EUREKA

Editor: *Martin Fieldhouse (Emmanuel)*

Business Manager: *H. K. Moffatt (Trinity)*

No. 22

OCTOBER, 1959

THE JOURNAL OF THE ARCHIMEDEANS

The Cambridge University Mathmatical Society; Junior Branch of the Mathematical Association

Contents

Please send subscriptions (see page 4) and advertisements to:

The Business Manager, "Eureka,"

and contributions and other communications to:

The Editor, "Eureka,"
The Arts School,
Bene't Street,
Cambridge,
England.

Editorial

We are happy to present another *Eureka* brim full of "the mixture as before" or, as our first Editorial gently put it, "some of our less orthodox or less mature researches." In fact, so full there is scarcely room for the Editorial.

We would like to thank the many contributors to this issue, and to invite you all to send us information and items serious or fanciful for future issues. In particular, we would be very glad to re-establish links with mathematicians in other universities and to publish some contributions from them.

It so happens that the present editorial staff cannot read German and have always been puzzled by the remarkable variety of German Bs and Gs which appear in some mathematics apparently just to confuse the reader. Hence it was with delight that we happened to discover a complete German alphabet whilst preparing this issue, and with some surprise that we find it contains actually only one B and one G. Perhaps others have been puzzled likewise?

$\mathfrak{A}\ \mathfrak{B}\ \mathfrak{C}\ \mathfrak{D}\ \mathfrak{E}\ \mathfrak{F}\ \mathfrak{G}\ \mathfrak{H}\ \mathfrak{I}\ \mathfrak{J}\ \mathfrak{K}\ \mathfrak{L}\ \mathfrak{M}\ \mathfrak{N}\ \mathfrak{O}\ \mathfrak{P}\ \mathfrak{Q}\ \mathfrak{R}\ \mathfrak{S}\ \mathfrak{T}\ \mathfrak{U}\ \mathfrak{V}\ \mathfrak{W}\ \mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}$

The Archimedeans

Our programme this year is full and varied, with a number of distinguished speakers at our evening meetings. As our speakers are anxious to be understood by all, members with limited mathematical background should not feel deterred. We are departing from our usual practice this year by holding our first meeting at the very beginning of term, and by inviting a very eminent member of our own University, Sir George Thomson.

This year there will be an emphasis on opportunities for mathematicians in business. We shall be having a talk by the Secretary of the Institute of Actuaries, and later a visit to the Stock Exchange. We shall also be visiting the National Physical Laboratory, and the Shell Head Office in London.

Tea meetings have proved popular in the past. We are holding several this year at which research students will discuss some of the lighter aspects of modern mathematics. There will be a Symposium of undergraduate speeches, and a meeting at which members will describe their vacation employment.

Our social activities have already included a Punt Party and a Ramble, and we shall arrange visits to the Gilbert and Sullivan

Opera in December, and to *My Fair Lady* in February. There will be a debate in the Lent Term, and we hope to hold a Christmas Party.

The Bridge Group will cater for both beginners and experts, the Music Group will meet weekly, and we shall revive the Play-Reading Group. It is hoped that members will support these activities.

The Committee hopes that the programme will be of interest to you all, but if you have any suggestions or criticisms do not hesitate to voice them, either to the Secretary or through the book in the Arts School. G. J. S. Ross.

Mathematical Association

President: Miss L. D. Adams, B.Sc.

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 21s. per annum: to encourage students, and those who have recently completed their training, the rules of the Association provide for junior membership for two years at an annual subscription of 10s. 6d. Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published four times a year and deals with mathematical topics of general interest.

At present the Association has twelve Junior Branches (of which The Archimedeans was the first). The members of a Junior Branch may attend all meetings of the Association as associate members.

Postal Subscriptions and Back Numbers

For the benefit of persons not resident in Cambridge we have a postal subscription service. You may enrol as a permanent subscriber and either pay for each issue on receipt or, by advancing 10s. or more, receive future issues as published at 25% discount, with notification when credit has expired.

Copies of *Eureka* Nos. 11, 13, 15, 16, 17 (1s. each), 18 (1s. 6d.), 19, 20, 21 (2s. each), are still available (postage 2d. extra on each copy). *Complete set of nine,* 10s., *post free.* We would be glad to buy back any old copies of Nos. 1 to 10 which are no longer required.

Cheques, postal orders, etc., should be made payable to "The Business Manager, *Eureka*," and addressed to the Arts School, Bene't Street, Cambridge.

Constructing Open Knight's Tours Blindfold!

By R. C. Read

THE problem of constructing open knight's tours on a chessboard, beginning at a given cell and ending on a given cell of the opposite colour has been studied for a long time, and many methods of solution have been given. Some of these, such as that due to Euler ([1], p. 176) require adjustments to be made to an initial trial tour until all cells previously omitted have been included; such methods therefore are not suitable if one wishes to construct a tour correctly the first time, or to construct a tour blindfold. Unsuitable, too, are methods, like that of Warnsdorff ([1], p. 180; [2], p. 259), which require careful check to be kept at each stage of the cells already traversed.

A method which would appear to have neither of these disadvantages is that due to Roget. A description of this method is given by Rouse-Ball ([1], p. 181), but unfortunately this description contains a fallacy, as will be seen later. In studying this I arrived at the following variation of Roget's method, which enables a knight's tour to be constructed blindfold with comparative ease.

Roget's method, as described by Rouse Ball, is briefly as follows. The board is divided into four *quarters* and the cells of each quarter are labelled with the letters *l*, *e*, *a*, and *p* as in Fig. 1. The cells labelled with a particular letter can then be combined into a *circuit*. If the initial and final cells are labelled with a consonant and vowel (or vice versa), say *l* and *a* respectively, then all the *l*-cells are used first, then all the *e*-cells, all the *p*-cells and finally all the *a*-cells, ending on the required *a*-cell.

For the case when the initial and final cells are labelled both with consonants, or both with vowels, the description of the method reads as follows: ". . . first select a cell, Y, in the same circuit as the final cell, Z, and one move from it, next select a cell, X, belonging to one of the opposite circuits and one move from Y. This is always possible. Then leaving out the cells Z and Y, it always will be possible, by the rule already given, to travel from the initial cell to the cell X in 62 moves, and thence to move to the final cell on the 64th move."*

This last statement is incorrect as can be seen from Fig. 2.

* *Editor's note:* The original description of this method by Roget in *Philosophical Magazine*, April, 1840, was not so explicit, and thus not erroneous.

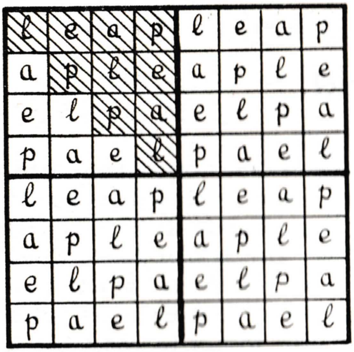


Fig. 1.

