Eureka Editor
archim-eureka@srcf.net
The Archimedeans
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WA
United Kingdom

Published by The Archimedeans, the mathematics student society of the University of Cambridge

Thanks to the Betty & Gordon Moore Library, Cambridge

EUREKA 25

THE ARCHIMEDEANS' JOURNAL

OCTOBER, 1962 PRICE 2/6

EUREKA

Editor: *R. W. Bray (Pembroke)*
Business Manager: *C. D. Rodgers (Gonville and Caius)*
Advertising Manager: *J. W. Prince (Magdalene)*

No. 25

OCTOBER, 1962

THE JOURNAL OF THE ARCHIMEDEANS
The Cambridge University Mathematical Society; Junior Branch
of the Mathematical Association

Contents

Please send subscriptions (see page 12) to:

The Business Manager, "Eureka,"

advertisements to: The Advertising Manager, "Eureka,"

and contributions and all other communications to:

The Editor, "Eureka,"

The Arts School,

Bene't Street,

Cambridge,

England.

Editorial

In the past, the image of the mathematician has been that of rather an aloof creature, perhaps not even entirely of this world: to justify this, we have but to recall those occasions when a lecturer was oblivious of his audience, the audience of the subject and the subject of the lecturer.* But how much longer is the mathematician going to keep up this pretence of self-effacement? Not for long, we fear, since his fellow-men are demanding that his subject be popularised. The powerful organ of the press has devoted much energy toward this end already, and now it has been joined by the more subtle techniques employed by politics (decimalisation of our currency) and economics (methods of operational research applied to the railways system).

Soon the atmosphere of the mystic will no longer surround mathematics, it will become *respectable*. To be able to walk in Hausdorff Spaces and seek seclusion in Boundary Layers will become a social necessity. Above all, the mathematician will receive the plaudits due to a star, his popularity will increase more and more, and finally he will gain the distinction of denigration usually reserved for royalty.

Although it is therefore not without a little hesitation, we present another issue of this journal, which may rightly claim to be one of the original "lighter offerings" which mathematics has provided.

We thank all those who have helped in any way to produce this issue. The number of contributions received far exceeded the number we could print. Many good articles have therefore not reached the printers: we apologise to their authors for this and ask them to try again next year.

The Archimedeans

Last year:

The Archimedeans had another successful year. The evening meetings were well attended, and to one of these we were pleased to welcome a party of undergraduates from King's College, London, who heard Professor Semple talk on "The Problems and Principles of Enumerative Geometry". The tea meetings again proved to be very popular, the subjects discussed varying from steering bicycles to mathematical ethics.

At the start of the year five groups were meeting under the auspices of the Archimedeans, the Computer, Mathematical Models, Bridge,

* For a finer analysis see *Eureka* 15, p. 2.

Play-Reading, and Music Groups, and during the course of the year we were pleased to welcome a sixth to their number, the "Puzzles and Games Ring". Also on a purely parochial level, the Society ratified the revised Constitution at its Annual General Meeting: our thanks go to those members who have given their time to this work.

This year:

There will be the usual evening meetings during the first two terms at which distinguished mathematicians will speak on a variety of topics of mathematical interest. We are pleased to have among the speakers two mathematicians who are visiting Cambridge from America, Professor Chew and Professor Dalitz. One talk on a slightly different note from usual will be given by Dr. Adams of Manchester University who will speak on some of the problems and difficulties which face people who intend to become research students.

There will again be a selection of tea meetings at which research students will discuss some of the interesting points of their research; and there will be the annual Problems Drive in the Lent Term. There will also be visits to places of mathematical interest including the Radio Observatory at Cambridge and places further afield.

On the social side there will be a Christmas Dinner of the Society, and the usual Punt Party and Ramble in the Easter Term. Throughout the year there will be meetings of the various Groups; the Music Group will listen to music both familiar and unfamiliar, the Bridge Group at its weekly meetings will continue to cater for both beginners and experts, and the Play-Reading Group will meet to read plays of all kinds. Visits will be organised to theatres in London, including, it is hoped, a visit to "Beyond the Fringe" this autumn.

The Committee has tried to arrange an interesting and varied programme of activities for the coming year. Any suggestions as to how it may be improved will be welcomed. They should be made to the Secretary either directly or through the suggestions book in the Arts School.

I. B. Taylor, *Secretary.*

Squares and Circles

1. An n-dimensional box in Euclidean space with sides of integer length a_r, $r = 1,2,\ldots, n$, consists of $a_1 a_2 \ldots a_n$ unit cubes. Through how many of these cubes does a diagonal of the box pass?

2. Two halfpennies, one penny and two florins, taken in any order, fit around a sixpenny piece.

Prove this statement to be false.

The Calculation of Partition Numbers

By J. A. Tyrrell

A partition* of a positive integer n is a representation of n as the sum of any number of positive integral parts. The number of different partitions of n is denoted by $p(n)$.

In the last issue of this journal, M. Rowan-Robinson gave a tabular method of calculating the $p(n)$ by the intermediate use of the double sequence $\{p_r(n)\}$, where $p_r(n)$ is the number of partitions of n in which the greatest component part is equal to r. The numbers $p(n)$ are then given by the formula

$$p(n) = \sum_{r=1}^{n} p_r(n).$$

In this note, we give a different and slightly more illuminating approach to this method of computing the $p(n)$ and give a proof of Rowan-Robinson's partially-proven formula (2)

$$p_r(n) - p_{r-1}(n-1) = p_r(n-r);$$

the proof we give is valid for all relevant values of r and n.

We begin by reminding the reader of the idea of conjugate partitions. Consider the diagram

$$\begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & & \\ \times & & & \end{array}$$

This contains seven crosses and, according as to whether it is read horizontally or vertically, indicates the two partitions $4+2+1$ and $3+2+1+1$. These furnish an example of conjugate partitions and it is scarcely necessary to describe the general process in detail. It is clear that the largest component part of a given partition of n is equal to the number of component parts in the conjugate partition. We conclude that $p_r(n)$ may be alternatively defined as the number of partitions of n into precisely r component parts and from now on we think of $p_r(n)$ as defined in this way.

Suppose now that we have a list of all the partitions $(a_1^{(i)}, \ldots, a_r^{(i)})$ of n into precisely r parts. Then $(a_1^{(i)}-1, \ldots, a_r^{(i)}-1)$ are (all) the partitions of $n-r$ into less than or equal to r parts, any zero components being omitted. This correlation leads at once to the formula

$$p_r(n) = p_1(n-r) + \ldots + p_r(n-r). \tag{1}$$

Rowan-Robinson's formula (2) is a simple consequence of this.

* For more information generally about partitions, reference may be made to chapter XIX of Hardy and Wright's *Introduction to the Theory of Numbers* and also to *Eureka*, No. 8, p. 10.

If we tabulate the $p_r(n)$ as follows

n \ r	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	1	1				
4	1 +	2 +	1	1			
5	1	2	2	1	1		
6	1	3	3	2	1	1	
7	1	3	4	3	2	1	1

the partition sequence $\{p(n)\}$ then emerges gradually from both the rows and the columns (every number to the right of the dotted line being a member of the sequence). In addition, the sum of the nth row is $p(n)$.

This table admits a simple stepwise construction which reflects the formula (1) as follows: to calculate $p_r(n)$, proceed north-westerly from the position (n, r) as far as the indexing column; in the row reached, the sum of the first r elements is $p_r(n)$. The table may also be constructed column by column by a rule under which the elements of any column are obtained by summing an appropriate sequence of elements from the previous column. The precise rule is summed up in the formula

$$p_r(n) = \sum_{k=o}^{-} p_{r-1}(n-kr-1),$$

the summation proceeding until the terms become devoid of meaning. The proof of this formula is not difficult and is left as an easy exercise for the reader.

* * * *

"I wish, finally, to commend for more general use the practice of providing lecture notes in advance. Among obvious advantages the chief is economy of time and energy; my course formerly consisted of twenty-three lectures; now, when it is fuller and more discursive, it consists of fifteen. It is possible that the art of lecturing has not yet recognized the full importance of the younger invention of printing."

