formulation.

Consider n distinct points in Euclidean space of dimension $\geqslant 2$. We define an *edge* associated with two of the points to be the set of all continuous curves having the two points as end points, and passing through none of the other points. An edge is thus uniquely specified by its two end points. Moreover, if we label the n points $A_1, A_2, \ldots, A_n$, then any edge is specified uniquely by two numbers i, j such that $i \neq j$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. Note that the edge specified by i, j is the same as that specified by j, i.

Two points A_i, A_j are *joined* if either

(1) $i = j$

or (2) There exists a subset of the edges such that in the numbers specifying them, i and j appear exactly once, and the others (if any) exactly twice each.

For example, if three edges are (i,k), (k,l), (l,j), then A_i and A_j are joined. It is clear that this enables us to put the points specifying the edges in a unique sequence from A_i to A_j. In the example this would be A_i, A_k, A_l, A_j. We shall speak of this as the sequence of points joining A_i to A_j. This does not exclude the possibility that there might be another such sequence, but if so it would correspond to another set of edges joining A_i and A_j.

We observe that joining is an equivalence relation. From the definition it is clearly reflexive and symmetric. We shall prove transitivity. Suppose both A_i and A_j are joined to A_k, as in Fig. 1.

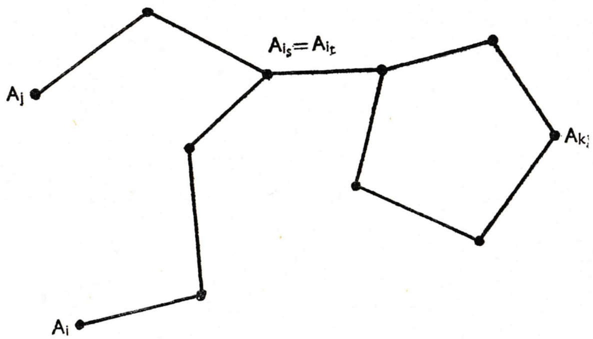


FIG. 1

Let the sequences of points joining A_i to A_k and A_j to A_k be $A_i = A_{i_1}, A_{i_2}, \ldots, A_{i_p} = A_k$ and $A_j = A_{j_1}, A_{j_2}, \ldots A_{j_q} = A_k$. Now these sequences certainly have a point in common, for $A_{i_p} = A_{j_q} = A_k$. Let the first common point be $A_{i_s} = A_{j_t}$. Then the sequence $A_{i_1}, \ldots, A_{i_s}, A_{j_t}, A_{j_{t-1}}, \ldots, A_{j_1}$ is a sequence joining A_i and A_j. Joining is thus an equivalence relation, and so divides the n points into equivalence classes, which we call *connected subsets*.

We define a *loop* to be a set of more than two edges such that in the points specifying them,

(1) Each point appears exactly twice,

(2) The set of points is a connected subset.

Thus if three edges are (i,j), (j,k), (k,i) then A_i, A_j, A_k are joined in a loop.

LEMMA: Given r edges associated with r points, then a subset (not necessarily proper) forms a loop ($r > 2$).

PROOF: The result is trivial for $r = 3$. We proceed by induction. Suppose the lemma true for $r = n$. Then we prove it for $r = n + 1$. There are 3 cases:

(1) There exists a point with no edge ending on it. Then the remaining n points have $n + 1$ edges, and disregarding one of them we have the result.

(2) There exists some point with just one edge ending on it. Then, disregarding this point and edge, there remain n points with n edges, and we have the result.

(3) Every point has at least two edges ending on it. Then each has just two, for this exhausts all available edges. So from the definition, each connected subset forms a loop.

The lemma follows by induction.

THEOREM: Given $n - 1$ edges associated with n points, then a subset of the edges forms a loop if and only if there exist two points not joined to each other. This is illustrated for the case $n = 5$ in Fig. 2, where the first two configurations have both loops and points not joined, and the last two have no loops and are connected.