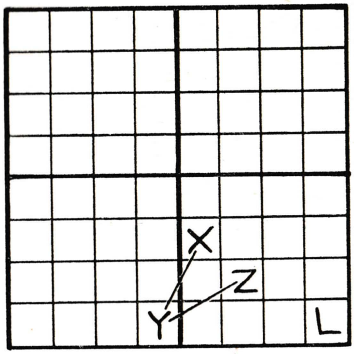


Fig. 2.

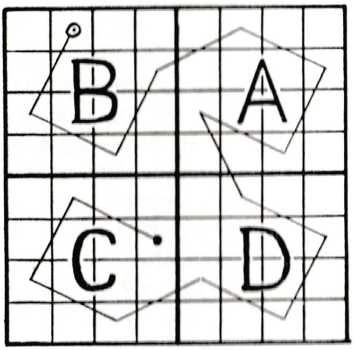


Fig. 3.

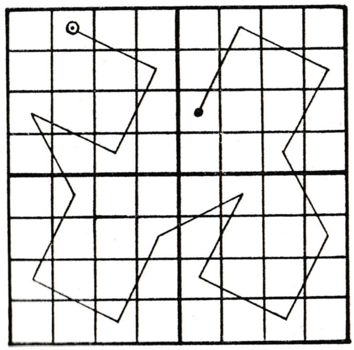


Fig. 4.

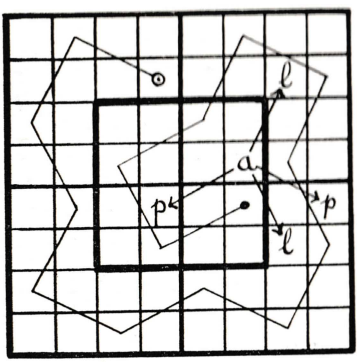


Fig. 5.

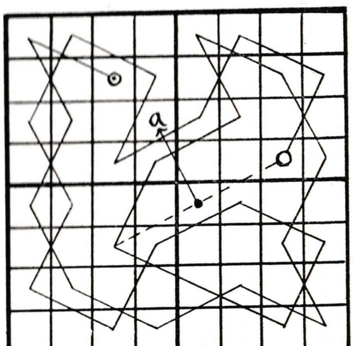


Fig. 6.

If the final cell is Z, the corner cell L cannot be chosen as Y since there would then be no cell of an opposite circuit available as X. On the other hand, we cannot go from the initial cell to X, leaving out Y and Z, since there is then no way of including L.

In the method described below, the general plan of Roget's method, viz. that of dealing with all cells having a given letter at one time, as far as possible, is kept. The tour is effected in stages, at most 16 cells being concerned at each stage.

Definitions

The cells will be said to be of two *types* according as they are labelled with a consonant or a vowel.

Adjacent quarters, such as B and A, or A and D in Fig. 3 are said to be *unlike*. A quarter *like* a given quarter is either the non-adjacent quarter, or the given quarter itself.

A *path* is a sequence of 15 moves taking in all 16 cells with a given letter. We shall define two special kinds of path.

An *ordinary path* is one which passes through all four quarters in turn, clockwise or anti-clockwise, and which takes in all the cells in a quarter before leaving that quarter.

An ordinary path can be made to end in either of the quarters unlike that in which the path began. This can be verified for the shaded cells in Fig. 1, and follows for all other cells by symmetry. A typical example is illustrated in Figs. 3 and 4.

A cell of the centre block of 16 cells (Fig. 5) will be called *central*. A central cell has the property that from it we can move to a cell of the other type in either a like or an unlike quarter. Thus from the *a*-cell shown in Fig. 5 we can move to either a *p*-cell or an *l*-cell in either a like or an unlike quarter.

A *spiral path* is one which takes in first noncentral cells, and then, when these are used up, central cells. If the first cell of the path is central, the first move is made to a non-central cell. Provided the first move is to a cell in the same quarter there is no difficulty in constructing spiral paths, and it is readily verified that there is a spiral path starting from any cell. A typical example is given in Fig. 5. Note that the last two cells of a spiral path are central cells.

We now give the method for the construction of knight's tours, and distinguish three cases.

Case I. *Initial cell of different type to final cell*

The tour will then consist of alternate vowel- and consonant-paths. Suppose the initial cell is a *p*-cell, and the final cell an *a*-cell. Construct a spiral *p*-path; move to an *e*-cell and construct a spiral e-path. Move to an l-cell and construct a spiral l-path. Now move to an a-cell in a quarter unlike the quarter containing the final a-cell (this is possible). There is then an ordinary a-path which ends in the quarter containing the final cell. Moreover, since we enter this quarter by a cell of the opposite colour to the final cell, we can always make our path end on the desired cell.

CASE II. *Initial cell of same type but different letter to final cell*

Suppose the initial cell is an l-cell, and the final one a p-cell. Construct a spiral l-path as far as its penultimate cell. Mentally continue the path to the last cell and on to a cell of the opposite type, say an e-cell. Make a note of this e-cell (Fig. 6). We now continue the tour from the penultimate l-cell (which is central) to an e-cell in a quarter unlike that of the noted e-cell. There is then an ordinary e-path ending at the noted cell, as in the last part of Case I. From this square we move to the last l-cell and so to an a-square (Fig. 6). The tour then proceeds as in Case I, via a spiral a-path, and an ordinary p-path started and ended in the appropriate quarters.

CASE III. *Initial cell* (I) *of same letter* (say l) *as final cell* (F)

Mentally construct a path beginning at I and ending at F. This is always possible and quite easy to do. Let P be any central cell of this path, and Q the next cell after P. Make a note of a cell of the opposite type which is a knight's move from Q. It is not necessary to remember how this cell was obtained. Let us suppose that the noted cell is an a-cell as in Figs. 7 and 8.

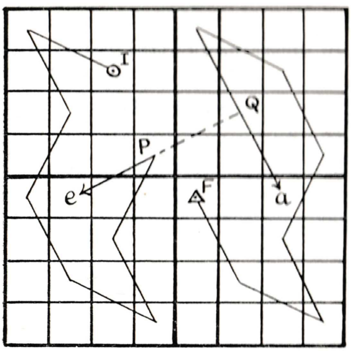


Fig. 7.

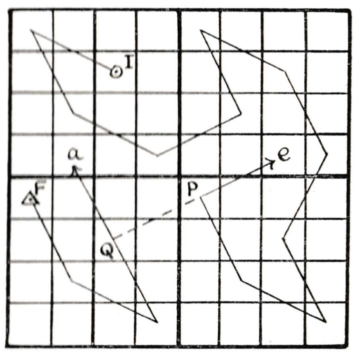


Fig. 8.

Construct the tour as follows: Construct the portion of an l-path between I and P, and, from P, move to an e-cell. Construct a spiral e-path, move to a p-cell and construct a spiral p-path. Now move to an a-cell in a quarter unlike that of the noted cell. There is then an ordinary a-path ending at the noted a-cell, and from there the tour is completed via the remaining l-cells.

NOTE

In constructing an ordinary path a certain amount of circumspection is necessary, but it is not necessary to look more than four moves ahead—sufficient to see that the first move after entering a quarter is one which will enable that quarter to be left, or the final cell of the path reached, as the case may be.

