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Eureka Editor
The Archimedeans
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WA
United Kingdom

archim-eureka@srcf.net

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The Cambridge University Mathematical Society; Junior Branch
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Contents

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Editorial

At the end of the Easter Term the Archimedeans are rather like Don Quixote who jumped on his horse and galloped off in all directions at once. As this year the editor did likewise, it is rather a surprise even to him to see the appearance of another *Eureka*. The editor found himself in ten countries, a number which he carefully counted on his abacus, newly acquired in Russia. The abacus is such a common sight on shop counters there that Russians are surprised to hear that we don't have them in England, for how can we manage without them?

Our contributors too are more widely scattered than has been usual. Mr. Cross writes from Birmingham, the Haselgroves from Manchester, and Prof. Menger and Dr. Miller from America, though the latter soon returns to Cambridge. The remaining articles are contributed by Archimedeans.

We regret that the rising costs of printing have at last made it necessary to raise the price of *Eureka* to 2s. 6d. Due mainly to our steadily increasing circulation we have been able to keep our price constant at 2s. since 1952, though the first two-shilling *Eureka* appeared in 1948. The first edition in 1939 cost only 6d. However, we hope that we shall continue to enjoy your support. You will find precise details of our new terms for postal subscribers on page 4.

The Archimedeans

This is the Jubilee Year of the Archimedeans. Since its first meeting in 1935, the Society has gone from strength to strength and we hope that this year will be the best ever.

We have as usual a full list of meetings and plenty of social activities. If you are a new member, I would like to stress that we ask all our speakers to make their lectures intelligible to everyone, including people with only a limited mathematical background. So please come along and support them.

Amongst our speakers this year we have the distinguished atomic scientist, Sir William Penney. Also coming to talk on subjects connected with Nuclear Physics is Professor Salam of Imperial College, London. To go with these meetings, there will be a visit to the Research Establishment at Harwell. We shall also be visiting I.B.M. in London, and the Observatory and the Cavendish Laboratory in Cambridge.

Because of their popularity, tea-meetings again form an important part of our programme. At these research students reflect on some

of the less serious branches of mathematics, and at one, the annual Problems Drive, Archimedeans can pit their wits against each other and last year's winners. On the social side we have a Punt party and a Ramble, and we shall arrange a visit to see "West Side Story" in London. We hope this year to hold a debate and also a Christmas Party.

The Music Group will meet again and the Bridge Group will be happy to have you whether expert or novice. The Play-Reading Group, restarted last year, will continue their meetings. You will again be able to buy and sell textbooks through the Archimedeans' bookshop.

The Committee has tried to arrange an interesting programme. If you have any ideas or complaints please tell them to the Secretary or write them in the book in the Arts School.

P. H. FROST, *Secretary.*

Mathematical Association

President: Dr. E. A. MAXWELL

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 21s. per annum: to encourage students, and those who have recently completed their training, the rules of the Association provide for junior membership for two years at an annual subscription of 10s. 6d. Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published four times a year and deals with mathematical topics of general interest.

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For the benefit of persons not resident in Cambridge we have a postal subscriptions service. You may enrol as a permanent subscriber and either pay for each issue on receipt or, by advancing 10s. or more, receive future issues as published at approximately 25 per cent. discount with notification when credit has expired. The rates this year are:

For those who pay on receipt 2s. 6d. (+ 2d. *postage*).

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Some copies of *Eureka* Nos. 11, 13, 15, 16, 17 (1s. each), 18 (1s. 6d.) 19, 20, 21, 22 (2s. each) are still available (postage 2d. extra on each copy). *Set of ten* 12s. 6d. *post free.* We would be glad to buy back any old copies of Nos. 1 to 10 which are no longer required.

Cheques and postal orders should be made payable to "The Business Manager, *Eureka*" and addressed to The Arts School, Bene't Street, Cambridge.

Gulliver in Applyland

By Karl Menger

Before "he married Mary Burton, second daughter . . . of a hosier in Newgate Street," young Gulliver, according to Swift,[1] "learned navigation and other parts of mathematics useful to those who intend to travel." Mathematics for its own sake began to interest Gulliver, a few years later, because of his adventures (not mentioned by Swift and only recently recorded in the *Mathematical Gazette*[2]) on an island that he called Land without One, Two, Three. Upon his return to England he began to study what then—in the early 1700's—was the most advanced branch of mathematics—analysis: the theory of functions and fluents. On a later trip to the Land without One, Two, Three,[3] Gulliver discovered a neighbouring isle called Applyland—a name that had originated in the preoccupation of the insular arithmeticians with scientific applications.

The Applylanders said that an object was *multiplicatively connected* with another object if the former was a multiple of the latter. If they wished to indicate that a pound was multiplicatively connected with an ounce but did not intend to specify the number establishing the said connection (i.e., the number expressing *which* multiple of an ounce was equal to a pound), then they wrote

$$(1) \quad lb. = lb. \times oz.$$

On the other hand, they would not write

$$(2) \quad yd. = yd. \times oz.,$$

since a yard is not multiplicatively connected with an ounce at all. In order to indicate that a yard is some multiple of a foot as well as some multiple of an inch, the Applylanders wrote

$$(3) \quad yd. = yd. \times ft. = yd. \times in.$$

"Suppose," Gulliver wrote in his diary, "someone back home wished to express the fact that the weight w of an object was functionally connected with its mass m, but did not intend to specify the function that established the said connection. Following the example of the Applylanders he would have to write

$$(1') \quad w = w(m).$$

[1] Swift, *A Voyage to Lilliput*, Chapter I.

[2] Menger, "Gulliver in the Land without One, Two, Three," *Mathematical Gazette*, 53, 1959, pp. 241–250.

[3] Menger, "Gulliver's Return to the Land without One, Two, Three," *Mathematical Monthly* (appearing shortly).

Denoting the velocity of the falling object by v he would not, on the other hand, write

$$(2') \quad v = v(m),$$

since v is not a function of m at all. In order to indicate that the velocity in ft./sec. is some function of the distance in feet travelled as well as some function of the time in seconds elapsed, such a man would write

$$(3') \quad v = v(s) = v(t).$$

Just as the symbol *yd.* in (3), the letter v in (3′) would have three discrepant meanings: a specific fluent (the velocity in ft./sec.); a specific function (since $v = 8\sqrt{s}$, the function $8\sqrt{}$); and another specific function (since $v = 32\,t$, 32 times the identity function)."

A few days later, Gulliver added a postscript:

"Rereading the preceding entry I am reminded of a little incident involving the older sister of my wife, Mary Burton. She is married to a man whose first name happens to be Lemuel like mine. Once, at a little gathering in Newgate Street, someone mistakenly introduced me as the husband of my sister-in-law, whereupon the latter exclaimed: 'Oh no. This Lemuel is the Lemuel of Mary Burton,' and, pointing to my older brother who was sitting next to me, she added: 'And the Lemuel of this here David." What reminds me of her way of talking is the formula $v = v(s) = v(t)$.

Newton never would write anything of that kind."

In this last remark, Gulliver was of course right but he was too optimistic to foresee that, 200 years later, formulae such as (3′) would become rather common.

Even though Gulliver considered the matter unimportant in itself, he recalled it as a symptom of a confusion of objects with numbers connecting objects, of which he found further evidence in the Applylandish theories of the predecessors of numbers and differences between amounts and lengths.