J. E. Littlewood in the preface of "The elements of the theory of real functions" published in *1926*! Tempus fugit.

The M-F Relationship as a Positive Feedback System

By C. D. Rodgers

Introduction

This paper is an attempt to deal with a system which, as far as is known to the author, has not been subjected to any degree of mathematical analysis. The treatment is intended to be qualitative rather than quantitative, as the variables involved are not precisely defined, nor are they quantitatively observable.

Only the two-body problem will be dealt with, as the general case for three or more bodies is extremely complicated. However, a third body can be introduced into the following analysis as a perturbation.

Definitions

$I_m(t)$, $I_f(t)$	Intensity of m or f's response as a function of time
$S_m(t)$, $S_f(t)$	Stimulus applied by m or f. This is the only information m has of f, and vice versa.
$M_m(t)$, $M_f(t)$	Memory function
c	Compatibility

The treatment is symmetric in m and f, except where the context indicates otherwise, and only one of each pair of equations will be printed.

Basic Equation of the System

As a first attempt, the following dependence of I on S is postulated:

$$I_m(t) = \int_{-\infty}^{t} cM_m(t-T)[aS_m(T)+bS_f(T)]dT$$

where a and b are self- and cross-response coefficients. This is more easily handled as a differential equation, but first some analytic form for $M(t-T)$ is required. An exponential function is probably as good as anything else, and we will use:

$$M_m(t-T) = \exp(-m(t-T))$$

This leads to:

$$\frac{dI_m(t)}{dt} = c[aS_m(t)+bS_f(t)]-mI_m(t) \qquad (1)$$

The term in square brackets is not the best description of the observed effects, but it is analytically convenient. It does not, for example, allow for the reduction in I_f induced by an S_m which is too large compared with I_f. This will be treated as a special case later.

These equations have the following properties:

(a) In the absence of stimulus, I decays exponentially with the same rate as the memory function (cf. the well-known proverb).

(b) In the absence of a return stimulus, I may increase or decrease, as is to be expected. The condition for increase is:

$$caS(t) > mI(t) \tag{2}$$

(c) In order to decrease I_m, S_f must be less than a certain critical value, which is negative if (2) and positive otherwise.

Influences Determining $S(t)$

The above equations describe how $I(t)$ depends on $S(t)$. We must now discuss the factors which determine $S(t)$. This is slightly more difficult, as it would appear that outside influences have a rather large effect. We must eventually take into account such things as perturbations produced by other bodies, the overload condition ($S_m \gg I_f$), discontinuities in real $S(t)$'s, and a function, so far not discussed, which might conveniently be referred to as "intention".

$S(t)$ is clearly determined to a considerable extent by $I(t)$, however, and for the simple case the following relation is convenient:

$$S(t) = kI(t)$$

This is probably a good description for the early stages, and external influences can be incorporated by making changes in the parameter k.

The Simple Case

A complete solution is available for the case k = constant. The equations reduce to:

$$I'_m = (cak_m - m)I_m + cbk_f I_f$$
$$I'_f = (c\alpha k_f - f)I_f + c\beta k_m I_m$$

The general solution is of the form:

$$I_m = A(1 + B \exp K_1 t) \exp K_2 t$$

where:

$$K_1 = -\sqrt{[(c(ak_m - \alpha k_f) + f - m)^2 + 4c^2 b\beta k_m k_f]}$$
$$2K_2 = cak_m + c\alpha k_f - m - f - K_1$$

As K_1 is negative, the important term is $\exp K_2 t$, positive feedback occurs if $K_2 > 0$. Thus the condition for a successful relationship reduces to:

$$cak_m + c\alpha k_f > m + f \tag{3}$$

$$\text{or} \quad (cak_m - m)(c\alpha k_f - f) < c^2 b\beta k_m k_f$$

The first of these conditions is basically the same as (2), it is left to the reader to put a physical meaning to the second.

Perturbations

As suggested above, external influences can be dealt with by means of a discontinuous change in k, solving the simple case on either side of the discontinuity. For example, the effect of an M' might be to make $k_f = 0$, after which I_m will increase or decrease according as $cak_m >$ or $< m$. There will doubtless be reasons which make $k_m \to 0$ eventually, but this shows the short-term effect.

If the overload condition is satisfied, k_f could be made zero, or even negative, and the effect is easily calculated. There is an interesting possibility if $-k_f$ is large enough, K_2 may become complex, and the system goes into oscillation. The inequality (3) states the condition that the amplitude increases.

Intention

This really belongs to another problem altogether, one of Game Theory, namely how to optimise S_m to produce the closest approach to a desired S_f. There are a number of difficulties associated with this problem, for example:

(a) M does not have complete control over S_m.

(b) I_f is unknown to M, and he must keep S_m/I_f outside the overload region.

(c) M does not know whether F is responding normally, or playing to some stratagem.

If any game theorist has any suggestions on this problem, the author will be glad to hear of them.

Conclusion

The equations which have been set up describe some aspects of the system in a satisfactory qualitative manner, though they are far from perfect. They do, however, suggest a basis from which future work may be carried out.

Acknowledgements

The author's thanks are due to Mr. M. J. T. Guy for many helpful discussions, and to Miss R. Miller for her co-operation in much of the experimental work on which this paper is based.

Have you noticed how dry and uninteresting mathematical literature is getting these days? Gone is the era when every paper, monograph or note was an absorbing bed-time story and every textbook a best seller. To revive and quicken the lost spirit of mathematical journalism, a specimen article (with comments) has been produced below in the hope that the dormant talent which lies below the sombre facades of so many of our readers may blossom forth in full glory. A new age, a new culture, awaits the genius of a twentieth-century Shakespeare.

Growth

By Ham

Note the snappy title, and a nom-de-plume to avoid the barbarous shafts of unenlightened criticism.

"Men may come and men may go, but I go on for ever."

—Elementary Anthology for Juveniles

Quotations give a subtle indication of the vast store of literature from which the writer can draw. Relevance to the subject matter is not essential.

One does not need to live on this earth very long before one discovers the phenomenon of growth. Everything from public service departments to newspaper empires establishes this phenomenon, and the only difference lies in the different rates of growth.

Take advantage of the current wave of interest in newspaper empires. The astute reader will have noticed the oblique reference to the revered name of Parkinson.

Every mathematician knows that the size of one of these "things" at time t should be called $f(t)$—positive and well behaved, of course, because it wouldn't do to consider anything irregular.

The intelligent reader will ask himself what is meant by "well-behaved". This is a well-known device for getting the reader to think for himself.

Some things grow at the same rate, so we might as well lump them together and put them into their own equivalence class:

$$F = \text{order} f = \{g : 0 < A \leqslant f/g \leqslant B < \infty \text{ for all } t \text{ sufficiently large}\}$$

Don't be afraid to introduce standard notation. We have shares in Mathematical Dictionaries, Ltd.

You have probably got the idea by now, so we will refrain from all but the most essential comments in order not to interrupt the natural flow of the article. In fact we recommend you to read the article twice—the first time without the comments to get the full impact, and the second time with them so that the writer's technique can be fully appreciated.

Now that we've put our growths into neat little classes, we just *have* to put them into some sort of order. Some things grow faster than others, so we say F is faster than G ($F > G$) if $f/g \to \infty$ as $t \to \infty$ (all $f, g : f \epsilon F, g \epsilon G$). But, alas and alack, some classes positively refuse to be put in order, and we can't really say which is the faster. Admittedly this is rather nasty, but, before you give up in disgust, may we remind you that partial ordering is a good enough excuse for introducing some sort of topology. First of all we say a set σ of equivalence classes is convex if it contains all the bits in between, i.e. for every $F, G \epsilon \sigma$, $F < H < G$ implies $H \epsilon \sigma$, and second of all we say that the convex set σ is open if, for every $F \epsilon \sigma$, there exists $F_1, F_2 \epsilon \sigma : F_1 < F < F_2$.

We would hate to disappoint you if you want some more topology, but we just can't hold ourselves back any longer from telling you that the set Γ of growth rates or equivalence classes is a vector space. Yes, it really is. All you have to do is define

$$\alpha F + \beta G = \text{order}(f^{\alpha} g^{\beta})$$
$$O = \text{order } 1$$
$$(\text{and } I = \text{order } t)$$

and the rest is easy. It must be admitted that Γ is rather a big vector space, perhaps too big to handle successfully. You begin to realise this when you look at the following theorems gleaned from a book by Hardy called "Orders of Infinity" (available from all lending libraries).