The path from I to F in Case III may sometimes not be immediately obvious; but it will be seen that it is not necessary for P and Q to be a knight's move apart, and this fact can be used to simplify the construction. For example, we can construct portions of paths from I to P and from F to Q, together taking in all the cells having the given letter. Provided only that P is a central cell, and Q is not a corner cell, the construction proceeds as before.

1. W. W. Rouse Ball. *Mathematical Recreations and Essays* (Revised by H. S. M. Coxeter). American Edition, 1947. Macmillan.
2. M. Kraitchik. *Mathematical Recreations*. Fourth Impression, 1949. George Allen and Unwin.

The Computer Revolution

By MARTIN FIELDHOUSE

ABOUT ten years ago *Eureka* (Nos. 10, 11 and 13) was reporting on the building of, and early experience with the remarkable high-speed electronic computer EDSAC in the Mathematical Laboratory. A great change has taken place in these ten years. A new computer, EDSAC II, nearly 60 times faster replaced EDSAC two years ago. Whereas there were then only two other computers in the world, there are now about 100 other computers in this country alone, and well over 1,000 in America. Ten years ago one authority thought there might never be a need for more than four or five large computers in this country! Soon in every scheme for automation a computer will be essential.

This summer an International Conference on Information Processing was held in Paris. About 1800 people from 37 countries attended, which alone demonstrates the increasing interest in computers. It might have been expected that such a conference would have looked back with satisfaction on a decade of great achievements and forward to a period of less rapid advance. But not at all. There can be little doubt that we are witnessing "the computer revolution," for the whole tenor of the conference was that even more remarkable developments lie ahead. I will try to indicate a few.

If we discuss computers by generations where each generation is approximately 100 times faster than its predecessor, then the present is the era of second generation machines of which EDSAC II is an example. These machines can perform on average about 10,000 elementary operations ("add," "multiply," etc.) per second. Third generation computers are being built. The first in this country will be the computer MUSE at Manchester University, which may be ready in about two years time. Its average speed will be at least 700,000 calculations per second.

In the more distant future lie the possibilities for even faster computers until at about 100 times faster than MUSE the speed of light becomes a severely limiting factor. Three types of very fast computing elements are being developed: cryotrons which make use of superconductivity, very thin magnetic films, and parametrons. These latter are circuits with resonating frequency (say) f, which can be made to resonate exactly in or out of phase with each other by using a pumping frequency $2f$. It is anyone's guess which of these devices or what other device will prove most successful.

A major advance will occur when computers can be made to read printed and even hand-written information directly instead of only punched cards or other media on to which all information must first be laboriously transferred. It is likely that reliable reading machines will soon be developed. There are already a few which work quite well as long as special type is used. But better mathematical theories of pattern recognition are needed.

To enable very fast computers to operate continuously and efficiently the designing principle of "time-sharing" has been introduced. This allows the central arithmetic unit of the computer to be interrupted in the middle of its calculations in order to carry out some more pressing work before returning to its calculations and carrying on from where it left off. Using this method a fast computer will be able to keep busy many subsidiary units such as printers and magnetic-tape decks. In some circumstances a computer will be able to work on several problems at once, or (as in one American system) several computers will work on one problem at the same time!

Much time and study is being devoted to the development of a common symbolic language for computers. Such a language will enable a programme written for any computer using it to be mechanically translated and run on any other computer using it. At present each computer has its own code and communication between programmers of different computers is very difficult. In its printed form this new language will look similar to mathematics. But whereas methametics consists mainly of statements describing relations between quantities, this new language is "algorithmic" and consists mainly of instructions to perform operations on quantities.

Perhaps the most exciting study is that of trying to teach computers to learn and think. The great difficulty is to know what we mean by "learn" and "think." If a human thought process can be explicitly described then the computer can be made to copy it, so that in certain simple senses computers can already "learn" and "think." It is recognized that the human brain has quick access to far more information and experience than any computer so far built, but apart from this it is rather surprising that no one has yet been able to say whether and how human thought processes differ essentially from computer thought processes.

What are the problems that require even bigger and faster computers? Here is just one example. In order to predict the heights of the tides at any place, it is necessary at present to analyse tidal reading taken at that place or some place close by over a period of about 30 years; in future it may be possible to calculate the tides directly for any place in the world given instead data regarding the shape and depth of the oceans.

A Question of Limits: II

By H. T. Croft

We generalize our previous results (*Eureka*, **20**, p. 11), which were: If $f(x)$ is a real, continuous, positive function,

then $\lim_{n\to\infty} f(na) = 0$, for all a, as $n \to \infty$ through integral values,

implies $\lim_{x\to\infty} f(x) = 0$, as $x \to \infty$ continuously:

but $\lim_{n\to\infty} f(n + a) = 0$, for all a

does not necessarily imply $\lim_{x\to\infty} f(x) = 0$.

Now, let f be as above; and let $g(n)$, $h_1(n)$, $h_2(n)$ . . . be polynomials. Let $G =$ degree $(g(n))$ and $H =$ max (degree $h_i(n)$) for all i. Then we have: $\lim_{n\to\infty} f(a.g(n)) = 0$ for all a, implies $\lim_{x\to\infty} f(x) = 0$.

And $\lim_{n\to\infty} f(g(n) + b_1h_1(n) + \ldots + b_kh_kn) = 0$

necessarily implies $\lim_{x\to\infty} f(x) = 0$, only if $G \leqslant H$.

For consider the case $\lim_{n\to\infty} f(a.g(n)) = 0$. If $f(x)$ does not tend to 0, there exists an infinite set of intervals (r_j,s_j) with $r_j,s_j \to \infty$, where $f(x) > \epsilon$, for some ϵ. We now exhibit an a for which $a.g(n)$ falls infinitely often in an (r_j,s_j) and the contradiction will be established.

Take a_1 in some arbitrary interval (α,β). Consider the systems of intervals (z_n, z_n') which $a_1g(n)$ fills as a_1 runs from α to β. As $\{g(n+1)/g(n)\} \to 1$, these intervals will cover the positive real axis from some point onward. Therefore, for some sufficiently large n, and suitable a_1, $a_1g(n)$ is an interior point of an (r_j,s_j), and so, by continuity, is $a_1'g(n)$ for the same n and a_1' sufficiently near to a_1, i.e. in an interval (α_1,β_1) say, interior to (α,β).

We repeat the argument, taking (α_1,β_1) for our new (α,β), and so on. For any a_q in (α_q,β_q), $a_{q+1}g(n)$ falls in at least q intervals (r_j,s_j). The descending system of intervals (α_q,β_q) has at least one common point. For such a point a, $a.g(n)$ falls infinitely often in the (r_j,s_j). Thus

$$\lim_{n\to\infty} f(a.g(n)) = 0 \text{ implies } \lim_{x\to\infty} f(x) = 0. \qquad \text{Q.E.D.}$$

The same argument applies in the other case if the degree of say $h_1 = H_1 > G$, for $\lim_{n\to\infty} \dfrac{g(n+1) + b_1h_1(n+1) + \ldots}{g(n) + b_1h_1(n) + \ldots} = 1$. If $G = H_1$, we may replace $g(n) + b_1h_1(n) + \ldots$ by $g'(n) + b_1'h_1'(n) + \ldots$ where $G' < H_1'$ (with an obvious notation) and the argument proceeds as before.