The *immediate predecessor* of a number was indicated by an asterisk after the numeral. The numerals, each of which included the symbol | (pronounced: stix), had been taken over from the Land without One, Two, Three (cf. l.c.[2]). Hence the Applylanders wrote: $(6|)^* = 5|$, $(5|)^* = 4|$, though, on conceptual grounds, some preferred $6^*|$ to $(6|)^*$, and $5^*|$ to $(5|)^*$. There were no numerals $3|$, $2|$, $1|$; the islanders wrote $|^t$, $|^p$, $|$ (pronounced: stixtriple, stixpair, stix). Hence everyone wrote $(|^t)^* = |^p$ and

$(|^{p})^* = |$; and they said: the predecessor of stixpair is stix. The *remainder* of an amount or a length B after the subtraction of a smaller A was denoted by B — A. For instance, they wrote *yd.* — *in.* and *yd.* — *ft.*

To Gulliver's bewilderment, the Applylanders considered the asterisk and the minus sign, as applied to lengths, as synonymous. "Why, those symbols refer to altogether different worlds," he said. "The former belongs to the logical realm of numbers, the latter to the domain of observables. Besides, the predecessor is associated with a single number, and the difference, with a pair of observables." He put these ideas in a little paper in which he also articulately connected predecessors and differences:

If an amount or a length B is connected with an amount or a length A by a number, then B — A is connected with A by the predecessor of that number; in a formula, if B = nA, then B — A = n*A. Thus, since ft. = 36 in., ft. — in. = 36* in. = 35 in. Even for numbers,[4] since $35\,| = 7 \times 5\,|$, $35\,| - 5\,| = 7^* \times 5\,| = 30\,|$; and $n\,| - |\, = n^*\,|$ for any n.

"It is distressing," wrote the Immortal recommending that the paper be rejected, "to see a man who in his youth mastered useful parts of mathematics waste his time on minutiae of terminology and trivia of notation. Certainly the paper is not up to the standards of the *Applylandish Achievements.*"

At the same time, Gulliver wrote in his diary:

"The Applylander's odd identification of predecessors of numbers with differences between two observables has no parallel in analysis back home. The closest analogue that I can invent is as follows: For each function, denote the derivative by a prime next to the function symbol[5] as in $(\sin x)' = \cos x$, although, on conceptual grounds, some might prefer $\sin' x$ to $(\sin x)'$. But unless they introduced a permanent symbol for the identity function[6] they all would have to write $(x^5)' = 5x^4$.

The rate of change of s, the distance travelled, with t, the time, is denoted by $\frac{ds}{dt}$. If s is connected with t by the sine

[4] It will be remembered (loc. cit.[2]) that in writing 4 ft. and $4 \times 5\,|$ the islanders used only the stix-free part of the numeral $4\,|$, just as in writing $\sin t$ and $\sin \log x$ we use only the x-free part of the traditional symbol $\sin x$ for the sine function.

[5] Here, Gulliver anticipated Lagrange's famous concept of *fonction dérivée* and Lagrange's notation for the derivative.

[6] In possession of a symbol, say, j for the identity function, one would of course write simply $\sin' = \cos$ and $j^{5\prime} = 5\,j^4$.

function, then $\frac{ds}{dt}$ is connected with t by the derivative of the sine function; in a formula, if $s = \sin t$, then $\frac{ds}{dt} = \sin' t = \cos t$. If they used j for the identity function, they might even write: $j^{15} = j^5(j^3)$ implies $\frac{dj^{15}}{dj^3} = j^{5\prime}(j^3) = 5j^4(j)^3$ and $\frac{df}{dj} = f'$ for any differentiable f. But would anyone in Europe consider $'$ and $\frac{d}{d}$ as synonymous? Could they overlook that the derivative is associated with a single function, and the rate of change, with a pair of fluents or functions? Could they ever confuse fluents introduced by Newton with functions introduced by Leibniz?"

And in his optimism, Gulliver concluded this entry in his diary by writing with a flourish:

"NEVER."

Squaring the Bishop

FROM Charles Babbage, "Passages from the Life of a Philosopher":

> Another amusing puzzle . . . is called squaring words, and is thus practised:—Let the given word to be squared be Dean. It is to be written horizontally, and also vertically, thus:—
>
> D E A N
> E . . .
> A . . .
> N . . .
>
> and it is required to fill up the blanks with such letters that each vertical column shall be the same as its corresponding horizontal column, thus:—
>
> D E A N
> E A S E
> A S K S
> N E S T
>
> The various ranks of the church are easily squared; but it is stated, I know not on what authority, that no one has succeeded in squaring a bishop.

We shall be interested to hear if any reader succeeds where Babbage failed.

Group Theory in the Sixth Form

By F. M. Hall

Mathematics teachers are becoming interested in the possibility of teaching elementary abstract algebra to their sixth forms. Already a start has been made in America, and it is probably only a matter of a few years before the subject is introduced into our own schools.

Recently during teaching practice, I had the opportunity of giving a course on group theory to four senior boys in their last year at school. All had obtained university scholarships, two in mathematics and two in classics but none of them intended to read mathematics at university, though they were interested in learning a little advanced work while waiting to go up.

I realised I must spend a lot of time on the preliminaries. I used as many examples as I could throughout, and avoided giving abstract definitions without preparation. I commenced with sets and suggested that we try to consider more general sets than they had previously done with only real and complex numbers. This led to the formulation of the laws of algebra, and by a detailed investigation of these we discovered that there was one fundamental process in algebra (addition or multiplication) and were then able to define a group.

I spent a long time on examples of groups, which I obtained from sets of numbers besides talking about some abstract groups. Rotation groups I found difficult to explain (even with the aid of a box which I became adept at spinning around various axes) but once understood they were a very useful and interesting source of examples. The cyclic groups were quickly appreciated, that of order two becoming popular when it was pointed out that "plus" and "minus" were the same when working with it, so that it is impossible to make mistakes in sign!

I was surprised at the ease with which the idea of a subgroup was assimilated, and I was soon able to prove Lagrange's Theorem. This was one of the highlights of the course, being an important theorem which can be proved rigorously and fairly simply, although it does involve the idea of a coset, a much more difficult concept than that of a subgroup. Lagrange's Theorem enabled us to investigate the structure of various groups, and in particular to obtain all those of orders up to six. This kind of work was concrete and so easily appreciated. I naturally attempted to follow this with a discussion of invariant subgroups and factor groups. This was found extremely difficult, and factor groups in particular

suffered from the impossibility of giving easy and convincing examples and applications. It was probably hardly worth the time and trouble spent on it.

With the next topic, homomorphisms, I again had trouble in justifying its importance. Near the end of term I just had time to give the results of some of the more advanced theorems. Most successful was the part concerned with Sylow's Theorems on prime power subgroups, it being easily understood, while important results such as the Jordan-Hölder Theorem were, I found, exceedingly difficult even to state simply.

I had intended to include some matter on rings and fields at the end of the course, but I had time only for a quick sketch of the definitions. I regretted this for ever since the definitions of groups the boys had been anxious to learn about other systems. I would not repeat the course without greatly extending this part of it.

It will be seen that difficulties always started, as might be expected, when the subject became too abstract. Thus subgroups were appreciated but factor groups, being strange and relatively complicated, were not. Investigations on particular groups always proved fruitful, and I was continually being asked to give counter examples, for instance of products which do not lead to groups, and of subgroups which are not invariant.

From this experience it would seem that we will have to select the subject matter in the teaching of abstract algebra carefully. The basic concepts of group, ring and field should be dealt with at length. Of further topics, subgroups form an obvious choice, while some work on homomorphisms should be possible. There is probably scope also in factorisation theory.

We will have to be careful generally to keep our work relevant and fairly concrete. Abstract ideas can be understood by schoolboys only if supported by numerous examples, and the work must appear useful in a mathematical, if not also in a practical sense. I found the main difficulty to be that of finding theorems which were simple enough to be understood, yet important enough to give the proving of them some point.

It may seem impossible to give a course in abstract algebra to a normal intelligent sixth form, but I would disagree with this. Experience in teaching projective geometry and calculus shows us that difficult subjects can be mastered successfully at school level, and much of the present work in algebra and trigonometry which is trivial and soon forgotten can profitably be replaced by more important and worthwhile studies.

Printing Mathematics

By Peter Basnett

Mathematical printing presents several problems to the compositor. The chief of these are the availability and the correct arrangement of the pieces of type for each of the symbols used by mathematicians, although the actual reading of a hastily written manuscript may present grave difficulties.