The first one says that given an ascending sequence of classes $F_1 < F_2 < F_3 \ldots$ you can find a class G faster than the lot: $F_n < G$ for all n. This theorem comes from Paul du Bois Reymond. Everyone knows the example of transcendental functions, e.g. $I < 2I < 3I < \ldots <$ ord exp t.

The next one says that you can always squeeze another class G between an ascending sequence F_n and a descending sequence H_n: i.e.

$$F_1 < F_2 < \ldots . < G < \ldots . < H_2 < H_1.$$

This extension comes from Hadamard, and the example here is of course the inverse of the first example: $O <$ ord log $t < I/n$ for all n.

Naturally this column isn't big enough to contain all the exciting theorems there are, some discovered and perhaps others waiting for you to discover them, but we can just spare enough room to include two more, from Pincherle and du Bois Reymond.

The third one says that even if f is ever so much faster than g, we can find a function h faster than 1 such that

$$\text{order } h[f(t)] = \text{order } h[g(t)]$$

which just goes to show how much the unenlightened can put the damper on real talent!

The fourth one, however, strikes a happier note, and we end on this: if $f-g > 0$ and g is faster than 1, there is a function h such that

$$\text{order } h[f(t)] > \text{order } h[g(t)]$$

and so even the tiniest bit of talent will blossom if given the right sort of encouragement.

EXERCISES

1. Show that if the fractional increase in the world's population each year is the same, the population doesn't grow, it explodes.

2. Show that if $y(x,t)$ is a uniform approximation to $Y(x,t)$ as $t \to \infty$ in every closed interval $0 < A \leqslant x \leqslant B < \infty$, then the same holds true for any closed interval $0 < A \leqslant x/f(t) \leqslant B < \infty$ for any order f in an open convex set containing order 1.

One final comment: Always throw in a couple of exercises. This serves the double purpose of making the reader think and saving future examiners the trouble of thinking up their own exercises. Put in an easy one first to lull them into a false sense of security, and then put in a real stunner, just to put them back in their place.

Postal Subscription and Back Numbers

FOR the benefit of persons not resident in Cambridge we have a postal subscription service. You may enrol as a personal subscriber and either pay for each issue on receipt or, by advancing 10*s.* or more, receive future issues as published at approximately 25 per cent discount, with notification when credit has expired. The rates this year are:—

For those who pay on receipt 2s. 6d. (+2½d. *postage*)

For those who pay in advance 2s. (*post free*)

Some copies of *Eureka*—Nos. 17 (1*s.* each), 18 (1*s.* 6*d.*), 19, 22 (2*s.* each) and 23, 24 (2*s.* 6*d.*)—are still available (postage 2½*d.* extra on each copy). Set of six, 10*s.* post free. We would be glad to buy back any old copies of Nos. 1 to 10, 12 and 14 which are no longer required.

Cheques and postal orders should be made payable to "The Business Manager, *Eureka*", and addressed to The Arts School, Bene't Street, Cambridge.

Some Notes on Solitaire

By J. D. Beasley

For those not acquainted with the game of solitaire, we start with a brief description. The board is as in Fig. 1 (ignore the figures in the squares). Each square, or "hole", may contain a peg, and the move is to jump one peg over an adjacent one (horizontally or vertically, not diagonally), removing the peg jumped over. The standard problem is to start with the board full, apart from the centre hole, and to move until there is only one peg left, situated in that hole; but a much wider range of problems may be considered.

There are two basic problems set by a solitaire board—the finding of elegant (in terms of symmetries, low numbers of moves, or long chain jumps with one peg) solutions to given problems if they are soluble, and the proving of their insolubility if they are not. The first has been well covered by Bergholt [1], some of whose results are quoted in the *Scientific American* [2], and we shall here do no more than give a useful technique by which solutions may be obtained if they exist. For impossibility we have the well-known method of Reiss [3], expanded by Hermary [4] and Charosh [5], which covers a large range of problems; for problems not so covered we shall give some further ideas, most of which do not seem to have appeared in print before.

1. *Tests for Insolubility*

For the theory it is convenient to extend the game to allow any integral (positive or negative) number of pegs in a hole. If a problem is insoluble in this extended game it is clearly also insoluble in the ordinary game; what is more important is that the converse is sufficiently near the truth to make the extended game profitable.

The basic idea in the methods for proving impossibility is that of "functions" of position. Consider a value attached to each hole, as in Fig. 1 (unmarked squares containing zeros), and sum these values for the holes with pegs present (with the obvious weighting for negative and plural pegs in the extended game). Thus the function of Fig. 1, for the position of Fig. 2, has value 4. Fig. 3 shows another function, and here we will add the relation $X+X = 0$; its value for the position of Fig. 2 is accordingly X. By taking such functions with various special properties we can obtain necessary conditions for problems to be soluble.

Reiss's theory depends on the function of Fig. 3. We note that *there exists no move which changes the number of pegs in the* X *holes by an odd number*; the reader may convince himself of this by experiment.

Hence if this function has value X for a given position, it will have value X for any position derivable from it by legitimate moves, since $X+X=0$. So we cannot reduce the position of Fig. 2 to a single peg in *e*3, since the function has value zero for this latter position.

By rotating the function through one, two and three right angles we obtain three similar functions, which together divide all positions into sixteen classes (no further such divisions are possible). *For one position to be reducible to another both must be in the same class.* This is the classical theory of Reiss.

The function of Fig. 1 has a different property; here *there is no move which will increase the value of the function.* (It is easily seen that the condition for this is that, if α, β, γ are three adjacent holes, $f(\alpha)+f(\beta) \geqslant f(\gamma)$; for the move of α to γ, jumping over β, "loses" $f(\alpha)$ and $f(\beta)$ and "gains" $f(\gamma)$.) This function was produced to settle the "five-cross" problem, which is to reduce the position of Fig. 2 to its reverse (i.e. the position with *c*4, *d*3, *d*4, *d*5 and *e*4 filled and the rest empty); it proves it impossible since the value of the function for the first position is 4 and the second 6, and there is no move which can make the increase.

Such functions have acquired the picturesque name of "pagoda" functions, on account of the appearance of one of them when produced on a large board; "p.f." is the standard abbreviation.

For problems which do not yield even to a p.f. we use the function of Fig. 4 (and other similar ones). Here the α's and π's, taken separately, form p.f.'s (α and π being assumed positive); and for δ we assume the relation $\delta+\delta=0$. It follows that *any move which changes δ by an odd amount makes a definite loss of either α or π.*

To see this function in practice, consider the position of Fig. 5, which we wish to reduce to a single peg in *c*3. We must make one and only one move losing π, and can afford none losing α. The ten moves given by *b*3-*b*5, *a*4-*c*4, *d*3-*c*5, *f*3-*f*5, *e*4-*g*4 and their reverses introduce an odd δ which, being unable to make a second move losing π, we cannot remove; hence these moves are barred to us. This still leaves the other ten moves which lose π; but we now turn the function through a right angle and repeat the argument. This proves the problem impossible; the unconvinced reader is recommended to try it on a board and actually count the δ's.

The above methods all work in the extended game; but if a problem is possible in the extended game we must find another criterion. Our only real weapon here, so far, is the apparently trivial "exit theorem", which states:—

(a) a block of pegs which starts completely full in the ordinary game cannot be partially emptied without a move outwards across the boundary (an "exit");

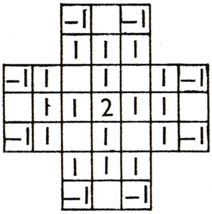


Fig. 1

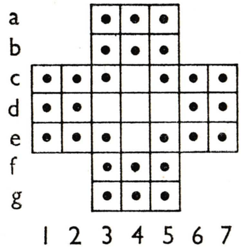


Fig. 2

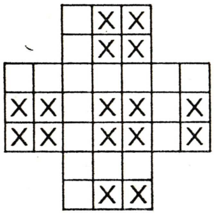


Fig. 3

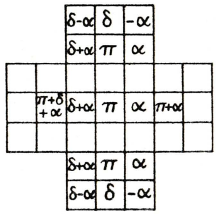


Fig. 4

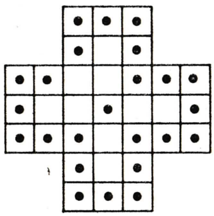


Fig. 5

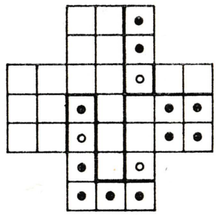


Fig. 6

(b) a block which starts partially full cannot be completely emptied without an exit;

(c) a block of three or more pegs which starts completely full cannot be completely emptied without two exits.