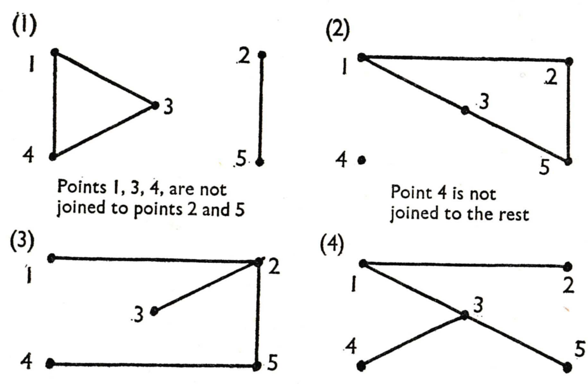


FIG. 2

PROOF: Suppose there exists a loop of r edges and r points, and suppose the theorem false, i.e. that all points are joined. We shall deduce a contradiction. Let the points in the loop be $A_1, \ldots, A_r$. By our assumption, A_1 is joined to some point not in the loop. Of the sequence of points joining A_1 to the point not in the loop, take the first point not in the loop and call it A_{r+1}. Let the edge joining this to the loop be called the $r + 1$ th edge. Then similarly a point in the set $A_1, \ldots, A_{r+1}$ is joined to a point A_{r+2} not in the set by the $(r + 2)$th edge, and so on to A_{n-1} joined by the $n - 1$th edge, leaving the nth point unjoined, and giving the required contradiction. So the existence of a loop implies the existence of two points not joined.

Conversely, if there exist two points not joined, then there are at least two equivalence classes of connected subsets. Let there be r points in one of them. If there is a loop in it, the case is proved. If not, then by the lemma it contains $r - 1$ or fewer edges. So the remaining $n + r$ points have $n - r$ or more edges, and so by the lemma contain a loop.

A *tree* is a set of $n - 1$ edges associated with n points such that no subset of the edges forms a loop (or equivalently, by the theorem, such that the set is connected). Thus in Fig. 2, the first two configurations are not trees, the last two are.

Perhaps a word about the history of the problem would be appropriate at this stage. I was originally concerned with programming an IBM 7090 computer to enumerate explicitly all possible trees associated with n points for particular values of n. This I did by the simple, if rather heavy-handed, procedure of generating all possible combinations of $n - 1$ edges and examining each one to decide whether or not it was a tree. This was how the theorem occurred to me, as it seemed easier to write a programme to decide whether there was more than one connected equivalence class than to look for loops directly.

The programme worked excellently for values of n up to 5, but ran into trouble for $n = 6$. It was not so much the limitations of the computer (the 7090 is a big, fast machine), but a machine operator tends to get a bit doubtful after the output printer has churned out 10 feet of paper covered solidly with apparently meaningless numbers. If it had not fallen by the wayside, the output for $n = 6$ would have run to over 50 feet of paper, with 1296 groups each of 5 pairs of numbers. It was at this time that I became interested in the less cumbersome problem of finding how many trees are associated with n points. This brings us, at length, to the recurrence relation.

Let the number of different trees associated with n points be p_n. We shall define $p_1 = 1$ for reasons which will appear later. Clearly $p_2 = 1$, $p_3 = 3$, and it is fairly easy to show that $p_4 = 16$. For higher values of n, however, it becomes extremely tedious to evaluate p_n by the simple expedient of drawing diagrams and counting. We shall derive an expression for p_{n+1} by a method illustrated diagrammatically in Fig. 3.

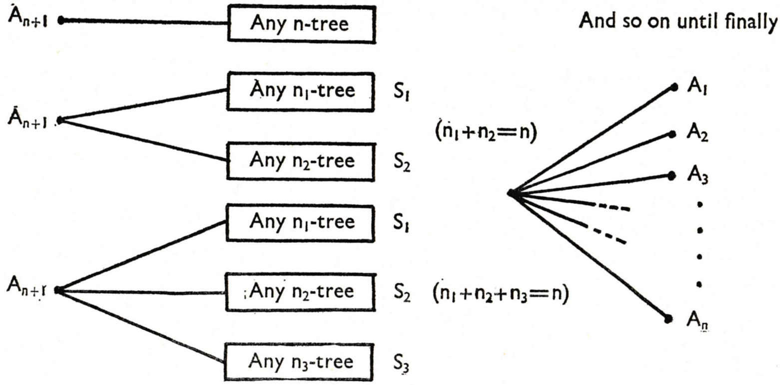


FIG. 3

Suppose we have n points $A_1, \ldots, A_n$. Add another point A_{n+1}, and another edge. At least one edge must end on A_{n+1} for the configuration to form a tree. Suppose just one does; join A_{n+1} to A_r say, for $1 \leqslant r \leqslant n$. Then any n-tree on $A_1 \ldots A_n$ will give rise to an $(n+1)$-tree on $A_1 \ldots . A_{n+1}$. So we have p_n new trees for each choice of r, and n possible choices of r, so in all np_n new trees.

Suppose now there are two edges ending on A_{n+1}, having say A_r and A_t as their other end points, where $r \neq t$. Now A_r and A_t must not be joined in $A_1 \ldots A_n$, or together with A_{n+1} they will form a loop. So, for the moment ignoring A_{n+1} and its two edges, we may consider the distinct equivalence classes of A_r and A_t. Call them S_1 and S_2, and let there be n_1 points in S_1 and n_2 points in S_2. Clearly $n_1 + n_2 = n$.