However, if $G = p > H_1, H_2 \ldots$, the opposite conclusion will be true. For we may replace $g(n) + b_1h_1(n) + \ldots$ by $g(n) + b_1'n^{p-1} + b_2'n^{p-2} + \ldots$ And by making a change of variable, $n' = n + k$, for some positive or negative integer k, we may write this as $g(n') + b_1''(n')^{p-1} + \ldots$ with $0 \leqslant b_1'' < p$. We may now drop the primes.

Now, take $f(x)$ equal to zero outside the intervals $(g(m), g(m) + \epsilon_m)$, $(m = 1,2\ldots)$ where $\epsilon_m \to 0$, and equal to a triangular peak function of height 1 in each such interval. Then $f(x)$ does not tend to 0, but I say $f(g(n) + b_1n^{p-1} + \ldots) \to 0$ for all $b_1, b_2 \ldots$

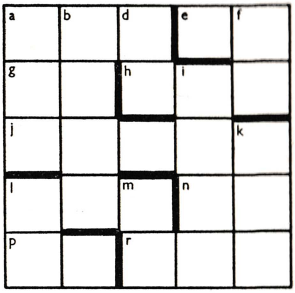
If not, for infinitely many n, $g(n) + b_1n^{p-1} + \ldots$ lies between $g(m)$ and $g(m) + \epsilon_m$. Because of the condition we saw we could impose upon b_1, viz. $0 \leqslant b_1 < p$, the above will only happen (for sufficiently large m and n) if $n = m$; and so $0 \leqslant b_1n^{p-1} + \ldots < \epsilon_n$. Hence successively $b_1 = 0$, $b_2 = 0 \ldots$ And as $f(g(m)) = 0$, by construction, for all m, we see $\lim_{n\to\infty} f(g(n) + b_1n^{p-1} + \ldots) = 0$.

Q.E.D.

As to the three possible directions of extension of these theorems, i.e. extending $g(n)$ to functions other than polynomials, weakening "for all a" condition in the hypothesis, and weakening the continuity condition, it is trivial from the proof that "$g(n) \to \infty$ and $\{g(n+1)/g(n)\} \to 1$" and again that "for all a in any (small) interval" are sufficient hypotheses; whilst the continuity condition cannot be entirely dropped, as we showed in our previous article.

Numerical Square

By Rev. A. H. Barrass



Small letters across, capitals down. Each number is of the form

$$x_i^2 \pm y_i$$

where x_i is a positive integer and y_i is a positive prime number.

$x_D = \sqrt{x_j}$, $y_F = \sqrt{x_m} = \sqrt{x_p}$,
$y_e = x_e = x_h - y_B^3$
$x_I = x_D^2 - y_A$
$y_F = y_p - y_B$
$20y_L = 5x_B + x_A$
$x_a - 1 = 2y_F y_p$
$x_h y_B = x_r + 1$
$x_A + 1 = x_a$
$y_I = y_L^2 + y_F x_g^2$

$y_B = \sqrt{x_g}$, $x_n = x_F = y_p$
$x_L^2 = y_F y_m - y_F$
$y_A y_m = y_p x_a - y_F^2 - y_F$
$x_g y_A = x_r + x_h$
$y_p y \;= x_a - 7y_F$
$3x_D = x_A$
$y_h y_K = x_a^2 - 2x_A$
$x_B = y_p y_m + y_p$
$y_F y_p = x_a - 4y_B^2$

Values of quantities not found in the clues are equal to numbers which can be found therein.

Two Matrix Problems

By Max Rumney

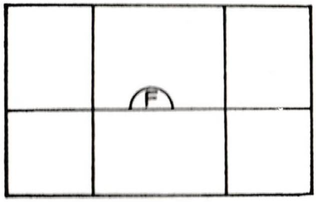
1. Devise a method of arranging the natural integers 1 to n ($= 2km^2$, m odd) into $2k$ square matrices of order m with no integer repeated, no matrix singular, and the integers in each matrix in ascending order of magnitude, so that the sum of the $2k$ determinants is divisible by $n + 1$.

2. For $n \geqslant 2$, arrange $n^2 - 2n$ integers of the form 2^k ($k = 1, \ldots n-1$) and any $2n$ odd integers into an $n \times n$ matrix in which no row or column has a factor and whose determinant has value 2^{n-1}.

Whewell's Ale Caucus Race

(1) The race will be run in Great Court at 2 p.m. on Monday, 3rd August, 1959, being a Bank Holiday.

(2) The course will be once round each of the 18 rectangular circuits which it is possible to describe in Great Court.



Great Court

(3) Competitors are not allowed to run round two or more of the circuits simultaneously and each circuit must be run *as a whole.*

(4) Each circuit is to be described in the same sense, undergraduates to run clockwise and B.A.s or research students anticlockwise. To avoid confusion undergraduates may not execute left turns nor B.A.s or research students right turns.

(5) It is not permissible to double back over one's track on any segment of the course.

(6) Competitors must wear their gowns unless they are natives of Sweden.

(7) The umpire will be at the Fountain from 1.45 onwards. Competitors will draw by lot before 2 p.m. the staircase door from which they are to start. They must engage in academic conversation with each other there until the last stroke of two.

(8) The finishing point will be by the Fountain.

(9) The prize will be a firkin of Whewell's Ale which all competitors will share and which will be dispensed by the winner.

(10) There will be an Umpire

NOTES

(*a*) Sample routes may be obtained from any mathematician in the College for a consideration.

(*b*) Swedish nationality may be obtained from either Q1 or Q2 Great Court also for a consideration.

COMMENT: One of the Junior Fellows of the College was nominated Umpire. The Race was won by an undergraduate, Adrian Rushton, in the very fast time of 13 minutes, 43 seconds. Andrew Paterson, B.A., last year's Business Manager of *Eureka,* required 13 minutes, 45 seconds, thus proving that it is almost as fast to run anti-clockwise as it is to run clockwise.

Readers will have immediately recognized that it is significant that the number of circuits is the product of two triangular numbers.

Problems Drive, 1959

By M. R. BOOTHROYD and J. H. CONWAY

1. In the year 1960, question 3.14159265 . . . of the Archimedeans' Problems Drive was: "Find the next term in each of the following series:

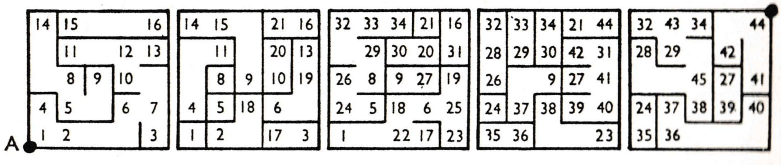
   (i) 5, 7, 9, 13, 17, 19, 21, 25, . . .

   (ii) 0, 1, 1, 3, 11, 43, 225, 1393, . . .

   (iii) 1, 3, 7, 13, 19, 31, 43, . . ."