Trivial as it sounds, this has proved enough to deal with most problems of this sort to have confronted us so far; in any case such problems are not very common.

2. *Techniques for obtaining solutions*

Having discussed methods for proving insolubility we now give a useful technique for finding solutions. What we shall do is to show that certain positions can always be reduced, either to a single peg or to zero, without affecting the rest of the board. The effect is to give us more complicated "moves" which prove to be more convenient for solving problems than the simple moves. (Any reader with experience of computer programming will recognise an immediate analogy with subroutines—which is exactly what our new "moves" will be.)

First, we consider Fig. 6. The ordinary move (here *a*5-*c*5) is shown at top right. The L formation at the bottom can be reduced to a single peg in *e*3 by the four moves *g*3-*e*3, *d*3-*f*3, *g*5-*g*3-*e*3; so we link these four moves together and call them the "L move". It can be used in any position, and proves valuable in clearing corners. The situation on the right is a little more complicated; here we make the three moves *d*7-*d*5, *e*7-*e*5, *d*5-*f*5, and reduce the square of four to a single peg (*f*5) outside it diagonally. We had *d*5 and *e*5 empty to start with; but, had they not been, the same three moves, in a different order, would have had the same effect on the square and would have left *d*5 and *e*5 as they had started. This set is known as the "four move".

In order to make complete removals of blocks we need some help from outside, which leads to the notion of a "catalyst". In Fig. 7, top, the squares X, X are supposed to be "unlike", i.e. one is full, the other empty. We can now remove the block of three: if *c*4 is full, *c*6 empty, we play *c*4-*c*6, *a*5-*c*5, *c*6-*c*4; the block of three is now gone but the rest of the board is unchanged. The play with *c*4 empty, *c*6 full, is similar. Note that the catalyst (X, X) is used in the play, but is restored at the end of it.

The above "three-removal" is given by Hermary; but there are several others which are useful. Fig. 7, bottom, shows the "L-removal", again X, X are assumed unlike. Fig. 8, top, shows the

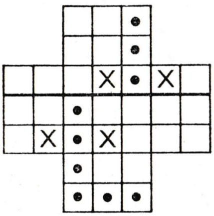


Fig. 7

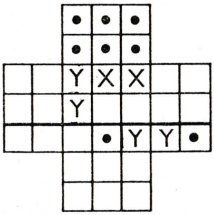


Fig. 8

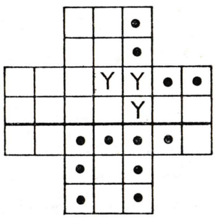


Fig. 9

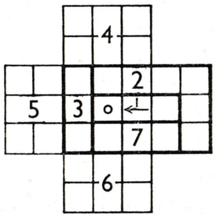


Fig. 10

"six-removal", with X, X as before; but here we may also require the use of *c*3 or *c*6, their original state however is restored before the end of play. In this removal, Y, Y unlike also forms a possible catalyst. A fourth useful removal (the "two-removal") is shown in Fig. 8, bottom; the pair *e*4, *e*7 can be removed provided Y, Y are unlike. A fifth (the "four-removal") is given in Fig. 9, top, the four being removable if any two of the three Y squares are unlike.

Another useful device is that of "double catalysis". In Fig. 9, bottom, *e*2 and *e*4 form a catalyst for the left-hand row of three, but there is apparently none for the right-hand one. However the first move of the removal of the left-hand row is *e*4-*e*2, after which *e*4 and *e*6 form a catalyst for the right-hand three; so we can break off from the left-hand removal, remove the right-hand three, and resume the left-hand removal. The reader can easily produce other examples. The important feature of all these removals is that the rest of the board is left unchanged at the end.

To see how removals work in practice we give a solution for the standard centre-peg game, starting with the board full except for the centre square. In Fig. 10, we make the ordinary move *d*6-*d*4 (shown by 1), then remove in turn the L(2), the three (3), three sixes (4, 5 and 6) and the last three (7). The appropriate catalysts are always present at each stage.

3. *Reversals*

We conclude with a few remarks on reversals. A reversal is the reduction of a position to its reverse, as in the "five-cross" problem given earlier; an "irreversible" position is a position which cannot be reversed, the five-cross being an example. There are many interesting results on reversals; two are:—

(i) If a position is irreversible in the extended game, then any position obtained from it by removing any peg other than an outside corner peg is also reversible in the extended game.

(ii) In any reversal, either the number of moves horizontally across *b*4 or *c*4 and *e*4 or *d*4 are both odd, and those vertically across *d*2 or *d*3 and *d*5 or *d*6 both even, or vice versa.

The reader may care to prove these. If he wishes to try out the above methods further, we give the following data on possible reversals:—

All one-peg reversals are possible.

Two-peg reversals are possible unless the pegs are:

(a) *a*4 and *b*4;

(b) any two from *b*4, *d*2, *d*4, *d*6, and *f*4. These last are known as "key-pegs".

Three-peg reversals are possible unless the pegs are:—

(a) $a4$, $b4$ and any third other than $c1$, $e1$, $c7$ or $e7$;

(b) two key pegs and any third other than an outside corner peg;

(c) any three pegs from rows b, d and f.

The proofs require only the theory given here and a certain amount of manipulation. Many other interesting results are given in the *Scientific American* [3], to which we refer the reader.

Acknowledgements

For the majority of this, to J. H. Conway, R. L. Hutchings and J. M. Boardman. Very little of any importance is not theirs; in particular the basic ideas all are.

[1] E. Bergholt, "*The Game of Solitaire*" (U.S.A., 1920).
[2] M. Gardner, *The Scientific American*, June 1962.
[3] M. Reiss, *Crelle's Journal*, Vol. 54, p. 343 ff. (Berlin, 1857).
[4] E. Lucas, "*Récréations Mathématiques*", Vol. 1, part 5 (Paris, 1882).
[5] M. Charosh, *The Mathematics Student Journal*, March 1961 (U.S.A.)

π in Four 4's

BY J. H. CONWAY and M. J. T. GUY

IN *Eureka* 13, p. 11, there appears the following interesting variant of the four 4's problem:—

"Arrange four 4's, and any number of the ordinary mathematical symbols, to give as good an approximation to π as you can find. For example,

$$\sqrt{\sqrt{\left(\frac{4!!+4}{4!!}\right)^{4!!}}}$$

is a very good approximation to e, and can clearly be modified to be as good as we please.

(π = 3·1415926535897932...; logarithms and trigonometrical functions may not be used.)"

We consider this problem here, together with certain related questions. We shall allow ourselves the symbols (,), $+$, $-$, $\times$, $\div$, the usual notations for roots ($\sqrt[4]{4}, \sqrt{4}$), powers (4^4), factorials ($4!$), and the decimal notation (44, $\cdot 4$, $\cdot\dot{4}$). Factorials are to be of integers only—otherwise $\pi = \sqrt{[(-\sqrt{4}/4)!]^4}$—and we shall not allow such monstrosities as $\cdot\sqrt{4}$.

Our first remark is that the example given above for e may be improved so as to use only three 4's, since, as $n \to \infty$, $n/\sqrt[n]{n!} \to e$.

We may derive similar "explicit" formulae for various other interesting numbers. Thus $n\sqrt[n]{a}-n \to \log a$, so that we obtain a sequence of approximations to log 2 by putting $a=\sqrt{4}$, $n = 4, 4!, 4!!, \ldots$. We have in this way found formulae (in four 4's) for log 2, log 3, log 4, log 5, log 6, and $\log_a b$ for a variety of rational a and b (e.g. $\log_{10}2$, $\log_{10}3$). Our best result of this kind for π has seven 4's, and is derived from

$$\pi = \lim_{n\to\infty}\left(\frac{2^n \,.\, n!}{\sqrt{\sqrt{n}\sqrt{(2n)!}}}\right)^4.$$

We can also find $\log \pi$ in seven 4's, but as yet we have not been able to find any formula of this type for Euler's constant γ.