When it is realised that a typical piece of type (see Fig. 1) is 0·918 in. high, 0·1383 in. deep, and varies in width between about 0·04 and 0·14 in., and that the compositor is a very highly-skilled, and comparatively highly paid, worker it will be seen that the more the job can be mechanised the better.

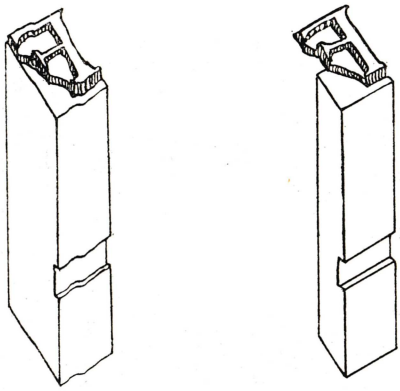


Fig. 1. The left-hand drawing shows an ordinary, 10 point, capital F. The right-hand drawing shows the same F, cast on a 6 point body for use with the four-line system.

There are two principal methods of setting type mechanically, neither of which is, as used for run-of-the-mill work, satisfactory for setting mathematics. The first is the "Linotype" system, which casts a complete line at a time, all in one piece of metal. Since practical considerations limit the number of characters available to some three hundred which must include the complete Roman and Italic alphabets, the impossibility of inserting mathematical signs by hand effectively rules out "Linotype" for setting mathematics. However, as the Americans are satisfied with a much lower standard of typography than we are in this country, some of their books have been set by a modification of this system.

The main alternative is the "Monotype" system, which has the

advantage that each piece of type is cast separately, so that many of the special pieces of type needed for mathematical work may be inserted by hand. The bulk of the job can be set on the machine with other characters of the same width in place of the mathematical signs so that all that remains to be done by hand is the replacement of the out of place letters by mathematical signs. With a special arrangement of the keyboard for mathematical use, although only 266 characters are available, remarkably little work needs to be done by hand. In this system the size of type which is normally used is 11 point (0·15217 in. deep), with $5\frac{1}{2}$ point superiors and inferiors, the principal reason for this is that when made up with a 2 point rule fractions which require two lines have a depth of 24 points, which is a standard type size, so that integral signs, spaces and so on are readily available. Most modern British books are printed with type set by this system.

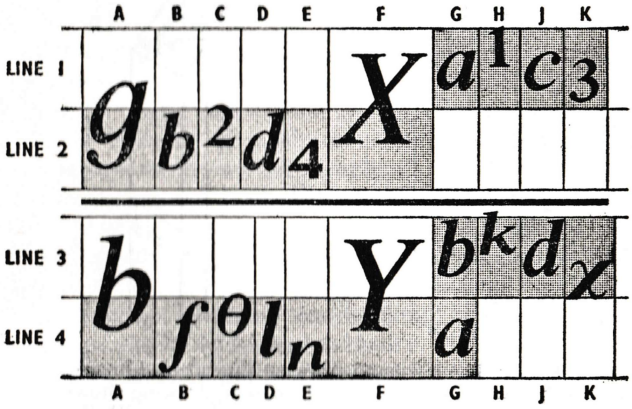


FIG. 2.

The latest technique is an adaptation of the "Monotype" machine to mathematical purposes known as the "Monotype four-line" system. It is so called because displayed fractions which would normally take two lines of 11 point type are cast as four lines of 6 point (0·083 in. deep). The principal characters are 10 point, but cast with an overhang on a 6 point body, and supported by a specially high space of exactly the same width (see Fig. 1). To avoid taking one character for each special space required away from the 266 normally available an ingenious device has been invented which stops the metal entering the mould of a letter, on depression of a key, known as the delete key, at the same time as the letter key.

Fig. 2 shows white those pieces of type which serve merely as supporting spaces, and stippled those which actually have characters on them. In line 1 we have, at A to F supporting spaces cast with the aid of the delete key from the letters and figures in the line below, G is a 6 point character used as a superior, H is a superior to G, J the same as G, and K an inferior to the superior *c* of J. In the second line A and F are 10 point letters, with an overhang protruding over line 1, B and D 6 point letters used as inferiors, C and E respectively superior and inferior to inferiors. Between lines two and three is a 2 point rule separating the numerator and denominator of the fraction. Lines 3 and 4 follow the same pattern, the 6 point *b* of line 3, G being cast from the same matrix as that of line 2, B and the *a* of line 4, G from the same matrix as that in line 1, position G. These two also demonstrate how easy it is to set a superior and an inferior to the same letter, in the correct relative position.

As four lines of 6 point plus a 2 point rule makes 26 points, integral signs and so on have to be cast specially to this non-standard size. The great advantage of the four-line system is that only the 26 point signs and the 2 point rule is left to be inserted by hand. So far as I am aware the only readily available book printed using this system is Rutherford's *Fluid Mechanics*.

The literature on this subject is very limited, but I can recommend the following: T. W. Chaundy, *The Printing of Mathematics* (Oxford, 1954) and A. Phillips, *Setting Mathematics*, in the Monotype Recorder, Volume 40, Number 4. I must thank the Monotype Corporation for much valuable assistance in preparing this article, and for the loan of the block for Fig. 2.

Pentominoes

By J. C. P. Miller

A domino consists of two adjoined squares; and there is only one basic shape. Similarly one may have trominoes, consisting of three adjoined squares; this time, however, there are two shapes, according as two of the squares are adjoined to adjacent or opposite sides of the third. The idea can be extended to *polyominoes* in general, having many adjoined squares. It is with five squares, forming the 12 *pentominoes*, that there first arise substantial possibilities for interesting puzzles—the 5 tetrominoes are too few, and the 35 hexominoes too many.

There is an excellent chapter on Polyominoes in *The Scientific American Book of Mathematical Puzzles and Diversions*, by Martin

Gardner—a book to be recommended. It will be published in this country in 1960 by George Bell. Most of this chapter appeared as an article in the *Scientific American* for Dec., 1957.

It is not my purpose to cover this ground again, but I give a diagram (Fig. 1) that exhibits the shapes of the 12 pieces, in an arrangement as a 6 × 10 rectangle, which is the shape of the box in which the puzzle has been on sale for some time. This is one of 2339 possible arrangements; this was demonstrated by Drs. C. B. and J. Haselgrove on the automatic electronic computer at the University of Manchester, after some 1500 or more solutions had been found and listed by hand methods.

The pentominoes can also be arranged in a 3 × 20 rectangle in just 2 ways, and in 4 × 15 and 5 × 12 rectangles in numbers of ways as yet unknown. The Haselgroves' machine program is capable of giving the answers. It might seem that the machine programs take the fun out of the puzzle. That this is not quite true will, I hope, appear below.

From Fig. 1 it is evident that new related solutions can be obtained by quite simple symmetries and exchange operations. For instance, we can replace Fig. 2 by Fig. 3, Fig. 4 by Fig. 5, and we can rotate Fig. 6 through 180°. After any of these has been done, new operations may disclose themselves, so that a long chain of related solutions may be obtained.

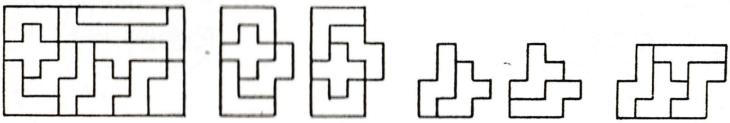


FIG. 1. FIG. 2. FIG. 3. FIG. 4. FIG. 5. FIG. 6.

We proceed to define suggested permissible operations. We first remark that any piece may be turned over and used in either position; thus (i) and (ii) in Fig. 10 are regarded as the same piece. This has been assumed in enumerating the number, 12, of distinct pentominoes. Next, complete rotation or reflection (turn-over) of the whole puzzle is not regarded as giving a new solution—only a *part* of the arrangement may be changed.