   Unfortunately, the printer had altered one number in each of the three series. Find his mistakes.

2. Certain inner walls of a $5 \times 5 \times 5$ array of cells are removed. The diagram shows the five horizontal layers. There is no wall between cells in adjacent layers bearing the same number. Find a route from **A** to **B** which passes through as few cells as possible.

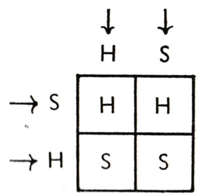


3. Express the integers from one to twelve using only three 1s and the usual arithmetic symbols. The symbol for integer part and trigonometric functions are not allowed.

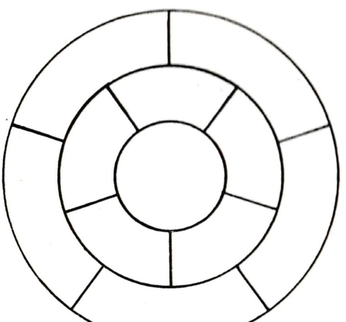
4. Bloggs & Co. grind corn at their A, B and C mills at the rate of 20, 15 and 35 sacks per hour. The flour is to be taken to four bakeries P, Q, R and S, who require 20, 20, 10 and 20 sacks per hour. The table gives the cost in pence per sack of taking the corn from each mill to each bakery. Determine how many sacks per hour each mill should supply to each bakery if the costs are to be as low as possible.

	A	B	C
P	3	6	3
Q	6	9	8
R	5	8	5
S	4	8	5

5. Each cell of a $2 \times 2 \times 2$ cube is to be filled with a single digit so that the whole forms a three-dimensional cross-number puzzle. No digit is repeated. S indicates a perfect square. H indicates half an even square, or the integer part of half an odd square. Clues are given for both directions in the upper plane; and for downward verticals, the latter being indicated by letters *in* the appropriate cells.



6. There are exactly twelve ways of colouring the edges of a pentagon with five given colours (one colour to each edge), provided we count rotations and reflections of any colouring as identical colourings. There are also twelve pentagonal faces to a regular dodecahedron. Colour each edge of a dodecahedron in such a way that every possible pentagonal-colouring appears as a face. Here is a diagram and table.



Possible Colourings

12345	equals	15432	A
12354	,,	14532	B
12435	,,	15342	C
12453	,,	13542	D
12534	,,	14352	E
12543	,,	13452	F
13245	,,	15423	G
13254	,,	14523	H
13425	,,	15243	J
13524	,,	14253	K
14235	,,	15324	L
14325	,,	15234	M

7. A. The total number of true statements in this problem is 0 or 1 or 3.

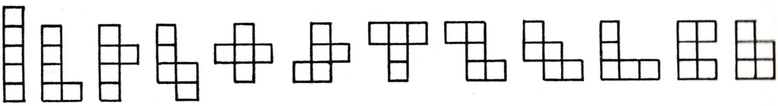
   B. The total number of true statements in this problem is 1 or 2 or 3.

   C. The total number of true statements in this problem (excluding this one) is 0 or 1 or 3.

   D. The total number of true statements in this problem (excluding this one) is 1 or 2 or 3.

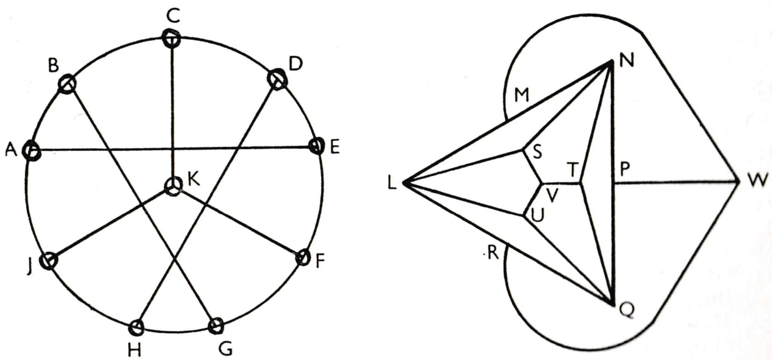
   Which of the above statements are true?

8. Using only the shapes shown below, or their reflections, build two 5×5 squares using no shape twice. (A shape and its reflection are counted as the same).



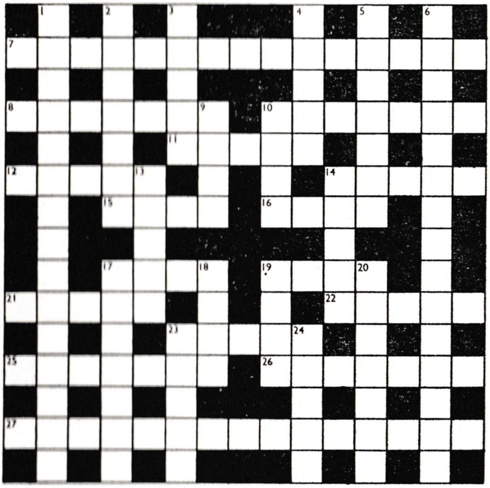
9. Five identical boxes originally contained the same quantity of sugar, but a little has been transferred from one to another. Given an uncalibrated balance and no weights, show how to identify both the heavy and the light box with the least number of weighings.

10. Prove that on neither of the accompanying networks is there a closed polygon containing all the vertices. In the first figure, the lines only meet at the ringed vertices, so that the network is really three-dimensional.



Crossword

By G. Berry



ACROSS

7. These should be in strict one to one correspondence (8, 7).
8. Of astronomical interest although a microscope sounds appropriate (3, 4).
10. Of hydrodynamical interest perhaps? (7).
11. They often form a criterion for consistency (5).
12. Solid figures indicating rough seas (5).
14. This confused surface suggests retribution (5).
15. It may be named after 2 down and obviously belongs to him when inverted (4).
16. Artist and direction give causes of suffering (4).
17. Finite set from over the border (4).
19. Meeting of two branches (4).
21. Groups with at least six members (5).
22. Raw material taken from chopped up theorems. Only

DOWN

1. Mixed colours and a news' bearer. Either is an adequate description of a curve (5,2,7).
2. His laws are related to wave theory (7).
3. Determinant part (5).
4. Solutions under 19 down may-be (5).
5. Surfaces and fields of action (7).
6. Homer made lament (anag.). The first may pave the way for the second (5,3,7).
9. Functions without the man are practically the same (4).
10. Lines are generally so in spaces of more than two dimensions (4).
13. Subtraction of fifty gives trader's action (5).
14. Solid. With dried plums, only a manner of speaking! (5).
17. Transformation of interest to fiddlers (7).
18. Ten again is the nearest (4).

ACROSS

fit for burning, colloquially (5).
23. Starting point often self-evident (5).
25. Euclid's ——s are geometrical (7).
26. Operator. Confused ancient poet sir! (7).
27. A door must be so set (4,2,6,3).

DOWN

19. Solid useless to the gardener (4).
20. 9 down has one of these, producing monotony (7).
23. Sphere of man or beast (2,3).
24. Large number lost in the —— of a fog (5).

The solution to this crossword will be printed in our next issue.