We shall now show that the above devices are unnecessary. In fact:

Theorem 1. Any real number may be approximated arbitrarily closely using only four 4's and the usual symbols.

It follows from the formula $n(\sqrt[n]{a}-\sqrt[n]{b}) \to \log(a/b)$ that for sufficiently large n we have:

$$2^m < 2^n(4^{2^{-(n-m-1)}} - 4^{2^{-(n-m)}}) < 2^{m+1}$$

for the limit of this expression as $n \to \infty$ is $2^m \log 4$, and $1 < \log 4 < 2$. If now m is any integer and $n > m$, both $n-m$ and $n-m-1$ are positive, so that we may write the expression as $2^n(\sqrt{}^{\,n-m-1}4-\sqrt{}^{\,n-m}4)$, the indices of the root signs indicating repetitions. Taking square roots k times, we have:

$$2^{m/2^k} < \sqrt{}^{\,k}(\sqrt{(4^n)}(\sqrt{}^{\,n-m-1}4 - \sqrt{}^{\,n-m}4)) < 2^{(m+1)/2^k}$$

Now we may take n to be of the form $4(!)^p$ so as to satisfy all the above conditions, and then the expression between the inequality signs will use only four 4's. Since the numbers $2^{m/2^k}$ for integers m and positive integers k are dense in the positive real numbers, we have proved our theorem. (For a negative number we need merely add another minus sign.)

Theorem 2. If we allow use of the integer part sign, every integer is representable with four 4's, and every rational number with five.

The first part is obvious, and the second part becomes a corollary of the first when we note that any rational p/q equals $m/(4(!)^n)$ for suitable integers m and n.

We may modify Theorems 1 and 2 so as to use other (positive integral) numbers instead of 4's. The only condition is that at most three of these may be 1's.

Finally we pose these questions:

(i) Is there an "explicit" form for π with less than seven 4's?
(ii) Is there an explicit form for γ?
(iii) Are the numbers $\sqrt{}^{\,n}(4(!)^m)$ dense in $x > 1$?

Problems Drive 1962

Set by R. L. Hutchings and J. D. Blake

A. In how many ways can an octahedron be coloured using two colours, each colour being used at least once?

B. Form twelve ordered pairs from the numbers 1 to 12, each number being used once as the first number in a pair and once as the second number, such that each possible difference (0, 1, . . ., 11) between the two numbers in a pair occurs exactly once.

[E.g. (1, 5), (2, 2), (3, 4), (4, 1), (5, 3) is a solution for the numbers 1 to 5.]

C. In the year 196*x*, there were eight candidates for the four places on the Archimedeans Committee. Each of these was also standing for one of the four offices, two standing for each of President, Vice-President, Secretary, and Treasurer. There was also a third candidate for each office. In how many ways can the new committee be formed?

D. Write down the next term in each of the following sequences:

(*a*) 1, 3, 4, 7, 11, 18, 29, . . .
(*b*) 1, 1, 4, 27, 256, . . .
(*c*) 4, 6, 9, 10, 14, 15, . . .
(*d*) 1, 10, 11, 101, 111, 1011, . . .

E. In the sum

$$\begin{array}{r} \mathrm{T\ H\ I\ S} \\ \mathrm{I\ S.S\ O} \\ \hline \mathrm{H\ A\ R\ D} \end{array}$$

the letters represent distinct digits in the octal system of numeration. Substitute digits for the letters to make this a correct addition.

F. Show how ten pawns may be placed on a chessboard to form five straight lines with four pawns in each line.

(A straight line is not necessarily an orthogonal or diagonal. The pawns are to be placed at the centres of the squares.)

G. Show how nine squares of sides 2, 5, 7, 9, 16, 25, 28, 33 and 36 may be arranged to form a rectangle without overlapping.

H. Represent the number 100 in as many ways as you can find using four identical digits and the normal arithmetical signs, not giving representations such as $(5\times5)/(\cdot5\times\cdot5)$ in which any other digit may be substituted for the 5.

K. Tom is twice as old as Dick was when Tom was half as old as Dick will be when Tom is twice as old as Dick was when Tom was a year younger than Dick is now. Dick is twice as old as Tom was when Dick was half as old as Tom was when Dick was half as old as Tom was two years ago. How old are Dick and Tom?

L. A and B are towns which have telephone exchanges, C is a village with 250 telephone subscribers. There are not more than 20 simultaneous calls between A and C, and not more than 5 between B and C. If A and B are 17 miles apart and C is one mile nearer to B than to A, will it require more wire to connect C's subscribers to the A or to the B exchange (including the necessary extra lines between A and B in each case)?

M. (*a*) At least two of these statements, apart from this one, are true.

(*b*) At least two of these statements, apart from this one, are false.

(*c*) At least one of these statements is false.

(*d*) x of these statements are true.

Given that, if you knew the value of x, you could determine uniquely which statements are true and which are false, determine the value of x.

Elastic Waves in Geophysics

BY C. C. L. SELLS

A STUDY of the phenomena which form the subject of this article is of great geological interest as it provides valuable information about the Earth's crustal structure. It is also very interesting theoretically since there is plenty of scope for argument, even after half a century of exploration.

Experimental evidence consists of seismograph records of earthquakes and explosions from large weapons being tested or from smaller charges fired deliberately for experimental purposes. These sources may all generate several different kinds of wave, and it is the classification and calculation of their travel times that engages attention. In solid media the two fundamental waves are compressional (P) waves of the form $\mathbf{u} = \nabla\phi$ (irrotational) and shear (S) waves of form $\mathbf{u} = \text{curl } \mathbf{A}$ (equivoluminal). The velocity of propagation of P-waves is always greater than that of S-waves, both being of the order of several kilometres per second.

In solid media with a plane boundary there also exists a surface wave, made up of combinations of ϕ and **A** satisfying the boundary conditions and decreasing exponentially with depth: the velocity of this wave (the Rayleigh wave, after Lord Rayleigh who discovered it in 1885) is smaller than the P- or S-wave velocity so that it is usually a late arrival on seismograms, and the motion is elliptical in the vertical plane containing the direction of propagation, a distinguishing characteristic on horizontal and vertical seismographs. The introduction of layers (the top layer may be a liquid as in the case of the ocean floor) generalises to Stoneley waves (after Stoneley who began a thorough investigation of their properties in 1924) with properties depending on all the media.

In addition, there is a wave which has a horizontal displacement perpendicular to the direction of propagation and so is of S type polarised horizontally (SH), and which again decays exponentially with depth from the deepest interface between two media. This wave (the Love wave) can exist only in a medium with at least one layer, but as it involves only one component per layer (the Stoneley wave has two) it is much easier to handle analytically.

What happens after a quake or explosion? Consider first the P wave emitted from the epicentre. When the wave front strikes the free surface or an interface, solution of the boundary equations always gives an S wave polarised vertically (SV), and usually a reflected P wave as well; in the case of an interface there will be a refracted P and SV wave also, and all these waves are found to obey laws like the laws of geometrical optics (Snell's law etc.). A Stoneley wave is also generated after a time lag depending on the depth of the source. On the seismogram we thus find the first arrival (the direct or once reflected P-wave), then a pattern of waves starting as P, reflected and finishing as SV, starting as P and refracted as P or SV into the next layer, reflecting back and forth in the layer, refracting back to the top layer and travelling to the receiver as P or SV, etc. Before 1900 many seismologists lost their way in the maze and identified the surface or Rayleigh wave (no layer) with S and missed the true S; this was first found by Oldham from the Assam 1897 earthquake records. (This delay was probably due to low damping of the early instruments; modern ones include a vane immersed in oil, or electromagnetic damping, to eliminate free vibrations excited by the generating disturbance which render later ground movement indistinguishable.)

Applying geometrical optics, equations for travel times of these waves may be set up, and if other station records of the same earthquake are available its time and the position of the epicentre may be found. Information about layer depths, and P and S velocities therein, may also be obtained.

The surface Love wave has also given much information directly. The period equation involves the layer thicknesses as parameters, and from an SH record with waves from the whole frequency spectrum a period curve may be plotted and these parameters found.