In enumerating connections, I suggest allowing:

(1) Symmetries, as exhibited by interchange of Figs. 2 and 3, or by rotation of Fig. 6. This is when any single block of pieces can be picked up *as a whole* and replaced in a different position.

(2) Interchange of two groups of pieces of the same overall shape.

(3) Interchange of two adjacent pieces to give the same shape. Figs. 4 and 5 give a sample; there are others.
(4) Rearrangement of *three* adjoining pieces.
(5) Rearrangement of *four* adjoining pieces.

As links in a chain or network, (3), (4), and (5) are successively weaker. The process can be continued to give weaker and weaker links, until finally *all* solutions are linked by rearrangement of twelve adjoining pieces. I have drawn the line at (5), i.e. at four pieces, because it is reasonably possible to recognise rearrangements of four pieces, but much less so with five. However, I have below given an indication about interchanges (4) and (5).

The enumeration of "symmetries" is by no means complete, and it is difficult to see how it could be accomplished on an automatic computer. I give a list of most of the best chains I have obtained, with sufficient lack of detail to make it a reasonable puzzle to repeat the enumeration, and maybe to improve on it!

Fig. 1 represents 63 solutions, the best I have. These are in groups as below, with links (4) and (5) marked

$$28\text{—}(4)\ 11\begin{cases}(5)\text{—}3\\(5)\text{—}15\text{—}(4)\text{—}2\text{—}(5)\text{—}4\end{cases}$$

Fig. 7 represents 48 solutions, with no links (4) or (5).

Fig. 8 represents 41 solutions, with two links (5).

Fig. 9 represents 37 solutions, with one link (4).

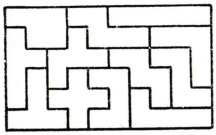


FIG. 7.

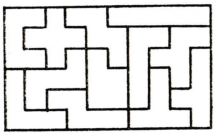


FIG. 8.

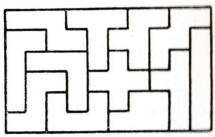


FIG. 9.

Other groups have respectively 35, 32 (twice), 26, 21 (twice), 20, 18, 16, 15, 14, 12, 11, 10 (three times) solutions, and there are, of course, many smaller groups. I should be very interested to hear of improvements on these; there must be such improvements, as my results are based on an examination of only about 1600 solutions.

It is clear that the enumerations—total, and of symmetries—can be done also for other shapes, e.g. 4×15 or 5×12 rectangles, and for 8×8 squares with holes. For example, a 2×2 hole in the middle makes a symmetrical shape, whose solutions have been enumerated by Mr. Dana Scott and by the Haselgroves. There are 65 solutions. Mr. Scott's method was published as Technical

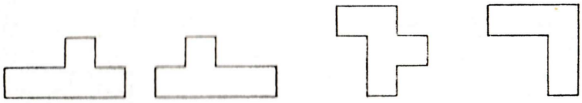
Report No. 1, 10th June, 1958, by the Department of Electrical Engineering, Princeton University.

Another version of the puzzle was suggested to me by Professor D. H. Lehmer, of Berkeley, California. If the basically two-dimensional pentominoes are made from five *cubes*, that is, with a thickness equal to the side of an individual square, the 12 varieties can be assembled into a block of dimensions $3 \times 4 \times 5$. This does not use the several essentially three-dimensional ways of putting together five cubes.

Yet another puzzle, suggested by Dr. van der Poel during a visit to Cambridge from Holland is to use the 12 arrangements of *six* contiguous equilateral triangles to make a single rhombus formed of two adjacent large triangles.

There is material in all these puzzles for those who like simple methods and also for those who can use an automatic computer.

Finally I should like to give a partial acknowledgement of the part played by others in assembling my list of solutions, though it is incomplete since my records are not now all available to me. The machine program by Drs. C. B. and J. Haselgrove has already been mentioned. The Haselgroves also played a considerable part in the earlier enumerations by hand, together with John Leech and Ken Johnson, and many others who contributed one or more new solutions.



(i) (ii) Piece B. Piece C.

Piece A.

FIG. 10.

A Computer Program for Pentominoes

By C. B. and JENIFER HASELGROVE

This is a short description of the method used on the Manchester University Mercury Computer to solve Pentomino problems. We describe it with reference to the 6×10 rectangle, but the procedure is the same for any shape.

We take the rectangle, which we call the board, give it a border of extra squares down the right hand side and along the bottom, and number the squares as shown in the figure. The squares are represented in the machine by 77 storage locations, each of which

contains the number 1 if it is occupied by part of a piece and 0 if it is empty. The locations corresponding to the bordering squares 7, 14, . . ., 70 and 71, 72, . . ., 77 always contain 1 and thus being "occupied" prevent any piece being placed so as to overlap them. The column 7, . . ., 70 in particular serves to define both sides of the board.

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22						
29						
36						
43						
50						
57						
64						
71	72	73	74	75	76	77

Information about the shapes of the pieces must be provided in the program. This is done as follows. Take for example piece B. If the top left hand square of this piece occupies square n on the board, then by reference to the figure we see that the remainder of the piece occupies squares $n + 1$, $n + 8$, $n + 9$ and $n + 15$. The four numbers 1, 8, 9 and 15 thus define the shape of the piece. A set of four defining numbers is given for every orientation of each of the twelve pieces, together with the number of possible orientations for each piece (ranging from 1 for the cross to 8 for the asymmetric pieces).

The pieces are "placed" on the board as follows. A record is kept of the pieces which have already been placed. The lowest numbered empty square is found and successive orientations of the available pieces in order are tested until one is found to fit. The testing consists of examining the four relevant neighbouring squares to see if they are empty. When a piece is found to fit the appropriate squares are filled in with 1's and a record is made of the piece number, its orientation, and the position of its top left-hand square. If an impasse is reached with no possible fit the previous piece is erased from the board and replaced by the next possible one (either another orientation of the same piece or a new one) in order.

When a solution is found it is printed out in the form of a list of piece numbers in the order in which they are placed on the board, each with a number to indicate the orientation in which it is used. The last two pieces are then erased and the process is continued as

if after an impasse. In this way a complete ordered set of solutions is found.

Our method of eliminating rotations and reflections of the whole board was to limit piece C to this one orientation. A better way would have been to insist that the cross had its centre in the top left hand quarter of the board. This would have saved a lot of time during which the machine was trying near-solutions containing all pieces but the cross—always the most difficult to fit in.

The procedure for a different shaped board is just the same. For holes, such as the 2×2 in the middle of the 8×8 board, the squares are filled in at the start like the border and can never subsequently be cleared. The sets of four numbers giving the shapes of the pieces will be different for different widths of board. The procedure for dealing with symmetries of the whole board may also need modifying.

The number 2339 of solutions for the 6×10 board took many hours of machine time and has not been checked. The authors hope to verify it when a much faster machine is available.

Connected Masses

By D. C. Cross

One of two masses connected by a short inextensible string and resting on a long smooth table is projected directly away from the other mass. Calculate the subsequent velocities.

Most textbooks in common use teach students to assume in such problems that, once the string has become taut, it remains taut in perpetuity and the two masses have a common velocity thereafter. The argument normally advanced is that an inextensible string can store no energy and that therefore, once taut, it will not slacken again. The snag here is the resulting loss of energy. If the masses are m_1 and m_2, and m_1 is projected away from m_2 with velocity v, the energy loss is $\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v^2$. As the ratio of m_2 to m_1 alters, the percentage energy loss varies between 0 and 100.

There have been various attempts to find a way around this difficulty. Professor A. G. Walker of Liverpool University, writing in 1943, commented:

"In my opinion the most important requirement is that energy should be conserved, and since this usually means that the string must slacken, it appears that the string must be allowed to have

elastic properties. This leads me to regard an inextensible string as a perfectly elastic string, with a very large modulus of elasticity. . . . In general, the string stretches and returns to its natural length in a very short time, and the calculated velocities are those existing just as the string slackens."[1]

The following extract is from a letter which I received from a mathematician working at the National Physical Laboratory at Teddington:

"The term 'inextensible' . . . is . . . intended to describe a string which is incapable of extending elastically but which is able to dissipate any amount of energy with negligible plastic extension."