S_1 must contain $n_1 - 1$ edges, and S_2 must have $n_2 - 1$, so that they are connected and contain no loops. With the two edges ending on A_{n+1} this gives the correct total of n edges.

Then we have an n_1-tree in S_1 and an n_2-tree in S_2. Each can be arranged independently to give $p_{n_1}p_{n_2}$ possible configurations. We observe that this applies for all possible pairs of positive integers n_1, n_2 such that $n_1 + n_2 = n$. In each case there are $\{n-2 : n_1 - 1\}$ ways of choosing the n_1 points to put into S_1, where for typographical reasons we are using the somewhat unorthodox notation $\{n : r\}$ $= n!/r!(n-r)! = \binom{n}{r}$.

There are also $\{n : 2\}$ ways of choosing A_r and A_t, so the number of trees in which two edges end on A_{n+1} is

$$\{n : 2\}\Sigma\{n-2 : n_1 - 1\}p_{n_1}p_{n_2}$$

where the summation is over all sets of positive integers n_1, n_2 with $n_1 + n_2 = n$.

Note that the pairs of integers $(n - r,r)$ and $(r,n - r)$ represent two different such sets, and must be counted separately in the summation (unless $n = 2r$).

Similarly if there are r edges ending on A_{n+1}, then we must find r positive integers such that $n_1 + n_2 + \ldots + n_r = n$. This will give rise to r connected equivalence classes $S_1 \ldots S_r$, where the possible number of ways of choosing the n_i points to go in S_i is

$$\{n - r - \sum_{j=1}^{i-1} (n_j - 1) : n_i - 1\} = \{n - r - 1 + i - \sum_{j=0}^{i-1} n_j : n_i - 1\}$$

adopting a convention $n_0 = 0$ to rationalise the summation. So the number of new trees when r edges end on A_{n+1} is

$$\{n : r\} \Sigma^* \prod_{i=1}^{r} \{n - r - 1 + i - \sum_{j=0}^{i-1} n_j : n_i - 1\} p_{n_i}$$

where the asterisk indicates summation over all sets of positive integers such that $n_1 + n_2 + \ldots + n_r = n$.

There is fortunately a major simplification to this singularly ugly expression. It may be shown that

$$\prod_{i=1}^{r} \{n - r - 1 + i - \sum_{j=0}^{i-1} n_j : n_i - 1\} = (n - r)! \prod_{i=1}^{r} 1/(n_i - 1)!$$

so the recurrence formula finally becomes

$$p_{n+1} = \sum_{r=1}^{n} (n!/r!) \Sigma^* \prod_{i=1}^{r} p_{n_i}/(n_i - 1)!$$

This formula agrees with previously known results

n	2	3	4	5
p_n	1	3	16	125

which were obtained from the computer. They strongly suggest the following

HYPOTHESIS: $p_n = n^{n-2}$.

The recurrence formula has been used to confirm this hypothesis for $n = 6$ and $n = 7$. Persistent attempts have so far failed to prove the result generally, and I intend (as a gesture of defeat) to programme the recurrence relation and so test the hypothesis for higher values of n. If anyone can produce a general proof I should be most interested to see it.

I should like to acknowledge the suggestions which I have received from Mr. John Triance, and the extensive help and advice of Dr. G. Allan of Churchill College, in the preparation of this article.

Partitions and Divisor-sums

By J. L. Dawson

We shall derive a formula connecting the following two number-theoretic functions:

$P(n)$ = number of partitions of n into positive integers, i.e., number of ways of expressing n as a sum of positive integers.

$\sigma(n)$ = Sum of divisors of n, including n and 1. We make the convention that $P(0) = 1$, $\sigma(0) = 0$.

$$\text{Let } \alpha(x) = \prod_{n=1}^{\infty} (1 - x^n) \qquad \ldots \qquad (1)$$

Then we have the well-known formula for the *Euler generating function*:

$$\frac{1}{\alpha(x)} = \sum_{r=0}^{\infty} P(r)x^r \qquad \ldots \qquad (2)$$

From (1),

$$\begin{aligned}
-\log\alpha(x) &= -\log(1-x) - \log(1-x^2) - \ldots \\
&= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) + \left(x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \ldots\right) + \ldots \\
&= x + \left(\frac{1}{2} + 1\right)x^2 + \left(\frac{1}{3} + 1\right)x^3 + \left(\frac{1}{4} + \frac{1}{2} + 1\right)x^4 \\
&\qquad + \left(\frac{1}{5} + 1\right)x^5 + \ldots \\
&= x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{7}{4}x^4 + \frac{6}{5}x^5 + \ldots
\end{aligned}$$

$$-\log\alpha(x) = \sum_{r=1}^{\infty} \frac{\sigma(r)}{r} x^r.$$

Differentiating with respect to x,

$$\frac{-\alpha'(x)}{\alpha(x)} = \sum_{r=1}^{\infty} \sigma(r)x^{r-1} = \sum_{r=0}^{\infty} \sigma(r+1)x^r \qquad \ldots \qquad (3)$$