Chessmas

By R. Schwarzenberger

Seven mathematicians were once shipwrecked on an island. They immediately set to work mining chalk, painting blackboards, weaving dusters and carving chessmen so that they would not have to abandon the mode of life which they found so congenial. Their housing problems were easily solved: around the cliffs of the island were seven caves each with access to a path along the shore, and easily reached from the Lone Pine which marked the centre of the island.

Each evening they would meet in one of the caves and decide where each should sleep that night by the following device: two by two they would play chess until a game was won or lost. Then the winner would leave (clockwise) for the neighbouring cave. The loser (anticlockwise) would do the same. As soon as more than one person arrived at a cave the procedure would be repeated. Every night, no matter in what order the games were played, or how long they took, fourteen games were played to a result before the seven mathematicians slept separate and undisturbed.

But one of the mathematicians (wiser than the rest) slept longer than the others. For each night as the chess started he would withdraw to a corner and let the other six choose their partners. Knowing that in a few hours everyone would have left the cave he went straight to sleep. Even when someone arrived eager to play chess, he slept on, knowing that soon a second would arrive and that they would play chess and both depart.

The younger mathematicians met by the Lone Pine to discuss this antisocial behaviour and to find ways of ensuring that *all* should play more chess. Surely the system could be modified so that more than a mere fourteen games were played each night.

And this is what they decided: Instead of starting all at the same cave each of the seven mathematicians would go to the Lone Pine and roll a stone to decide at which cave he should begin. As soon as two arrived at a cave chess would begin and the winner and loser would move as before. By this method, it was hoped, more than fourteen games would sometimes be played. If by any chance no games were played a special holiday called Chessmas would be declared the next day.

In high hopes of Chessmas the stone rolling began. Alas! That night the chess continued until the dawn and would have gone on forever had they been faithful to their plan. If only, they said, we had been shipwrecked on a shore of infinite extent and unlimited equidistant caves. Then we would have variety but never more than twenty games each night.

The younger mathematicians were not put off. They constructed an eighth cave and worked the plan just as before. Chessmas now occurs several times a year and a special holiday of seven days has been promised should there be a recurrence of the disaster which caused the building of the eighth cave. So far this hasn't happened, nor indeed has a night with more than twenty games. Can either occur?

The reader is invited to check the statements made and to solve the questions raised. Will it do him any good? Yes, if ever he is shipwrecked with $2n$ other unfortunate mathematicians.

Vipers, Logs and All That

By G. J. S. Ross

It is widely believed that the only mathematician in the Bible was Noah. Nobody else would have had a hope of passing the Eleven Plus. Admittedly, Moses' Book of Numbers is frankly disappointing, but I hope to show in this article that the Bible contains evidence of a higher standard of mathematics than is generally supposed.

Arithmetic is, of course, mentioned most frequently, and we are told that men sometimes worshipped figures.[1] At a very early stage "men began to multiply,"[2] and Abraham was familiar with division.[3] Some writers have pointed out that the arithmetic in Ezra[4] is faulty, but this is explained where it reads "certain additions were made of thin work."[5] The approximation for π is reasonable,[6] considering the fact that Moses destroyed the tables,[7] which were not replaced until Solomon's time.[8] Elsewhere we read "he shall not extract the root thereof,"[9] and "we wrestle against powers."[10]

The first attempts at Geometry were, of course, Euclidean. We read that "great rulers were brought down,"[11] "from Syracuse they fetched a compass,"[12] and Noah constructed an arc[13] and Ezekiel described a line.[14] Further progress was made when they took axes,[15] culminating in David's success with the calculus.[16] David, incidentally, was the first to refuse to accept what he had not proved.[17] St. Paul was familiar with four dimensions,[18] and Joshua continued with the arc along a Jordan path.[19]

Algebra, although thought to be an invention of the Arabs, was only too familiar to the Jews. For instance, Moses gives instructions about a matrix[20] and Ezekiel knew enough about rings to describe them as "dreadful."[21] Peter was kept half the night by four quaternions,[22] and the Jews were described as "a generation seeking after a sign."[23]

"As for the Pure, his work is right" said the writer of Proverbs,[24] and this attitude is reflected in the few existing references to Applied Mathematics. "I have seen thy abominations in the Fields" cried Jeremiah,[25] and the Psalmist complained "Thou hast afflicted me with all thy Waves."[26] Later the Father of Publius was "sick of the bloody Flux."[27]

It is easy to understand why they disliked mathematics. Apart from the deacons "who purchase to themselves a good Degree,"[28] they had to be examined, as was St. Paul.[29] We know that Elisha passed,[30] and Solomon was able to answer all the questions,[31] but Peter was much troubled when he saw the sheet,[32] and Job cried "My kinsfolk have failed, and my friends."[33] Perhaps Jehoiakim was an examiner, for "when he had read three or four pages he cast it into the fire."[34] As for St. John, all that he knew was "the Second woe is past, the Third cometh."[35]

[1] Acts vii. 43. [2] Gen. vi. 1. [3] Gen. xv. 10. [4] Ezra ii. [5] 1 Kings vii. 29. [6] 2 Chron. iv. 2. [7] Exod. xxxii. 19. [8] 2 Chron. iv. 8. [9] Ezek. xvii. 9. [10] Eph. vi. 12. [11] Ps. 136. 17. [12] Acts xxviii. 13. [13] Gen. vi. (archaic spelling). [14] Ezek. xl. [15] 1 Sam. xiii. 21. [16] 1 Sam. xvii. [17] 1 Sam. xvii. 39. [18] Eph. iii. 18. [19] Joshua iii. [20] Exod. xxxiv. 19. [21] Ezek. i. 18. [22] Acts xii. 4. [23] Math. xvi. 4. [24] Prov. xxi. 8. [25] Jer. xiii. 27. [26] Ps. 88. 7. [27] Acts xxviii. 8. [28] 1 Tim. iii. 13. [29] Acts xxviii. 18. [30] 2 Kings iv. 8. [31] 2 Chron. ix. 2. [32] Acts xi. [33] Job xix. 14. [34] Jer. xxxvi. 23. [35] Rev. xi. 14.

Solutions to Problems

LAST YEAR'S PRIZE CROSSWORD

Across: 1. Sinecure. 5. Cut. 8. Octagon. 9. Iota. 10. G–in. 12. poINT Of. 13. Set (3 meanings). 16. E-uler. 18. Group. 19. Age. 21. Eyes. 23. One. 25. Ball. 26. Laplace. 27. Lie (Sophus). 28. Integral (anag.).

Down: 1. Sloping. 2. Notation. 3. C–age. 4. Range. 6. Tram (Trammel of Archimedeans = linkage for describing ellipse). 7. Sin. 11. infinITE Mass. 14. Playfair. 15. Span. 17. Russell. 20. Gelon (King of Syracuse). 22. Abel (= 10 decibels). 23. gOLDen. 24. collAPSE.

The first correct entry was received from C. D. RODGERS and J. H. CONWAY.