Here are two diametrically opposed views. Perhaps because Professor Walker's interpretation involves no permanent energy loss and corresponds more closely with experimental evidence than its rival, it will eventually find a more general acceptance.

Whatever the outcome, there remains the problem of masses connected by strings not described as inextensible. Many textbook questions taken from old examination papers are of this kind. While the examiner's intention may be sometimes obscure, most textbooks provide answers appropriate only if the idea of a continued common velocity and a permanent loss of energy is accepted. Consider the following typical example:

"Three small bodies of mass 4, 5, and 6 oz. respectively lie in order in a straight line on a large smooth table, the distance between consecutive bodies being 6 in. Two slack strings, each 2 ft. in length, connect the first with the second, and the second with the third. The third body is projected with a speed of 15 ft./sec. directly away from the other two. Find the time which elapses before the first begins to move, and the speed with which it starts. Find also the loss of kinetic energy."[2]

Possibly the student is intended to assume no permanent energy loss in this example. In any event, general agreement on problems of this kind should not be difficult to achieve; there can be no question here of the three masses having a common velocity when the last mass starts to move.

When strings connecting masses suffer some slight permanent deformation, the velocity of approach of the two connected masses will, of course, be slightly less than their original velocity of separation. The term "coefficient of inextensibility" may be useful in these circumstances, but there is no reason for not using the term "coefficient of elasticity." Apart from the weight of the string,

there is a complete parallel between the phenomenon of connected masses and that of colliding bodies. Indeed, in his Third Law, Newton makes no distinction.

There is perhaps need for some such summary as the following:

> If two unhindered masses are moving with velocities such that the string connecting them becomes taut, then the change in momentum produced in the one will equal the change in momentum produced in the other: for the tension in the string will act on both masses for the same small interval of time and add to the momentum of the one that which it subtracts from the other. The two masses will continue to move independently and, if the directions of their motion be along the same line, presently collide.

No new theory is being promulgated here. The parallel between the two phenomena mentioned above is one recognised by some mathematicians both in Britain and abroad. Its more general acceptance should provide a way out of a difficulty of long standing.

REFERENCES

1. The Inextensible String: from the Edinburgh Mathematical Notes No. 33 of the Edinburgh Mathematical Society.
2. *Intermediate Mechanics* (*Dynamics*) by D. Humphreys, B.A., B.Sc., Longmans Green and Co., 1949, p. 148. (Originally set in an Oxford and Cambridge Higher Certificate examination.)

The Long Count

By N. SEUDOPYM

I'd been lying there for about a couple of hours trying to integrate the most uncooperative function I'd found for years. I'd made fourteen consecutive substitutions and couldn't remember the eighth—you must have done the same sort of thing yourself. I was desperate: I couldn't stand it any longer. I dug my wife in the ribs:

"I can't get to sleep."

She woke: "Uh?"

"I can't get to sleep."

"Oh . . . count something."

"What?"

"Oh, sheep!" she sounded peeved, and turned over with a most unfriendly air. So I left her alone.

Now I don't know if you're the same, but sometimes when an idea gets into my head I just can't get rid of it. This was one

of those times. I fought against this sheep-concept. I tried to remember the eighth substitution and even integrated the fifteenth, but to no avail. I kept drifting away, and eventually found myself standing beside a hedge near a gap just the right size to let sheep through. There's not much you can do in a situation like that, is there? I just waited for them to come.

3 came through the gap and ran away to the right. No, not three sheep, 3! The number, you know. Surprised and excited, I looked through the gap: on the other side was a real field,—*the* real field! I was so thrilled I was nearly knocked down by 15 and 76 who followed. Here was something worth counting! That was four. "Five, six, seven, . . .

". . . twenty-three, twen— that one's strange!" It was p/q. I was unprepared for this, and only noticed that $2p < q$—I didn't see what number it actually was. In quick succession came $(p + 1)/(q + 1)$, $(p + 2)/(q + 2)$, . . . $(p + 54)/(q + 54)$. That made seventy-eight.

Then $(5p + 3)/(4q + 3)$. "Seventy-nine." But suddenly I wondered if I'd seen it before. "Now $(5p + 3)/(4q + 3) = (p + n)/(q + n)$ only if $n = 4p + 3 = 3q + 3$, i.e. $4p = 3q$; but $2p < q$, so this is impossible. Seventy-nine!"

"$(6p + 8)/6q + 13)$. If $n = 5p + 8 = 5q + 13$, $5p = 5q + 5$; impossible if $q > 2p$. Eighty. p/q^2. Now (i) $p/q^2 < p/q \leqslant (p + n)/(q + n)$ for all n; (ii) $p/q^2 < p/q < (4p + 3)/(4q + 3) < (5p + 3)/(4q + 3)$; (iii) (Wait till I've finished this one!) $(6p + 8)/(6q + 13) < p/q$. Oh. Now what shall I say? If $(6p + 8)/(6q + 13) = p/q^2$, $6pq^2 + 8q^2 = 6qp +$ —— No, I can't go on. STOP!

"Look, you rationals! Form yourself into a square array. That's right. Good. Now then, peel off according to diagonals. Good. Eighty-two, eighty-three, eighty-four, . . .

". . . a hundred and six, a hundr— Hey! that's a funny little thing. Jumping around, up and down, up, down. Wonder what it is? Oh, it's π. Fancy π being energetic! There it goes, up and down, a hundred and seven . . .

"Ah, well, a hundred and eight, a hundred and— What? 3? We've had it before! 1? Oh, it's 3·1. Wait! It hasn't finished yet: 3·1 . . . 4 . . . 1 . . . 5 . . . 9 . . . 2 . . . It's π! It's that π again!

"Though I suppose it mightn't be π; it might terminate soon. I'll take a look over the hedge and see. Oh, it goes on and on . . . Of course, it might differ from π in the next place, mightn't it? 6. Oh. It doesn't. Mightn't till the millionth. I hate it, hate, hate!

"Ah good. Here's another number coming. Call that one a hundred and nine and hope. What's this one? 3·14159—No! Go away! There's a third. 3·1415. . . . . And a thing on the hedge 3·1415 . . . Thing behind. Thing above. Thing below. 3·1415 . . . Things all over the place. 3·1415 . . . Thousands of 'em 3·1415 . . . help! I CAN'T COUNT THESE."

Some Little Naggers

By H. T. CROFT and S. SIMONS

1. A milk bottle of the usual shape contains the usual mixture of milk and cream in equilibrium, and stands in the usual vertical position. The pressure exerted by the liquid on the base of the bottle is measured. The bottle is now shaken up so that the milk and cream are evenly mixed, and the pressure on the base is again measured with the bottle in the vertical position. Is the new pressure greater than, equal to, or less than the old pressure?

2. A lump of ice is floating in a tumbler of water, the whole system being at 0° C. Heat is now supplied and the ice melts, but the temperature remains the same. What happens to the level of the water in the tumbler?

3. Some cargo is jettisoned overboard from a boat floating on a lake; the cargo sinks to the bottom of the lake. What happens to the level of the water in the lake (relative to the land)?

4. A bucket full of water is falling freely under gravity. (Neglect air resistance.) A table-tennis ball is fastened inside the bucket at the bottom. If the ball is now released so that it is free to rise in the liquid, does it in fact do so?

5. A problem of Lewis Carrol's with a long and tangled history: A rope passes over a mathematical pulley (i.e. frictionless and without inertia). On one end of the rope is a monkey, and on to the other is tied a weight equal in value to the weight of the money. The monkey starts to climb. What happens?

6. A hollow ring made of square section metal is heated. Does the interior hole expand or contract?

7. O is the centre of a cube and A is one vertex. What is the shape of the section of the cube by the plane through O perpendicular to OA?