The general structure of the earth is now fairly well known from these studies. Firstly there is a sedimentary layer with P- and S-wave velocities about 5·6 and 3·1 km. per sec. respectively, then after a few kilometres there follows a granite or basalt layer about 10 to 20 km. thick with velocities about 8·0 and 4·4 km. per sec. An intermediate layer is separated by the Mohorovicic discontinuity (Moho for short) at 35 to 40 km. from a lower layer with velocities gradually increasing to about 13 and 7·5 km. per sec., and finally about half way to the earth's centre is the core in which P drops to about 9 km. per sec., but S has yet to be found. Research into local features such as continental shelving, layering of the sub-oceanic crust and jumps in the Moho near the American continental margin proceeds; much has been done, notably by Sir Harold Jeffreys in England and by W. M. Ewing and F. Press in America, among others, and much more remains to be done.

Letter to the Editor

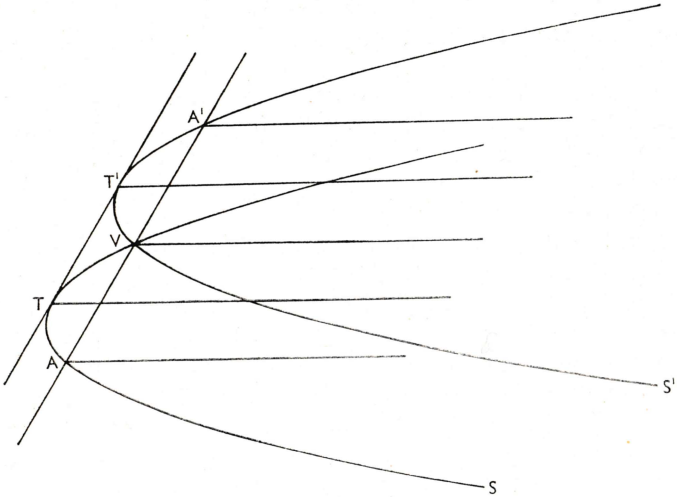
Dear Sir,

May we remark that a more elegant proof of Sells' Lemma (*Eureka* 21, p. 26) is obtained by that unjustly neglected method, reciprocation. The Lemma is:

"If X be a point of a conic, Y the point where the normal at X cuts the conic again, M the mid-point of XY, C the centre of curvature at X, F the Frégier point of X, then $(X, F; C, M) = -1$."

On reciprocating with respect to a circle centre X, the conic and its circle of curvature become parabolas, S and S' respectively, with three-point contact at infinity, i.e. with parallel axes (in the same sense) and equal latera recta (see below). F and C become the directrices of S and S' respectively, Y becomes the tangent at the vertex of S, and the point at infinity on XY becomes the latus rectum of S'. Hence in the transformed figure we have to prove that the following four lines form an harmonic pencil: the line at infinity, the directrix of S; the directrix of S', and the reflection of the latus rectum of S' in the tangent at the vertex of S. A small amount of simple algebra reveals this to be true.

It is perhaps not quite trivial that parabolas with three-point contact at infinity are parabolas "of the same size". Let TT' be their finite common tangent, V their finite common point, α their common point at infinity, and let AVA' be a straight line parallel to TT' cutting the parabolas in A, A' respectively. Consideration of the involution cut out on TT' by conics through α, α, α, V gives: a "horizontal" through V bisects TT'; but a "horizontal" through T



bisects AV (by the diameter property); thus we have five equally spaced "horizontals" through A, T, V, T', A'. If we shift the parabola S rigidly through $\overrightarrow{TT'}$, we see that it coincides with S' at V, T', A'; since parabolas with parallel axes are uniquely determined by three points, it follows that S, S' are congruent, i.e. have equal latera recta.

It is not hard to show that parabolas with four-point contact at infinity have both equal latera recta and also the same axis.

We should like to acknowledge the help of J. Faulkner with the latter part of the argument.

Yours truly,

H. J. Croft

Peterhouse,
Cambridge

P. V. Landshoff

Princeton,
New Jersey

Book Reviews

Mathematical Scholarship Problems. Compiled by J. C. BURKILL and H. M. CUNDY. (Cambridge University Press.) 7s. 6d.

It is a matter of common experience that a far better idea of the requirements of a particular examination is to be obtained from working through old papers than from a study of the syllabus: how much more valuable is this practice when—as in the case of the Cambridge Open Scholarship Examinations—the published syllabus is either very sketchy or non-existent! This method of preparation, however, does make it a tiresome matter to concentrate on topics, when several bookfuls of papers may have to be searched to find suitable material. This collection of problems taken from recent papers is particularly welcome, therefore, in that the problems are classified in groups of five, a detailed table of contents making it a simple matter to locate any desired topic. The content of Cambridge Scholarship papers changes but slowly—no doubt the very custom of using them for preparation for the next exam is a partial cause of this—but changes there are, and the compilers' selection reflects current trends; and besides the routine types of question, there is a refreshing section of teasers headed "Miscellaneous". Hints for solutions are provided, adequate enough to be helpful, not too full to spoil the fun. Perhaps this is a strange place to review a book that the undergraduate might be expected to have put behind him; but then the scholar is generally a two-years Part II candidate, and the less brilliant freshman who works through this book will find it a useful refresher course, while even the Major Scholar will find quite a few things to think about.

A. R. PARGETER.

Topology of 3-Manifolds and Related Topics. Edited by M. K. FORT, Jnr. (Prentice-Hall, Inc.) 80s.

In August, 1961, a "topology institute" took place in the University of Georgia. This book is the proceedings of that conference and, as such, contains papers by some thirty different authors. Every article is concerned with research, with new ideas and results, but the whole atmosphere of the book is informal and occasionally even friendly. Misprints, some of them quite puzzling, are scattered throughout, and there are even a few false theorems included, but, as it is essential that a report of this kind should be published as soon as possible, many of them are excusable.

Chapter I is devoted to decompositions and subsets of three-dimensional Euclidean space. In principle this consists of discussion of the weird spaces that can be obtained from Euclidean space by shrinking various cunningly selected subsets to single points. Here we have a careful explanation, by G. T. Whyburn, of the vocabulary involved, discussions of several specific spaces, and even an exposition of the celebrated Bing dogbone space written by its own creator.

The title mentions "related topics". These comprise all forms of topology that are inspired or motivated by geometric imagination, including much work on n-manifolds which are usually considered from the combinatorial point of view. There is quite a long section on Knot Theory, most of it by R. H. Fox, who eventually poses forty unsolved problems for consideration. The section on the Poincaré Conjecture, while still leaving the famous problem unsolved, suggests possible methods of attacking it, and includes E. C. Zeeman's solution of the generalised conjecture in dimensions greater than four. Zeeman's theory on "The Topology of the Brain and Visual Perception" is given in the last chapter; we take a look at the hand that fed us.

There is not space here to mention all the subjects discussed, and for further information the book should be read. It will not help the undergraduate in Tripos, but it will be a necessity for those contemplating topological research with a flavouring of geometry. W. B. R. LICKORISH.

Linear Differential Operators. By C. Lanczos. (Van Nostrand Co. Ltd.) 80s.

This book is a very clear and readable account, from the applied mathematician's point of view, of the analytical properties of linear differential operators and several related topics. It is not intended as a first introduction to the subject, but could be read with profit by any Part II student or research worker in mathematics or physics.

Professor Lanczos begins with chapters on interpolation and Fourier analysis and deals at some length with difficulties such as the Gibbs oscillations; much of this work is subsequently used for illustrative purposes. The theory proper begins with a résumé of matrix calculus. The analysis is extended from matrices to differential operators by means of a particularly lucid consideration of a function as a vector, and then an exhaustive treatment is given of Green's function. There follows a short chapter on the application of these methods to some communications problems, including the difficulties of telephone conversation in a foreign language: the author is Hungarian by birth, has spent many years in the United States, and now lives in Ireland.

The next chapter, entitled "Sturm-Liouville Problems", introduces most of the usual special functions of mathematical physics. Considerable use is made of the "K.W.B." approximation for solving differential equations. It is not until the next, and penultimate, chapter that homogeneous equations with inhomogeneous boundary conditions are discussed in detail, the emphasis having been on Green's function and therefore the opposite type of problem. Separation of variables is dealt with briefly, and a valuable discussion is given of "ill-posed" problems and parasitic spectra. The last chapter gives numerical methods for solving initial value problems, and estimates of their errors.

Because of its clarity of exposition and stimulating presentation of material, this book is highly recommended. J. F. HARPER.