NUMERICAL SQUARE:

a.	$948 = 31^2 - 13$		A.	$911 = 30^2 + 11$
e.	$42 = 7^2 - 7$		B.	$4902 = 70^2 + 2$
g.	$19 = 4^2 + 3$		D.	$81 = 10^2 - 19$
h.	$172 = 15^2 - 53$		F.	$22 = 5^2 - 3$
j.	$10053 = 100^2 + 53$		I.	$7512 = 89^2 - 409$
l.	$126 = 11^2 + 5$		K.	$344 = 19^2 - 17$
n.	$14 = 5^2 - 11$		L.	$17 = 6^2 - 19$
p.	$76 = 9^2 - 5$		M.	$68 = 9^2 - 13$
r.	$824 = 29^2 - 17$			

PROBLEMS DRIVE, 1959.

1. (i) Correct 17 to 15 (primes +2).
   (ii) Correct 11 to 10 (recurrence relation $u_{n+1} = n\,u_n + u_{n-1}$).
   (iii) Correct 19 to 21 ($n^2 + n + 1$).

2. A plane map can be drawn with as few as two arched crossings. Shortest routes, of which there are four, pass through 33 small cells.

3. $1 \times 1 \times 1$, $1 + 1 \times 1$, $1 + 1 + 1$, $\sqrt{(1/\cdot\dot{1})} + 1$, $\{\sqrt{(1/\cdot\dot{1})}\}! - 1$, $\{\sqrt{(1/\cdot\dot{1})}\}! \times 1$, $\{\sqrt{(1/\cdot\dot{1})}\}! + 1$, $1/\cdot\dot{1} - 1$, $1/\cdot\dot{1} \times 1$, $11 - 1$, 11×1, $11 + 1$.

4. Solution in sacks per hour:
   P to C 20, Q to A 5, Q to B 15,
   R to C 10, S to A 15, S to C 5.

5. Top layer:

1	6
2	4

Bottom layer:

8	0
5	9

6. There are three essentially distinct solutions. A ring of ten pentagons is given for each solution. Each pentagon of a ring touches the previous one along a line of the colour shown between the corresponding letters. Thus in the first solution C touches M along a line coloured 2 and B along a line coloured 4. (C will also touch K and H.)

(i) G 5 K 3 M 2 C 4 B 1 H 4 J 3 L 2 E 5 A 1 G.

(ii) B 2 M 5 K 2 F 5 A 1 H 3 L 5 E 3 D 5 G 1 B.

(iii) E 2 L 3 D 1 G 3 H 1 F 3 M 2 C 1 J 3 K 1 E.

7. B and D true, A and C false.

8. There are several solutions.

9. Weigh (i) AB against CD.
(ii) AC against BD
(iii) AD against BC.

Then AB $>$ CD, AC $>$ BD, AD $>$ BC $\rightarrow$ A heavy, E light, and similarly for all other combinations with no balancing at any stage. If, say, AB $=$ CD, then E must be normal, and neither of the other two weighings can balance. Then if AC $>$ BD, AD $>$ BC, A would be heavy and B light. All possibilities are of these two types.

10 (i). We may suppose CK not in the path, by symmetry.

Then KJ, KF, CB, CD are all present.

If now AE were not present, the AB, AJ, ED, EF would all be members of the path, and these form a complete path not containing GH.

Therefore AE is present. Now not both AB and DE can be present, as that we may suppose AB absent, and therefore AJ and BG are present. Then JH is absent and therefore DH and HG present.

Now GHDCB is a complete pentagon, all of which is present, and this is a contradiction.

(ii) Call LNQVW black vertices and the others white. Then any complete path must traverse alternately black and white vertices. But this is impossible since there are 5 black vertices and 6 white.

Explorer's Problem

Prof. R. L. Goodstein of Leicester has written to say that a proof that the solution of the Explorer's Problem given by I. C. Pyle in *Eureka* 21 is the best possible is given by G. G. Alway in the *Mathematical Gazette*, XLI (1957), p. 209.

The *Scientific American* posed this problem to their readers in May, 1959, and quoted *Eureka* on the solution in their June issue.

Book Reviews

Trigonometric Series. By A. ZYGMUND. Volume I, 383 pp. Volume II, 354 pp. (Cambridge University Press, 1959.) $10\frac{1}{2}$ in. 84s. per volume.

This work is an entirely new edition of the great classic on Trigonometric Series and Fourier Analysis by Professor Zygmund. It has been completely rewritten and greatly expanded to reflect the growth in this subject in recent years, so that the work is now more than double the size of the first edition. The central topic is the representation of functions by trigonometric series of the type

$$\tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x),$$

and Fourier series of the type

$$\sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu x}$$

but the work covers in detail a wide range of topics in the field of convergence and divergence of series, Lebesgue integrals, and summability in general.

Volume I contains the classical theory of trigonometric and Fourier analysis, whilst volume II contains the newer developments of the subject such as the concept of generalized derivatives and various topics in the field of multiple Fourier analysis.

Much of the material in the second volume is based on new work by Professor Zygmund and Professor Salem which has not previously appeared in print. The price, £4 4s. for each volume, is high enough to make the average undergraduate or research student pause before buying, but the work is a fundamental text for anyone hoping to do serious work either in general analysis or in the Fourier field in particular.

In Volume One the principles of the theory are analysed and explained in a clear, detailed and rigorous manner. There are many theoretical examples at the end of every chapter, so that the student can test his grasp of the text, but it is a pity that there are no actual numerical examples in the text nor in the exercises. In these days, such a work will be read not only by the theorist, but also by the practical worker in other fields, who will wish to make use of these theorems and techniques as tools in his work. A chapter outlining the special numerical methods or actually using these methods would not have been out of place here, with perhaps some mention of the wide applications in, for example, crystallography, or other branches of physics, chemistry and engineering. Many times in the earlier chapters a few more practical examples might have thrown great light on the points so ably and rigorously discussed in the text, and might have been very helpful to the not strictly mathematical reader. It is rather a pity also, in a work intended primarily for fundamental reference purposes, that volume one, containing as it does the more elementary parts of the theory, does not contain any separate index.

In Volume Two the modern extensions of the classical theory are explained. Dr. Zygmund devotes a chapter to the differentiation of Fourier series, and generalized symmetric and unsymmetric derivatives, and the central theorem of Littlewood-Paley is treated in detail. It should be noted that the formulae 2.36,37 on p. 253 are stated in an incomplete form, a laxity of presentation not found elsewhere in a work of such high standard and quality. There are no lists of the more popular transforms, since presumably the author expects the student to know that these are readily available elsewhere.

The strong differentiability of multiple integrals and the restricted summability of multiple series are considered, proofs being given in detail, in many cases for the first time.

The unsolved problems in the theory, such as convergence and divergence almost everywhere, the structure of the sets of uniqueness and the structure of functions with absolutely convergent Fourier series together with the many extensions possible in the multiple field are outlined and the way signposted for other research workers in these fields to make future discoveries and developments in the subject.

The binding and printing of the books are of the uniformly high standard which we have come to take for granted in the work of the Cambridge University Press, and this work, with its careful, rigorous and detailed proofs, and its high quality of scholarship, is an essential for any mathematical library, and for any serious student of analysis.