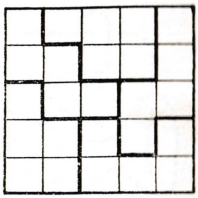
The answers to these problems will be found on page 36.

Problems Drive, 1960

Set by G. J. S. Ross and M. Westwood

A. In a three-cornered fight in a recent election, candidate A was elected, but without an absolute majority. Candidate C lost his deposit, i.e. failed to obtain one-eighth of the total number of votes cast. It was observed that the combined number of votes cast for any pair of candidates was a perfect square, and that the numbers involved were the least for which this was so. How many votes did each candidate receive? The other candidate was called B.

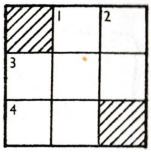
B. The cells in the accompanying diagram are to be filled with letters A, B, C, D, or E, such that no row, column, or outlined block contains the same letter twice.



C. The clues and solutions to the following cross-number puzzle are in the scale of seven.

Across: 1. ab (i.e. $7a + b$) where $ba - ab = 6$.
3. $\sqrt{2}$ correct to 2 decimal places.
4. 16th prime.

Down: 1. def where def (scale 7) $= efd$ (scale 10).
2. $p \times q$ where p, q satisfy $14p - 10q = 13$.
3. gh where $3gh = hg$.



D. Find the next term in each of the following series:

(*a*)	2	10	12	21	102
(*b*)	2	10	12	21	50
(*c*)	2	10	12	22	34
(*d*)	2	10	33	88	245

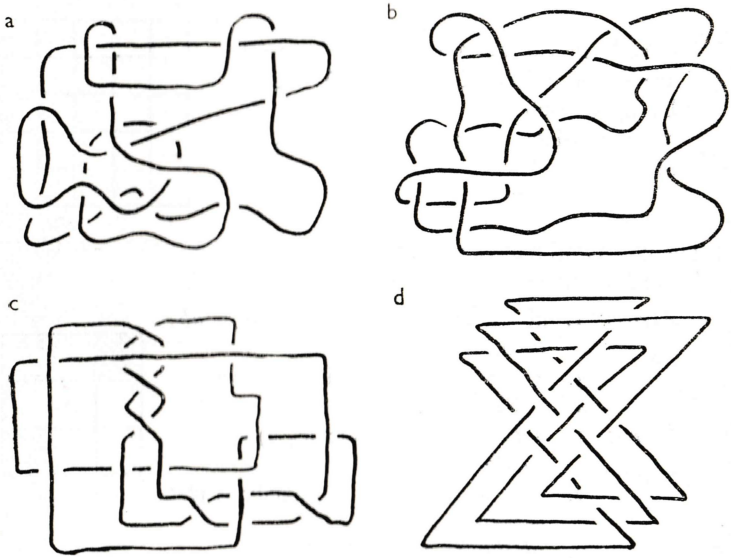
E. Given a point P on one of the sides of a general triangle ABC, show how to construct a line through P which will divide the area of the triangle into two equal halves.

F. TWO + TWO = FOUR.

If the letters in the above equation represent distinct digits (not necessarily in the scale of ten), find the minimum scale in which a solution exists, and give such a solution.

G. A convoy is approaching a town at 15 m.p.h. A despatch-rider leaves the rear of the convoy, then 23 miles from the town, and travels at 60 m.p.h. to deliver a message at the head of the convoy. He returns to the rear immediately and arrives back at the same time as the head of the convoy reaches town. How long is the convoy?

H. Which of the following knots are knotted and which not?



I. Show how to dissect a general triangle into four pieces which can be reassembled to form a quadrilateral, such that the boundary of the new figure contains no portion in common with the boundary of the old.

J. The cuboctahedron is bounded by six square faces and eight equilateral triangles, and its vertices are the midpoints of edges of a cube.

In how many distinct ways may it be coloured using seven colours such that opposite faces have the same colour?

K. Find a set of solutions in unequal positive integers for each of the following equations:

(*a*) $x^3 + y^5 = z^4$
(*b*) $x^3 + y^4 = z^2$
(*c*) $x^5 + y^4 = z^4$

L. Of four sets of twins from different families, one was a pair of boys, one was a pair of girls, and the other two sets were mixed. One happy day they all met, and soon a quadruple wedding was arranged. No set chose spouses from the same family. Bob and Nan each now have two sisters-in-law. Peg's husband is Dick's brother-in-law. Mrs. Jim Brown (*née* Smith) baby-sits for Mrs. White (formerly Liz Jones). If Liz and Sue are unrelated, what is the maiden name of Dick's wife?

/

Solutions to Problems

LAST YEAR'S CROSSWORD PUZZLE by G. Berry.

Across: 7. Courting Couples. 8. Sun Spot. 10. Streams. 11. Ranks. 12. Cones. 14. Penal. 15. Lens. 16. Wars. 17. Clan. 19. Cusp. 21. Overs. 22. Meths. 23. Axiom. 25. Element. 26. Divisor. 27. Open or closed set.

Down: 1. Locus or envelope. 2. Fresnel. 3. Minor. 4. Roots. 5. Spheres. 6. Lemma and theorem. 9. Tans. 10. Skew. 13. Sells. 14. Prism. 17. Cremona. 18. Next. 19. Clod. 20. Periods. 23. An orb. 24. Midst.

PROBLEMS DRIVE, 1960

A. (1) A 158, B 131, C 38.

B. (4) The solution is unique.

C. (2) *Across:* 1. 56; 3. 126; 4. 56. *Down:* 1. 526; 2. 66; 3. 15.

D. (3) (*a*) 111. Primes in ternary scale.
(*b*) 112. Cubic series.
(*c*) 56. $u_n = u_{n-1} + u_{n-2}$.
(*d*) 836. $u_n = n! + n^3$.

E. (5) If P is on BC, and A′ is midpoint of BC, draw A′X parallel to AP, meeting AC (or AB) at X. Then PX is the required line.

F. (4) 523 + 523 = 1346 in scale of seven.

G. (7) 15 miles.

H. (6) (*a*) Not knot, (*b*) Knot, (*c*) Knot, (*d*) Not knot.

I. (3) Several solutions. For example, drop perpendicular from midpoints of two sides onto third. From a rectangle, and place lower half on upper half.

J. (3) 210. There are 35 ways of selecting 3 colours for the square faces, and 6 ways of arranging the remaining 4.

K. (2) (*a*) $x = 32$, $y = 8$, $z = 16$. Solutions in powers of 2.
(*b*) $x = 2$, $y = 1$, $z = 3$; or 9, 6, 45; 6, 5, 29.
Solutions if $y^2 = \frac{1}{2}x(x-1)$, $z = x + y^2$.
(*c*) $x = 65$, $y = 130$, $z = 195$. Put $y = kx$.

L. (2) Brown.

Numbers in brackets indicate the average mark (out of 10) obtained by the 27 pairs competing.

A CORRECTION

Mr. D. E. Daykin has written to point out that in *Eureka* 9 (April, 1947) the formula on page 18 should read:

$$\Sigma x_1 x_2^2 \ldots x_n^{2^{n-1}}$$

and not $$\Sigma x_1 x_2^3 \ldots x_n^{2^n-1}$$

(Reference: *Combinatory Analysis*, by P. McMahon, C.U.P. (1915), p. 253, § 213.)

Book Reviews

The Unity of the Universe. By D. W. SCIAMA. 186 pp. (Faber & Faber, 1959.) 8¾ in. 20s. 0d.

Dr. Sciama's book is something much more than the average popular work on astronomy. Many books merely instruct but this also stimulates the imagination.

The book is divided into two parts. In the first the author gives a lucid account of the observational picture of the universe. Starting from a description of the measurements made by the Greeks, he progresses to modern times with brief accounts of Shapley's model of the Milky Way, Oort's work on radial velocities and developments in radio astronomy. The section concludes with chapters on the external galaxies and the expanding universe.