Differentiating (2),

$$\frac{-\alpha'(x)}{\alpha^2(x)} = \sum_{r=1}^{\infty} rP(r)x^{r-1} \qquad \ldots \qquad (4)$$

Combining this with (2) and (3),

$$\sum_{r=0}^{\infty} (r+1)P(r+1)x^r = \left(\sum_{r=0}^{\infty} \sigma(r+1)x^r\right)\left(\sum_{r=0}^{\infty} P(r)x^r\right)$$

And equating coefficients of x^n we obtain

$$nP(n) = \sum_{r=0}^{n} P(r)\sigma(n-r) \quad \ldots \quad (5)$$

NOTE: For the purposes of this article, we assume that all the infinite series involved are uniformly and absolutely convergent in some neighbourhood of $x = 0$; this can easily be checked.

Hence

BY DUSTCARTES

THE word "hence" often crops up in mathematical writings. The time has now come to reveal the true story behind its use.

In the town of Clangalot, in Medieval England, a crisis was arising. King Arthur (no relation to Ben Hur) had become so annoyed with knights armed with compass and dividers trampling over his table trying to square the circle that he was seriously considering changing it back to a rectangular one. But Merlin, the court magician (the King court him pinching Queen Guinivere), had other ideas. He advised Arthur to send the knights out on long and difficult quests, to keep them out of the way. Since there were many knights (one for every day of the year), this was not easy, but with Merlin's help it was accomplished.

Now one knight, Sir Chancealot (so called because of his practical interest in probability theory), was sent to search for a fabulous long-trunked beast, the *Eliminant*, found in the equilateral jungles of darkest (or at any rate darker) Africa. As he rode through the forest on his milk-white steed (well, you try drinking your daily pinta on the back of a horse), he suddenly heard a scream coming from a clearing ahead. He spurred on his horse, and arrived just in time to see a great dragon (a species of *Ptetrahedron*, now extinct) bearing down on a beautiful young vector function, with long golden curls and a zero div. Quickly drawing his semi-infinite broadsword (they hadn't invented sixth-rooters in those days) Sir Chancealot rode to the attack. When he reached the spot, the dragon had retreated 8 feet. Once more he approached.

The dragon retreated 4 feet. Again he tried. The dragon retreated 2 feet. "You can't fool me with Zeno's paradox!" cried the knight, and quickly summing the geometric series, he advanced 4 more feet and struck!

The dragon crumpled to the ground, pierced through the origin. Sir Chancealot then rode over to where the damsel lay on the ground in a dead faint. He picked her up, and set out for the nearest invariant subspace, where he was received with great joy, for the damsel had been feared lost.

The villagers inquired the purpose of his travels. When they heard that he was searching for an Eliminant, they suggested he try the gnomes—they might know. So Sir Chancealot set out for Gnomeland without delay.

At the gateway, over which hung a sign saying "Gnome sweet Gnome," he was accosted by a raggedly-dressed gnome who wanted him to buy some eels. "Why on Earth should I?" he asked. "Because of the Buy Gnome-eel theorem!"

He rode past the gate, and over a bridge with parabolic arches. Through the foci of these were the remains of cords, frayed and broken. He asked how this had happened. It appeared that one gnome called Quintus Latus (meaning Take Five), occasionally went on a pub crawl, and, like castor oil, was nasty when drunk. "Cords through the foci? Why, Latus wrecked 'em!"

This, of course, was no help in his quest, but luckily a wiser gnome than the others suggested he try asking an ogre, who lived in the mountains, and was reputedly very clever. As he rode along, he heard the ogre approach, singing "Ogre the hills and far away." Now the ogre was a nice ogre really, and only devoured damsels in distress because oxen were out of fashion, so when he heard what Sir Chancealot wanted, he was as helpful as he could be. Unfortunately, the only Eliminant he knew of was degenerate, and this was no use.

Discouraged, Sir Chancealot decided to give up his quest, retire from knighthood, and take up chicken-farming. So he went to Baron Humpdinger for advice, saying "I come not as a knight, I come for hens!"

When King Arthur found out what a rotten cowardly knight he was, he changed his name to "Sir Come-for-hens," which, in the course of time, became "Circumference."