Foundations of Geometry and Trigonometry. By H. Levi. (Prentice-Hall, Inc.) 72s.

The object of this book is to give a "modern precise algebraic approach to geometry—covers trigonometry and analytic geometry." It is designed to be used as a textbook in a mathematics course which, in England, would commence in the first year after "O" Level. Thus it would be given to students who should already possess a sound algebraic knowledge that includes some familiarity with set language, equations and the real number system (notions which in a modern mathematics course would have been introduced at an early stage).

The keystone of the book is a modern approach to geometry based upon five axioms. They are:—

Axiom 1: If A and B are any distinct members of a non-empty set θ, there is one, and only one, given 1-1 correspondence which pairs A with 0 and B with the number 1.

Axiom 2: If f and g are any two numbers of θ' then there is a linear expression $ax+b$ such that if X is any member of θ, if f pairs X with the number x and if g pairs X with x', then $x' = ax+b$.

Axiom 3: If A and B are any distinct points of a set θ there is one, and only one, line of θ which contains them.

Axiom 4: If p is any line of θ and if P is any point of θ then there is one, and only one, line of θ which is parallel to p and which contains P.

Axiom 5: If p and q are lines of θ then any parallel projection from p to q is an affinity from p to q.

With the help of these five axioms, the geometry of the line, the affine plane and the Euclidean plane are investigated with a very sound algebraic approach. The concepts of angle, direction, congruency and similarity are all treated algebraically with a rigour whose overpowering logic would have flattered Euclid—axiom, definition, lemma, theorem, corollary, exercise! Pythagoras's theorem is overcome by this mathematical sledge-hammer. Undaunted, the book now introduces trigonometry algebraically, and concludes with the geometry of the circle.

The impression left on one by this book is that as a precise analytical study of geometry it is an excellent work, but to what purpose? The abstract approach would appeal to a professional mathematician, but to the schoolboy it would seem to be an enormous work requiring thought beyond—far beyond—his powers of reasoning and which would appear to him to achieve results by a most arduous route which could be more easily obtained by a traditional approach.

P. DUNNE

Geometry of Complex Numbers. By H. SCHWERDTFEGER. (Oliver and Boyd, Ltd.) 30s.

The first section of this book is devoted to the study of the "Analytic Geometry of Circles". The Euclidean plane is parametrised by the complex numbers and it is pointed out, immediately, that to every circle in the plane there corresponds a set of Hermetian matrices. Not every Hermetian matrix represents such a circle, so the idea of an "imaginary" circle, with an imaginary radius and real centre (i.e. a complex number) is introduced. In the ensueing discussion the reader might sometimes be tempted to wonder if he is not studying Hermitian matrices rather than "circles". The author apologises in his introduction for his ambiguous usage of the words "real" and "imaginary", giving warning that there are even non-real points on the real axis; he feels bound to adhere to the conventional terminology. In this introduction it is stated that "Primitive concepts and facts of topology will be used." They are. The topologist should omit the sections which introduce the point at infinity.

Once the terminological difficulties of the first part of the book have been mastered, the remaining three-quarters of the book present a thorough discussion of Moebius transformations and of two-dimensional non-Euclidean geometries. A complete account of these subjects, written in English, will certainly be a useful work of reference, and it should be noted that a comprehensive bibliography is included. The book aims at showing an inter-relation between geometric and algebraic notions. Practically no prior knowledge of group theory is assumed, and many examples, with hints on how to solve them, are given throughout.

W. B. R. LICKORISH.

A Book of Curves. By E. H. LOCKWOOD. (Cambridge University Press.) 25s.

This book is a useful addition to the literature on mathematical recreations. It gives simple directions which enable anyone to draw numerous curves and Mr. Lockwood includes some useful instructions about the sizes of paper and appropriate dimensions for drawing many of them.

The book has two parts; the first, "Special Curves", describes thirteen curves, such as the conic sections and the cardioid, with eight or more pages for each. At least two methods, illustrated by clear diagrams, are given for drawing every curve. The production of a curve as an envelope of straight lines or circles leads to particularly attractive pictures. Many geometrical properties are proved and summarized, which may make it of some use as a reference book. There are also short historical notes which, not surprisingly, are brief in most cases.

The second part is entitled "Ways of finding new curves". It is shorter than the first part but contains more material for those constructing curves for pleasure. The ten chapters in this part often reveal the examples of Part I in their correct perspective as members of more general sets of curves.

The book is well described by a statement on the dust cover: "This book opens up a new field of mathematics at elementary level, one in which the element of aesthetic pleasure, both in the shapes of the curves and in their mathematical relationships, is dominant". D. H. PEREGRINE.

Mathematics in Your World. By K. MENNINGER. (G. Bell and Sons Ltd.) 21s.

This is a well-written mathematics book for the layman which does not use any calculus. It caters more for someone who is interested in seeing how mathematics is applicable to everyday problems than for one who is interested in the ideas of pure mathematics.

Probability, with examples from insurance and gambling, and the interplay between surface area and volume (why can't an elephant jump like a flea?) are emphasised. There is also an interesting chapter on highway construction. In pure mathematics brief mention—too brief to say much—is made of countable sets and topology. Each section is complete in itself.

Mathematicians at all levels above first-year sixth form would like the book to be more advanced, especially the sections in small print, but all could learn some interesting facts. It has been translated from the German, so some of the examples are not very British, i.e. the Nuremberg Ring is mentioned rather than the T.T. Course, but this adds to the interest without removing the clarity. G. B. TRUSTRUM.

Playing with Infinity. By R. PÉTER, translated by Z. P. DIENES. (G. Bell and Sons Ltd.) 25s.

This book is not written for the mathematician, it is intended primarily for the layman whose knowledge of mathematics is a dim memory of his schooldays. It describes the structure of the subject rather than its use, starting, perhaps too slowly, with elementary arithmetic. The language is deceptively simple, leading the reader gently onward to comparatively sophisticated concepts such as symbolic logic and Gödels theorem. The author is a teacher, and many of her examples were provided by her own class, though others were taken from all walks of life, including catching lions in the Sahara desert . . ..

The subtitle (*Mathematics for Everyman*), though rather well worn, is a much better description of the contents than "Playing with infinity". There are many books of this type on the market now, and this is well worthy of its place amongst them. It might make a useful present for that relative who doesn't understand what you're talking about, but don't forget to read it yourself first! C. D. RODGERS.

In the past year, we have received a large number of reprints and new editions, many in soft covers. As they may now reach a new buying public, we include the following reviews of some of them. The reference to a former Eureka *is to a previous review of the same book.*

Partial Differential Equations of Mathematical Physics. By H. BATEMAN. (Cambridge University Press.) Paper Edition. 27s. 6d.

The aim of this book is to discuss methods of solution of Partial Differential Equations; the author does this not only in bookwork but also through a large number of practical examples drawn from all branches of applied mathematics. The result is that there is a vast amount of information within the book, but it is in a form not entirely suitable for quick reference. The first, and longest, chapter in particular contains a comprehensive survey of elementary applied mathematics and might well prove rewarding reading to a Tripos student.

The book was written over thirty years ago and is still used as a standard work of reference by many research workers, however the inference should not necessarily be made that it is worthy of such a distinction.

R. W. BRAY.

The Theory of Ordinary Differential Equations. By J. C. BURKILL. (Oliver and Boyd, Ltd.) Second Edition. 8s. 6d. (*Eureka* No. 21.)

Appendices on the Laplace Transform and on the equation

$$P\,dx+Q\,dy+R\,dz = 0$$

have been added. In them the transform is defined, consequent properties are stated and a proof of the theorem "A continuous function is uniquely determined by its Laplace Transform" is given; techniques of solving $P/dx = Q/dy = R/dz$, as well as the above equation, are also discussed. The book contains proofs of many useful theorems, clearly set out and given in fair detail; it should prove useful to those who have a reasonable knowledge of mathematics.

L. TODD.

Cartesian Tensors. By H. JEFFREYS. (Cambridge University Press.) Students' Edition. 8s. 6d.

The object of the book was to illustrate the simplicity of many of the equations in mathematical physics written in tensor notation. Consequently most of the book is devoted to selling the technique and requires a knowledge of mathematics which exceeds that of most students meeting tensors for the first time. However chapters I and VII provide a good explanation of the method and the book is worth buying if only for this.