L. J. Slater

Multivalent Functions. By W. K. Hayman. viii, 152 pp. (Cambridge Mathematical Tracts No. 48, 1958.) 8½ in. 27s. 6d.

This is the first expository account of a subject which has grown up in original papers. The subject does not seem to be fundamental to any other part of mathematics—any interest it has is intrinsic. The author develops results giving bounds and asymptotic expressions for certain functions (e.g. modulus, coefficients) associated with a complex valued function f of a complex variable analytic in a domain D when various restrictions are placed on the number of solutions of the equation $f(z) = w$ in D. For example, if it has at most p solutions in D for each $w \in C$, then f is called p-valent. We then have the unsolved problem "If f is 1-valent in $|z| < 1$ and reduced to the standard form $f(z) = z + a_2 z^2 + \ldots$, then $|a_n| \leqslant n$." More general types of p-valent functions are considered in which the "average" number of solutions of $f(z) = w$ is restricted.

The author claims that most of the background for reading the book would be contained in an undergraduate course. This includes elementary theorems on complex power series, contour integration, the elements of conformal mapping and the maximum modulus theorem together with its immediate consequences. However, since the subject proceeds by intricate results of a classical nature, there is a very real need for the reader to have both a taste for such mathematics and an appreciation of the value of the results. This is particularly necessary as, while the first paragraphs in each chapter give a very good guide to the methods of that chapter, there is no general motivation.

The subject does not seem to afford much scope for a "bright" or original tract. One would perhaps wish for a more geometrical approach to some of the earlier confromal problems but it is good exercise for the reader to provide the required diagrams. The more difficult ones are provided by the author. B. E. Johnson.

Fallacies in Mathematics. By E. A. MAXWELL. xi, 95 pp. (Cambridge University Press, 1959.) $8\frac{1}{4}$ in. 13s. 6d.

Dr. Maxwell's new book is written "to instruct through entertainment . . . to amuse the professional, and help to tempt back to the subject those who thought they were losing interest." It is to be recommended to both these groups of readers, and the schoolteacher who has to do daily battle with fallacies in the classroom should certainly read it. The author's "general theory is that a wrong idea may often be exposed more convincingly by following it to its absurd conclusion than by merely denouncing the error and starting again."

Having illustrated in chapter I the distinction between a mistake, a howler ("an error which leads innocently to a correct result") and a fallacy (one which "leads by guile to a wrong by plausible conclusion"), the author uses chapter II to state four geometrical fallacies. These are discussed carefully in the next three chapters where it is shown that the usual axioms of elementary geometry are insufficient foundation for Euclidean geometry, for they take no account of ideas which underlie such words as inside, outside and between. The analysis of the fallacy that every triangle is isosceles is surprisingly far-reaching and we even find that Ptolemy's theorem and determinants have some bearing on the problem.

Chapters VI to VIII cover fallacies in algebra, trigonometry and calculus, many of which arise in connection with multiple-valued functions. Thus in algebra a failure to realize that -1 had two square roots $\pm i$ can lead to a "proof" that $+1 = -1$, and in trigonometry a number of fallacies stem from equating two distinct values of $\sin^{-1} A$. Again the substitution $y = \sin x$ in an integral is only valid if $\sin x$ is monotonic over the range of integration. The explanation on p. 60 of the fallacy on p. 55 might have been clearer: the phrase "turning values of x" has no meaning when x is considered as the independent variable.

Circular points at infinity and some "limit" fallacies take up chapters IX and X and the book closes with a collection of astonishing howlers which the author assures us are genuine.

In less than a hundred pages Dr. Maxwell has written a charming and lively book. The fallacies that there are no points inside a circle (p. 18) and the two in chapter VI which hinge on the fact that the equation $\tan z = i$ has no complex solutions are particularly neat. Finally, the author's expositions are generally so clear that although the book contains four "proofs" that every triangle is isoscles, the reviewer still doesn't believe it. K. R. MCLEAN

An Analytical Calculus. Volume IV. By E. A. MAXWELL. xiv, 288 pp. (Cambridge University Press, 1957.) $8\frac{3}{4}$ in. 22s. 6d.

This is the last of four volumes by Dr. Maxwell on Analytical Calculus. Dr. Maxwell found himself compelled to include in this volume more analysis than he originally intended; as a result the book is nearly a hundred pages longer than the other volumes.

The first section of the book deals with ordinary differential equations. The importance of linear independence, in connection with their solution is emphasised. Early use is made of the notation $D \equiv \dfrac{d}{dx}$ although it is never used formally for the evaluation of particular integrals as in $(D - p)^{-1}$. A good chapter on the solutions in integral form for linear differential equations with constant coefficients is included.

In the second section the theory of convergence is taken as far as the concept of uniformity of convergence and its application to the integration and differentiation of infinite series. Theorems in analysis, such as that an increasing function which is bounded above tends to a limit are quoted where necessary. Difficulties such as this are never glossed over although they may be relegated to a note at the end of a proof. There are good chapters on series solutions and Fourier series in this section. The last section is concerned with Laplace's equation and related equations; these are treated briefly but quite adequately for the level at which the book is written.

In common with the other volumes in this series arguments are given in great detail and there are plenty of illustrations and examples. The printing and lay-out of the book is to be commended highly.

J. E. ROBERTS.

General Homogeneous Coordinates in Space of Three Dimensions. By E. A. MAXWELL. xiv, 169 pp. (Cambridge University Press, 1959.) Paper Edition, 13s. 6d.

As this text-book first appeared eight years ago, it is already well-known. This new edition, apart from being paper-backed instead of bound, is identical with earlier copies. But as the paper-back brings down the price nine shillings, it will be interesting to see whether this trend finds favour among undergraduates. It is said that some have bought this book solely because of its concluding chapter on the use of the vector and matrix notation in geometry! But the preceding chapters too are worth reading. Both the presentation and the printing maintain the high standards of the author and the Cambridge University Press.

M. F.

The Two Cultures and the Scientific Revolution. By C. P. SNOW. 52 pp. (Cambridge University Press, 1959.) $7\frac{1}{2}$ in. 3s. 6d.

This penetrating and thought-provoking essay was presented as the 1959 Rede Lecture in Cambridge. The author begins by sketching the two cultures, scientific and "intellectual," separated by a great gulf. Each culture loses greatly by its lack of contact with the other. The intellectuals have not yet begun to understand even the old industrial revolution, much less the new scientific revolution. It is essential to bridge the gap between them. Living, as we do, in a densely populated island with few natural resources our real assets are our wits. But unless we educate ourselves to the best of our ability we may watch a steep decline in our fortunes within our lifetime, as the Venetian Republic declined in the past. This is our local problem, but the Venetian shadow may fall over the entire West unless we realize the great need to bring about the scientific revolution throughout the world. He ends "For the sake of intellectual life, for the sake of this country's special danger, for the sake of western society living precariously rich among the poor, for the sake of the poor who needn't be poor if there is intelligence in the world, it is obligatory for us . . . to look at our education with fresh eyes." The careful argument of the essay is well-supported by reasons.

M. F.