He became so infatuated with his hens that he began to address them as if they were human (he thought they were). "Hens," he would say, "we are going to square some roots today," or "Hens, we have solved our problem." This manner of speaking was rapidly adopted by the mathematicians of the time, who, being only ignorant mathematicians, couldn't spell, and wrote it "hence."

Book Reviews

Linear Programming. By An-Min Chung. (Prentice-Hall.) 63*s.*

This book is designed for use by mathematically-minded economists or businessmen. The treatment is thorough, and illustrated by many excellent worked numerical examples. Exercises are provided at the end of each chapter (no answers, unfortunately). The algebraic solution of the general linear programming problem is preceded by a geometrical argument showing in graphical terms the actual meaning of the mathematics. A chapter on optimality analysis supplies a practical guide to the changes in the solution due to changes in the formulation. The dual problem, with its applications to simplifying the solution of the primal problem, and to optimality analysis, is discussed in detail. After a chapter on the important "Transportation problem," the book ends with a brief account of non-linear programming.

J. L. Dawson.

The Axiomatic Method. By A. H. Lightstone. (Prentice-Hall.) 48*s.*

This book is in many ways similar to most introductory texts on symbolic logic. It explains, first of all, the basic ideas of symbolic logic and set theory and later develops the propositional and predicate calculi formally, concluding with a discussion of completeness theorems and related topics. A feature of this book is the particular emphasis placed on the axiomatic method as applied to algebra, and on its relation to formal systems of logic. Nevertheless, the reader is expected to have some knowledge of the fundamental notions of axiomatic abstract algebra. It is therefore somewhat doubtful whether chapter 3, *The Axiomatic Method*, serves any purpose other than to give further examples of the expression of mathematics in symbolic logic.

On the whole, however, this book provides a good introduction to the elementary theory of symbolic logic.

P. G. Dixon.

Philosophy of Mathematics: Selected Readings. Edited by Paul Benacerraf and Hilary Putnam. (Prentice-Hall.)

This book contains a valuable selection of extracts from the writings of the most important thinkers on the philosophy of mathematics of recent years. The first part opens with a symposium on the foundations of mathematics consisting of three essays, one by Carnap from a logicist standpoint, one by the intuitionist Heyting, and one by Von Neumann putting the formalist case. This is followed by selections from the works of Brouwer, Russell, Hilbert, *et al.* which helps to clarify these positions (while showing that they are by no means monolithic).

Part two is concerned with ontology in mathematics, and contains essays by, amongst others, Quine, Gödel, and Carnap. In part three a number of authors, including Ayer, Nagel, and Poincare, discuss the meaning of mathematical truth, while extracts from Wittgenstein's *Remarks on the Foundations of Mathematics* and from the writings of his critic are to be found in part four.

For anyone wishing to know something about current ideas on the philosophy of mathematics, this book can be recommended as a relatively painless and comprehensive introduction.

P. M. Lee.

Elements of Point-set Topology. By John D. Baum. (Prentice-Hall.) 48*s.*

This book deals with the topological basis of analysis, and no mention is made of algebraic topology. The reader is spoonfed at first with fundamental definitions and properties, which occupy a third of the book. The discussion gathers pace in sections on the countability axioms, compactness and connectedness, and the final chapter deals with metric spaces, going as far as Urysohn's metrisation theorem and Baire's category theorem.

It is strange that quotient spaces are not mentioned, though product spaces and equivalence relations are discussed.

The mathematics is rigorous and easy to follow, but the author's prose style is weak. For example, we read: "Natural though this path may seem, it is somewhat rocky, until we have a few further tools at our disposal" and "The main sequence of topics is, of course, the set of those which are of importance in analysis." The words "of course" could with profit be struck out wherever they occur.

A large number of exercises are included, all important results in their own right, and judicious references are given for further reading. The student who has mastered the text and exercises of this book will be well-equipped for a study of advanced analytic topology and abstract analysis. J. H. Webb.

Topological Vector Spaces. By A. P. Robertson and W. J. Robertson. (*Cambridge tracts in Mathematics and Mathematical Physics no.* 53.) 30*s.*

There are few books on this subject, and none in English so cheap and concise as this one. The chapter headings are: Definitions and elementary properties; Duality and the Hahn-Banach theorem; Topologies on dual spaces and the Mackey-Arens theorem; Barrelled spaces and the Banach-Steinhaus theorem; Inductive and projective limits; Completeness and the closed graph theorem; Some further topics (strict inductive limits, tensor products, the Krein-Milman theorem); Compact linear mappings. Each of the early chapters has a supplement containing material to illustrate and motivate the text; integration and distribution spaces are thus mentioned. There are no exercises, but the energetic reader may occupy himself with amplifying the proofs, of which many are highly condensed. The spirit of the book is geometrical, the Hahn-Banach theorem, for instance, being proved in terms of hyperplanes. This book is recommended both as an introduction to the subject of topological vector spaces and as a source of useful background knowledge for the study of modern analysis. C. L. Thompson.