K. JOHNSON.

The Methods of Plane Projective Geometry based on the use of General Homogeneous Co-ordinates. By E. A. MAXWELL. (Cambridge University Press.) Students' Edition. 13s. 6d. (*Eureka* No. 9.)

This well-known textbook first appeared in 1946; the latest reprint is now available in a paper cover and this should make it even more popular. As the original reviewer pointed out, it contains an outstanding chapter on (1-1) algebraic correspondences; however, its special merit is the emphasis laid on methods likely to be of value in more advanced work.

M. COCKRILL.

Elements of the Topology of Plane Sets of Points. By M. H. A. Newman. (Cambridge University Press.) Students' Edition. 18s. 6d. (*Eureka* No. 15.)

The aim of the book is "to provide an elementary introduction to the ideas and methods of topology by the detailed study of certain topics, with special attention to the parts needed in the theory of functions"—and it achieves just that.

The book has two parts, a fact which is by no means clear from the general layout of the book. The first part deals with the topology of a general metric space: after two introductory chapters on sets and metric spaces, it has a chapter each on continuous mappings and connection. The second part introduces gratings with integers mod 2 as coefficients: it uses this to prove Alexander's Lemma and Jordan's Theorem. The book contains several separation theorems and, in particular, one chapter is on domains in a plane and one on homotopy properties.

The book serves well its purpose of being an introduction to Analytic Topology. It proves several interesting theorems, for instance Cauchy's Theorem in its strongest form, and on the whole it is a clear and readable account. There are instructive examples and exercises scattered throughout the book. J. D. P. Meldrum.

Volume and Integral. By W. W. Rogosinski. (Oliver and Boyd, Ltd.) Second Edition. 10s. 6d.

This is a very readable account of the theory of the Lesbesgue Integral and its foundations in set theory. The Riemann Integral is also treated from the same standpoint so that the advantages of the former are clearly seen. A geometrical approach has been adopted throughout the book, making it easily accessible to Part II students.

The second edition, apart from a few corrections, is no different from the first. R. J. Jarvis.

A First Course in Mathematical Statistics. By C. E. Weatherburn. (Cambridge University Press.) Students' Edition. 18s. 6d.

This is a paperback edition of a book first published in 1946 which seems to be enjoying a new lease of life in its new form. Rightly so, for the treatment is thorough and assumes the knowledge of very little mathematics. An understanding is only acquired the hard way, i.e. by working through a wide selection of examples, and these are provided at the end of each chapter. It seems a pity, though, that room could not have been found for some brief solutions thereof. A. M. J. Davis.

*Principia Mathematica to *56.* By A. N. Whitehead and B. Russell. (Cambridge University Press.) Students' Edition. 17s. 6d.

This book is a paperback edition consisting of Part I and Section A of Part II, together with Appendices A and C, the Introduction and the Introduction to the Second Edition.

The Introduction to the Second Edition consists mainly of suggestions for the improvements which could be made. These come mostly from work done by mathematicians after the appearance of the first edition. The two most important are the introduction of the Sheffer stroke symbol and Nicod's reduction of the required number of primitive propositions. This section also contains explanations of the authors' ideas and difficulties. The Introduction proper consists of the authors' philosophical concepts of their subject and they deal with notation and the theory of logical types (which was needed to avoid the numerous paradoxes that had arisen in set theory). This Introduction is an invaluable help in understanding the rest of the book.

The remainder, in fact, develops Mathematical Logic. Part I deals with the theory of deduction, of apparent variables, classes, relations and products and sums of classes, whilst Part II deals with unit classes and couples. The appendices contain the modifications brought about by the stroke symbol and a discussion on truth functions.

The presentation is good, although one line is missing at the bottom of page 147, and the book itself is a well-planned abridgement of a classic on the foundations of mathematics. For anyone interested in the subject, this provides a very good introduction, detailed and thorough—at the price, it makes a very good buy. J. D. P. MELDRUM.

Solutions to Problems

Squares and Circles

1. Number of cubes =

$$\sum_{p=1}^{n} \sum_{r_j\text{'s}=1}^{n} \frac{(-1)^{p+1}}{p!} \times \text{H.C.F.}\,(a_{r_1}, a_{r_2}, \ldots, a_{r_p}) \times \epsilon\,(r_1, r_2, \ldots, r_p)$$

where $\epsilon(r_1, r_2, \ldots, r_p) = 1$ if r_j's all different,
$= 0$ otherwise.

Thus, in the case of a rectangle of sides a and b, the number of "cubes" is $a+b-\text{H.C.F.}(a,b)$.

2. The statement requires the radii of the florin and the half-penny to be equal, which is obviously incorrect. The easiest way to deduce this condition is to consider the two cases when the order of the circumferential pieces is FFHHP and FHFHP, and then employ elementary geometry.

Problems Drive 1962

The average marks (out of 10) obtained by the 18 pairs competing are given in brackets. The highest total was 59/110.

A. (2·6) 20.

B. (3·1) (1,12), (12,2), (2,11), (11,3), (3,10), (10,4), (4,9), (5,1), (8,5), (9,7), (7,8), (6,6).

C. (0·7) 886.

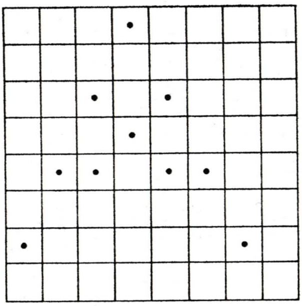
D. (7·2) (*a*) 47. $u_{n+2} = u_{n+1} + u_n$.
(*b*) 3125. $u_n = n^n$.
(*c*) 21. Ordered products of pairs of primes.
(*d*) 1101. Primes in the binary scale.

E. (1·6)

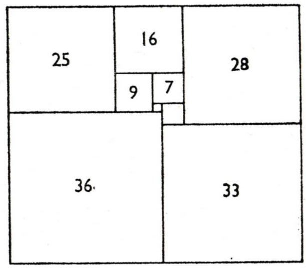
$$\begin{array}{r} 4612 \\ 1225 \\ \hline 6037 \end{array}$$

(There are three other solutions)

F. (1·7)



G. (3·4)



H. (4·9) $99+9/9$, $5!-(5\times5)+5$, $5\times[(5\times5)-5]$, $4(4!+4/4)$, $4\times4!+\sqrt{4}\times4$, $33{\cdot}\dot{3}\times3$, $99{\cdot}\dot{9}$, $(1/{\cdot}1)^{1+1}$, etc.

K. (0·8) Tom is 30, Dick 26.

L. (5·3) More wire is needed to connect C to B.

M. (2·6) $x = 3$.

Mathematical Association

President: Prof. V. C. A. Ferraro, Ph.D., D.I.C.
(Queen Mary College, London)

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of mathematics, in schools, colleges and industry.

The subscription to the Association is 21*s.* per annum; to encourage students, and those who have recently completed their training, the rules of the Association provide for junior membership for two years at an annual subscription of 10*s.* 6*d.* Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published four times a year and deals with mathematical topics of general interest.

ADVERTISE

In "Eureka" at these moderate rates.

Full page	£5 : 0 : 0
Half page	£3 : 0 : 0
Quarter page	£2 : 0 : 0

How District Bank helps you make the most of your money

As soon as you have money to manage and your own life to live, you need a bank account. That means now—and District Bank welcomes students' accounts. Friendly advice on money matters is always at your disposal. And we believe you will find a bank account particularly helpful in these four principal ways:

1. **Money keeps best in the bank.** You can draw it out as you want it . . . but it's less likely to "vanish into thin air."
2. **You know where you stand.** At any time, you can see exactly what you've paid in, paid out, and drawn out. Budgeting is much easier.
3. **Cheques are the safest** way to pay bills—you have a permanent record. And the Bank will make regular payments (subscriptions, etc.) automatically if you wish.
4. **When you're away,** at home or on holiday, you can arrange to cash cheques at any convenient branch. When you travel, the Bank will supply travellers' cheques and foreign currency.

The manager will be pleased to see you at:

DISTRICT BANK 1829

LIMITED

9 Trinity Street, Cambridge

Manager: Mr. C. W. Dodgson

OVER 570 BRANCHES . . . AND STEADILY EXPANDING

PRINTED BY
W. HEFFER & SONS LTD.,
CAMBRIDGE, ENGLAND