The remaining hundred pages are devoted to the theoretical aspects of the universe in which Mach's principle and the steady state theory occupy central positions. An exposition of the connection between inertial and gravitational forces follows, and then a thorough chapter on the origin of inertia, containing an account of some of the author's own work, in which the gravitational constant is related to the density of matter at great distances together with Hubble's constant. Einstein's general theory of relativity receives rather a brief treatment. More detailed information about observational tests of the theory and the possibilities opened up by satellite research would have been welcome, without risk of digression from the main argument of the book.

One of the most impressive chapters is that on the difficulties of studying a unique universe. Its essence is that a theory which describes the behaviour of a unique system ought not to possess any arbitrary features. The author regards this theory as so compelling that he says "the theory of the universe which best conforms to it is almost certain to be right." There are concluding chapters on the formation of galaxies and of the elements.

Dr. Sciama must be congratulated for having produced such a readable and enjoyable book which should be a valuable addition to the shelves of all interested in astronomy. J. IRELAND.

Confluent Hypergeometric Functions. By L. J. SLATER. 247 pp. (Cambridge University Press, 1960.) 11¼ in. 65s.

This treatise will no doubt become recognised as and remain for many years a standard reference book on the confluent hypergeometric functions. Most of the text is devoted to formulae. In the first chapter are the differential equations satisfied by these functions, in the second their differential properties and in the third their integral properties. The fourth chapter lists asymptotic expansions and approximations, whilst the fifth gives the relations between these functions and other well-known functions such as the Bessel functions, the Coulomb wave function, the Laguerre polynomials and the incomplete gamma functions. Only in the final chapter is the terse style relaxed a little to give diagrams of the positions of zeros, some fascinating three-dimensional drawings of the shape of one of the functions, and

also some brief details of how some of the calculations were performed on the computer EDSAC I.

The second half of the book is a set of tables. As these functions are specified by three parameters (whereas exponential functions have only one) it is obviously a difficult task to tabulate them satisfactorily One wonders how useful even these fairly extensive tables can be.

Certainly not a book to be read! But perhaps quite a few of those who do research will have cause to use this book and to be thankful for the extreme care which Dr. Slater has clearly taken in the preparation of her treatise in order to achieve the perfect accuracy required in a reference book of this nature. The printing and layout maintain the customary high standard of the University Press.

M. FIELDHOUSE.

Toeplitz Forms and Their Applications. By U. GRENANDER and G. SZEGÖ. 245 pp. (University of California Press and Cambridge University Press.) 45s.

A Toeplitz form is of the type $\sum_{\nu}\sum_{\mu} C_{\nu-\mu}\zeta_\nu\zeta_\mu$, i.e. a quadratic form for which the matrix elements $C_{\nu-\mu}$ depend only on the difference of the two subscripts. The book under notice is concerned solely with Hermitian forms, although extension of most of the results to the case of non-Hermitian matrices and associated bilinear forms is feasible and natural.

The key observation is this: if the order of the matrix is n, and the sequence C_ν has period n, then the matrix $(C_{\nu-\mu})$ has eigenvalues $\lambda_s = \sum_1^n C_\nu e^{2\pi i\nu s/n}$, $(s = 1, 2 \ldots n)$ and corresponding right and left eigenvectors $X_s = (e^{-2\pi i\nu s/n})$ and $Y_s = (e^{-2\pi i\nu s/n})$ respectively. However, if instead of assuming periodic variation in the sequence one supposes that the terms C_ν converge to zero sufficiently regularly with increasing $|\nu|$ (e.g. if one supposes them to be the Fourier coefficients of a function $f(x)$ which is Lebesgue integrable in $(0, 2\pi)$ and has period (2π) then the same results are found to hold asymptotically in some sense when n becomes large. For example, the eigenvalues of the matrix are approximated by the ordinates $f(2\pi s/n)$ $(s = 1, 2 \ldots n)$ in the sense that the two sets of values are asymptotically equally distributed. A further typical consequence is that $\lim[\det(C_{\nu-\mu})]^{1/n}$ equals $\exp[(1/2\pi)\int \log f(x)dx]$, the continuous geometric mean of $f(x)$.

A first, and to a large extent final, treatment of this topic was given by Szegö (1920–21), who found an extremal characterisation of the problem (in the Hermitian case) and introduced for the first time the idea of a reproducing kernel. Kac (1954) later showed that Szegö's results could be applied to give a very elegant treatment of certain problems concerning the distribution of partial sums of random variables, a line of investigation that has since been considerably extended by Spitzer (1956). However, the most natural application in the Hermitian case is to be found in the theory of stationary stochastic processes, where the matrices are all of Toeplitz type, and where Szegö's extremal characterisation together with almost all of his results have an immediate statistical interpretation. Statisticians had made

a certain amount of progress here in ignorance of Szegö's work: Koopmans (1942) derived the asymptotic eigenvalue distribution for certain special cases, while the reviewer (1951) recognised this and other results in the general case. Grenander (1952) demonstrated the relationship of this work to Szegö's investigation, which then appeared all the more remarkable in view of the fact that there had been no guiding application. To Grenander also goes the credit for giving the statistical results a rigorous formulation, and for recognising that the theory of prediction for stationary processes could be based upon known results concerning Toeplitz forms.

The present book provides a unified and logically complete account of the subject from both theoretical and applied aspects. The original problem is also generalised extensively: for example, the authors discuss forms for which the matrix elements are

$$C_{\nu\mu} = \int_X \phi_\nu(x)\,\phi_\mu(x) f(x)\,dx$$

where the $\phi_\nu(x)$ form an orthogonal family with respect to some weighting function on the set X.

It will be clear from what has been said that the subject is of potential interest to a wide range of readers. The two authors present their material in a lively and easy fashion, and one that demonstrates their complete mastery of analysis. The book is warmly recommended.

P. WHITTLE

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Proceedings of the International Congress of Mathematicians. Ed. J. A. TODD. lxiv, 573 pp. (Cambridge University Press, 1960.) 65s.

This contains texts of 16 one-hour, and 33 half-hour addresses given at the 1958 Congress in Edinburgh. There are six papers each in French, German and Russian. The longer papers are generally surveys of developments since 1954 (when the previous conference was held in Amsterdam). Broadly speaking, all of the papers are on pure mathematics: there is one on the mathematical problems of quantum field theory, dealing with distributions and generalised functions: one on statistical physics, on the Boltzmann equation: one on the foundations of statistics "directed mainly at non-staticians who may have wandered in": and two are on the solution of differential and difference equations and eigenvalue problems. The book ends with an enigmatic paper on Mathematical education, written in terse note form. The rest of the papers are undeniably pure.

Papers by the two Fields Medalists are included: K. F. Roth on

rational approximations to algebraic numbers, and R. Thom on differentiable varieties; and appreciations of their work by H. Davenport and H. Hopf respectively.

It is unlikely that anybody will want to deal seriously with more than two or three papers in the collection, and for this the cost would be excessive. However, it will make a most attractive library book, where people will probably find that the most interesting papers are not those in which they should be most interested. As is usual with books from the Cambridge Press, the layout and printing are impeccable.

D. H. FOWLER.

Fluid Dynamics. By D. E. RUTHERFORD, D.Sc., Dr.Math. ix, 226 pp. (Oliver & Boyd, 1959.) 10s. 6d.

In Cambridge in recent years it has become customary in the introductory courses on Fluid Dynamics to place more emphasis upon a physical understanding of the problems that arise than upon the mathematical devices that are at our disposal to solve them. Indeed, such an attitude is essential, if we are to avoid the conceptual pitfalls leading to the paradoxical situations that so puzzled our ancestors, as for example D'Alembert's paradox that the net force on a body in a uniform stream of inviscid fluid is zero.