Linear Topological Spaces. By J. L. Kelley, I. Namioka, and co-authors. (Van Nostrand.) 62*s.*

A linear topological space is a linear space endowed with a topology with respect to which the linear operations of addition and scalar multiplication are continuous. Many classical problems in analysis are best penetrated by means of linear topological spaces, and great insight is given for example into the structure of Banach Space, Hilbert Space, and the theory of distributions.

This book is a committee effort—eight other names appear on the title page below the names of the two main authors. Yet the layout and style is unmistakeably that of Kelley's already classic *General Topology*.

The first two chapters are of an introductory nature, discussing the basic theory of linear spaces and of linear topological spaces. Chapter 3 gives a brief account of the role of completeness and category in relation to the closed graph, open mapping, and Banach-Steinhaus theorems.

Scalar multiplication distinguishes the linear topological space from the topological group, and from chapter 4 onwards problems are discussed which have no parallel in topological group theory. It is here, for example, that the combination of convexity and compactness produce the powerful Krein-Milman theorem (the proof given of this result is unnecessarily complicated at the beginning).

In chapter 5 the notion of duality is made central to the study of locally convex spaces. The account is systematic and very good.

There is an appendix on ordered linear spaces and Banach Lattices.

There are many excellent exercises throughout the book, but the student who gets stuck is given no reference to a fuller treatment. The bibliography is trivial, and I detected only two references in the text to other sources. Both were to Kelley's *General Topology*, one of a general, the other of a cynical nature. This lack of any reference to original sources is most unfortunate, for while the book is one which every functional analyst should read, it cannot be classed as a reference work.

J. H. WEBB.

Introduction to Probability and Statistics from a Bayesian Viewpoint. By D. V. LINDLEY. (Cambridge University Press.) Part 1, *Probability*, 40*s*. Part 2, *Inference*, 45*s*.

These volumes represent the first attempt to present the Bayesian (or "degree of belief") approach to the subject at an undergraduate level. Part 1 is necessarily very similar to more orthodox treatments of probability theory (save for the emphasis placed on the fact that all probabilities are conditional) and those with some knowledge of probability could read part 2 with only occasional reference to part 1. Quite a number of proofs are omitted, which is sometimes annoying, but the style is very easy and a considerable variety of topics is covered in the first part.

Part 2 is more controversial. The author considers statistics as a process whereby we modify our degrees of rational belief in various hypotheses in the light of experimental evidence (and perform estimation, etc., accordingly). It has often been alleged that this procedure is unscientific since the conclusions drawn from an experiment depend on the personal prejudices of whoever analyses the data. To this there are several replies, notably (i) that as historians of science will confirm this *is* in fact how scientific theories progress, (ii) that if the analyst has very vague *a priori* knowledge it is possible to choose prior distributions which represent this state satisfactorily, and it can be shown that the precise form of the prior distribution becomes less important as the wealth of experimental evidence grows, (iii) there is no consistent method of analysis which does not involve *a priori* assumptions. However, probably the majority of statisticians would not whole-heartedly uphold the Bayesian position.

The approach seems, despite the disagreements among the statisticians, of considerable pedagogical value, for the unified treatment it makes possible should appeal to mathematicians. Because of the author's ingenuity in deriving tests virtually identical with the classical tests in many cases (for suitable prior distributions), it would be easy for a reader to pass on to more orthodox tests after reading this book.

The whole book is well written and includes considerable numbers of exercises and interesting examples (such as "... 'The Tories will probably win....' In none of these examples are natural repetitions conceivable....").

P. M. LEE.

Group Theory. By W. R. SCOTT. (Prentice-Hall.) 80*s*.

To give an adequate review of this book would require a book in itself. It has 480 pages, packed with information on all branches of group theory.

The presentation is clear, rigorous, and well-organised, and the book is as self-contained as is reasonably possible.

Chapter 1 gives an introduction to the subject (and includes most of the Tripos part 1 group theory course). From here we proceed to, among other topics, Isomorphism Theorems, p-Groups, Supersolvable groups, Extensions, Representations, the Multiplicative Group of a Division Ring. A fair quantity of material included has never before been published in book form. And on page 456 we are assured "... we have only scratched the surface of group theory....".

There are over 500 exercises, many extending the text (but no answers), an extremely comprehensive bibliography, an index of notation, and an index.

In short, this is an extremely good book, both as a reference work and as a text for those who wish to study group theory at research level.

I. N. STEWART.

Introduction to Lattice Theory. By D. E. RUTHERFORD. (University Mathematical Monographs. Oliver & Boyd.) 35*s.*

This is one of the first books to appear in Oliver & Boyd's new series of *University Mathematical Monographs,* designed to supplement the well-known *University Mathematical Texts.*

There are very few books on Lattices written in English, notably that by BIRKHOFF, so this book is a welcome addition to the series.

Its aim is to provide an introduction to this extremely interesting subject, and to give some of its many applications—such as the Jordan-Hölder Theorem, Boolean Algebras, Switching Circuits (with an excellent section on Boolean Matrices), Brouwer Algebras, and Topology. It succeeds admirably.

51 exercises are included, and there is an adequate index. I. N. STEWART.

Royal Society Mathematical Tables. No. 8. *Natural and Common Logarithms to* 110 *decimals.* By W. E. MANSELL. Edited by A. J. THOMPSON. (Cambridge University Press.) 40*s.*

Why? I. N. STEWART.

Calculus of Variations. I. M. GELFAND and S. V. FOMIN. (Prentice-Hall.)

This text, by two eminent Russian mathematicians, provides a thorough and stimulating introduction to the calculus of variations, emphasising, as any such book must, the physical applications of the theory. The early chapters cover material for a short course in the subject and the latter part of the book develops necessary and sufficient conditions for the different types of extrema and leads on to problems involving multiple integrals and to recent developments in the theory.

Some students of physics may prefer the type of discussion given in texts of mathematical physics (e.g., *Smirnov, Courant and Hilbert*), but it is felt that the modern and rigorous approach of the present authors will be most appealing to students closely connected with the field and to mathematicians generally.

The book has been translated and revised by the series editor, R. A. SILVERMAN. Once again, a fine job has been done, but the feeling may exist that the text does not read quite so well as some others in the Prentice-Hall series.

R. JONES.

Variational Methods in Mathematical Physics. By S. G. MIKHLIN. (Pergamon.) 100*s.*

This book is a very thorough, logically written discussion of variational methods used in physics. The author consistently adopts an approach which makes use of the basic ideas of Hilbert spaces, which are introduced explicitly in chapter 6. Consequently it is very easy reading for those familiar with these ideas.

The author's method is to introduce each topic with physical examples in order to provide motivation. A more general discussion in terms of Hilbert spaces follows and finally the results obtained are illustrated by further physical examples. These examples are all taken from the fields of classical physics and mostly deal with the problems of elasticity. This can be counted as a defect, for quantum mechanical examples are surely likely to be of at least equal interest to the majority of readers and could easily have been included. The main methods discussed by the author are Ritz's method, Trefftz's method, the Bubnov-Galerkin method, and the method of least squares. These are of interest to all mathematical physicists despite the narrow field from which the illustrations are chosen.

The main virtue of the book is that it brings coherence to problems from a wide variety of fields which are normally encountered in a disconnected and unsatisfactory way. The book uses a consistent theoretical framework and is thus clear and straightforward throughout. The defects of the book are those common to most Pergamon books, namely high price and poor proof-reading.

R. L. JACOBS.

A Course of Higher Mathematics. Vol. 4 : *Integral Equations and Partial Differential Equations.* By SMIRNOV. (Pergamon.) 126*s.*

This text is a translation of the fourth edition of one which has been a standard work in Russia and Eastern Europe for some years. Broadly entitled "Integral Equations and Partial Differential Equations," it also presents detailed discussions of the applications of this theory to the calculus of variations and to boundary value problems, with many references to problems in mathematical physics.

The treatment of integral equations is clear and readable and, apart from some references to original Russian work, deals with the subject in a fairly standard fashion. The discussion of the calculus of variations is more from the point of view of the mathematical physicist than that of the mathematician, and contains many applications to physical problems, including elasticity theory. The basic theory of partial differential equations is described with particular reference to characteristics of systems of equations and the final chapter gives a fine discussion of boundary value problems and Green's functions.

Covering basically the same ground as a number of other volumes (for example *Courant and Hilbert*), this book is lucid and interesting, and the translator and editor are to be congratulated on making it read as an English text, rather than a translation from the Russian.

As one has come to expect, Pergamon have held to their high standards in both format and price. It is the latter aspect which makes the volume an unlikely purchase for the private student, but it should certainly form part of any mathematical library.

R. JONES.

Boundary Value Problems. By A. G. MACKIE. (Oliver & Boyd.) 50*s.*

As the title indicates, Professor Mackie's book gives methods of solving boundary value problems in linear ordinary and partial differential equations. However, the book is chiefly an illustration of the use of Green's functions and integral transforms. Laplace transforms and Green's functions are introduced right at the beginning and then applied to initial and two-point boundary value problems, making free use of the adjoint operator. The extension of transforms to integrals is thoroughly covered, introducing the Mellin and Hankel transforms. The final chapter deals with Riemann's method and its connection with the ordinary Green's function.