It is perhaps inevitable that an author who seeks to fill a gap in the series of University Mathematical texts should revert to the more old-fashioned attitude of exploiting mathematical expertise to derive solutions of limited or unspecified scope. Thus the first three of the five chapters of *Fluid Dynamics* are devoted to inviscid, incompressible fluids, and, although the mathematics is clearly developed, physical insight is conspicuous by its absence. The situation is partly redeemed in chapter 4, which deals at some length with the effects of compressibility, and in chapter 5, where the effects of viscosity are rather belatedly and hurriedly disposed of.

In spite of these adverse criticisms, however, the book will be of use to Tripos candidates, especially those reading the course on inviscid fluids. It is simply and directly presented, and makes accessible many formulae and results useful in examinations. Perhaps the low price will be the deciding factor in its favour.

H. K. MOFFATT.

The Theory of Homogeneous Turbulence. By G. K. BATCHELOR. xi, 197 pp. (Cambridge University Press, 1959.) $8\frac{1}{2}$ in. × $5\frac{1}{2}$ in. Students' Edition, 18s. 6d.

This is a reprint of the 1956 edition of Dr. Batchelor's book, which was written in 1952; the new edition has cardboard covers, and costs much less, but otherwise shows no change from 1956, which version only differs from the original in having two extra footnotes and one extra reference.

To achieve any understanding of a complex phenomena it is best to start with the aspects that are most easily treated. Homogeneous turbulence is the "easy" part of turbulence, and this book discusses fully all that was known by 1952—since when not a great deal has been added to our knowledge of this subject. The treatment is both mathematical and physical, using whichever is more suitable for the understanding of the particular aspect under discussion. The result is a

clear exposition of turbulence which has become a standard reference work for all whose interests include the theory or effects of turbulence. It is not, perhaps, a book for "students" in the way that Hardy's *Pure Mathematics* is a book for students, but a determined undergraduate—if he had the time and energy to spare from examinations and rowing—could read it easily and with profit. To assimilate it fully takes rather longer, as the present writer knows to his cost. A. R. PATTERSON.

Ordinary Difference-Differential Equations. By EDMUND PINNEY. xii, 262 pp. (University of California Press; Cambridge University Press, 1958.) 37s. 6d.

Here is the first book in English devoted to a subject which is gaining in interest because of its applications to the analysis of, for examples, high-speed reciprocating machinery, control systems, traffic dynamics, economics models, and the irradiation of cells. A simple difference-differential equation describes the behaviour of a motor which is intended to be held to a constant speed by a governor. Let $y(t)$ be the error in the motor's speed; then the governor applies a corrective acceleration $-y'(t)$ which would be proportional to $y(t)$ but for a delay of (say) one unit of time caused by the elasticity of connecting shafts and other effects. Consequently the relevant equation of motion is $y'(t) + \gamma y(t-1) = 0$, where γ is a constant and $y(t)$ is specified initially for $-1 \leqslant t \leqslant 0$. Several methods for solving simple equations like this one are described in the book's first chapter; some methods involve the Laplace transformation

$$Y(z) = \int_0^\infty e^{-zt} y(t)\,dt$$

which when applied formally to the equation above yields

$$Y(z) = \{y(0) - \gamma \int_{-1}^{0} e^{-z(t+1)} y(t)\,dt\} / \{z + \gamma e^{-z}\}.$$

The inverse transformation provides a solution

$$y(t) = \int_{c-i\infty}^{c+i\infty} e^{zt} Y(z)\,dz / 2\pi i$$

(for any sufficiently large real number c) which can be represented more conveniently in several ways of which the one favoured by the author, because it exhibits most directly the behaviour of $y(t)$ as $t \to \infty$, involves the calculus of residues (see R. U. Churchill, *Operational Mathematics*, pp. 186–193). If the roots of the "characteristic equation," in this case $z + \gamma e^{-z} = 0$, are denoted by z_ν and are all distinct, and if C_ν is the residue of $Y(z)$ at z_ν, then the solution takes the form $y(t) = \Sigma_\nu C_\nu \exp\{z_\nu t\}$.

The validity of this sort of solution for more general linear difference-differential equations with constant coefficients is demonstrated in Chapter II, and Chapter III contains an analysis in great detail of the important roots z_ν. The theory is applied in the next three chapters to a host of special equations drawn from the literature.

Chapters VII and VIII constitute a digression mainly to equations with derivatives in one independent variable and differences in the

other(s); e.g. $2y'_h(t) - y_{h+1}(t) - y_{h-1}(t) = w_h(t)$. The treatment differs slightly from the usual matrix approach (F. R. Gantmacher, *Theory of Matrices*, Ch. XIV) in that the subscript h may vary continuously.

The last three chapters deal with non-linear equations which differ from those of the first six chapters by the addition of a small non-linear term like $\epsilon y^3(t)$. The author's main concern is with the behaviour as $t \to \infty$ of small-amplitude solutions, and in the last chapter he concentrates on Minorsky's equation, an equation frequently associated with time-lags in non-linear systems.

This is a useful book mainly by virtue of its attention to non-linear problems, which cannot easily be treated by the usual operational methods; but it is marred by a large number of (mostly small) misprints of which most occur in the thickets of microscopic and nearly illegible superscripts and subscripts. W. KAHAN.

Elementary Co-ordinate Geometry. By C. V. DURELL. xvi, 341, xxiii pp. (G. Bell & Sons Ltd., 1960.) 17s. 6d.

The book covers the work in co-ordinate geometry required for the General Certificate of Education. Part I, also available separately, is for the Advanced level, Part II for Scholarship level. The author has used the recommendations of the Mathematical Association made in the "Teaching of Higher Geometry in Schools."

Great emphasis is placed on directed number and the length of a "step." Circles are brought in very early in the book in an example

on the distance formula, and discussed more fully in a later chapter. The Parabola and Rectangular Hyperbola are both defined parametrically and their geometrical properties developed from their equations; the Ellipse and General Hyperbola are considered as orthogonal projections of the circle and rectangular hyperbola. There is a separate chapter on focus and directrix and particular results connected with these.

In Part II of the book, more general methods are set forth and the reader is shown how to handle systems of conics without going through involved algebraic calculations.

The book is a pleasure for the teacher to read because of the masterly use of varied methods of solution displayed by the author in the worked examples. D. T. JACK.

Cartesian Geometry of the Plane. By E. M. HARTLEY. xi, 324 pp. (Cambridge University Press, 1960.) 8½ in. 20s.

Dr. Hartley's book is intended as a first course in co-ordinate geometry, suitable both for mathematical specialists in their first sixth-form year and for others taking mathematics to "A" level. In the words of the Mathematical Association Report, "The purpose of the first course is to give the pupil a new tool for investigating properties of the geometry of physical space," and Dr. Hartley has succeeded admirably in this purpose. The treatment is elementary, being entirely metrical, real and finite, but within these limits is fairly rigorous. Pure methods are used whenever they give a simpler solution, and care has been taken to make the work as elegant as possible, avoiding the tedious proofs and formulae which adorn the older books. An elementary approach is well suited to the basic properties of the parabola, ellipse and hyperbola, but is less appropriate to the latter part of the book, dealing with line pairs and the general conic: this work would be less laborious if use could be made of polar properties and the line at infinity.

Dr. Hartley has born in mind particularly those students working alone, and has given prominence to illustrative examples, besides providing hints on the solution of the ordinary examples. She keeps their interest by giving methods of drawing the different conics, and by a section on some cubic and trigonometric curves. It is likely that the book will be used most by such students, but it should also be found useful by those teachers who do not know, or have forgotten, how best to obtain the standard results. F. M. HALL.

Topology. By E. M. PATTERSON. viii, 128 pp. (Oliver and Boyd.) 7½ in. Second Edition, 8s. 6d.

This new edition is identical with that reviewed in *Eureka* 21, except that errors and misprints have now been corrected.

R. SCHWARENBERGER.

Answers to the problems on page 22.

1. The new pressure is greater. 2. The level does not change. 3. The level falls. 4. No. 5. The monkey and the weight rise with the same speed. So if they both started "on the floor," they arrive at the pulley together. 6. It expands. 7. A regular hexagon.