The presentation is clear and proceeds with the use of several physical illustrations. The exercises at the end of each chapter help both to practice the methods described in the chapter and to further the theory.

This new series of *Mathematical Monographs* from Oliver & Boyd, dealing with rather more advanced topics than their well known and useful *Mathematical Texts,* is greatly welcomed. The price is perhaps rather high for the use an undergraduate would make of this volume, but it would be a useful addition to college libraries.

A. E. BARNSHAW.

Numerical Methods: 2, *Differences, Integration, and Differential Equations.* By B. NOBLE. (Oliver & Boyd.) 12*s.* 6*d.*

Dr. Noble's second volume on Numerical Methods is almost independent of the first, with chapters on finite differences, numerical integration, and ordinary and partial differential equations. Although the treatment is by no means too advanced for the average undergraduate the author pays particular, and most welcome, attention to the error analysis and stability of the methods he describes. He does not attempt to cover every aspect of the subject but contents himself with dealing in detail with typical methods, well chosen to illustrate the general principles. There is a liberal use of worked numerical examples, particularly in the last chapter on partial differential equations, and a selection of exercises at the end of each chapter. The reader is left with the impression that numerical analysis is a rather more logical and unified study than some texts lead one to believe.

For those interested in numerical analysis as a technique as well as those who want a simple introduction to the underlying theory, this book should prove very useful.

A. E. BARNSHAW.

Linear Representations of the Lorentz Group. By M. A. NAIMARK. (Pergamon.) 100s.

This is another of the very useful translations from the Russian published by Pergamon. It contains a very full and comprehensive account of the theory of linear representations applied to the rotation group and the Lorentz group.

The first chapter is a very brief introduction to the theory of the rotation group, the Lorentz group, and invariant integrals. The second chapter develops the theory of group representations to the level needed and then applies the theory to the rotation group. It is rather a pity from the physicist's point of view that spherical harmonics, the most commonly met basis functions for odd dimensional representations of the rotation group, are not discussed. Nevertheless this application follows a course which will be familiar to most quantum physicists as a parallel account is given in most books on quantum mechanics.

The third chapter, which is by far the longest, contains an account of the theory of representations of the Lorentz group. This chapter can be recommended with confidence to all who are interested in relativistic quantum mechanics. Important topics which are dealt with in the sketchiest outline even in advanced books are treated with precision and clarity here.

The last chapter is an account of equations invariant with respect to the Lorentz and rotation groups. Since so much of modern physics is concerned with finding and interpreting such equations, the reviewer thinks that this chapter should form part of the background of every quantum physicist.

The treatment throughout is elegant and clear, and the book is only marred by the high price which will probably ensure that it is found more often in libraries than on the student's bookshelf.

R. L. JACOBS.

Solutions to Problems

PROBLEMS DRIVE

1. If the order of the determinant is greater than 3, subtract the first column from the third, the second from the fourth, giving two columns of even integers. Hence the determinant is divisible by 4 or is zero. If less than or equal to 3, direct calculation gives the result.

2. 12. Use Euler's theorem, that for a simply connected polyhedron, $V - E + F = 2$, where V = number of vertices, E = number of edges, and F = number of faces.

3. For any n, $p = (1 + 1/n)^n$ and $q = (1 + 1/n)^{n+1}$ is a solution, e.g., when $n = 2$, $p = 2\frac{1}{4}$, $q = 3\frac{3}{8}$.

4. 10π m.p.h.

5. "Honesty is the best policy."

6. A solution (not unique) is 0123456789 = ESROVNTWUF.

7. 123356,124456. (223456,123455 are excluded by "rearranged").

8. At least 3 of the 4 get right the numbers of red and of blue counters, also the following combinations: green + blue, red + orange, yellow + orange. If all 4 made the same guesses you could not tell which was which.

9. (i) Correct 17 to 15 (primes + 2).
   (ii) Correct 11 to 10 (recurrence relations $u_{n+1} = nu_n + u_{n-1}$).
   (iii) Correct 19 to 21 ($n^2 + n + 1$).

10. (*a*) 8 by counting cases.
    (*b*) 75. One colour must be repeated: if on adjacent faces gives $4!/2$ ways, if on opposite faces gives $3!/2$ ways.

THREE

$a = 101$, $b = 1100$, $c = 1$, $d = 2001$, $e = 1200$, $f = 10$, $g = 210$, $h = 20$, $i = 1001$, $j = 120$.

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