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Eureka Editor

archim-eureka@srcf.net

The Archimedean

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Contents

Editorial	3
The Archimedean	3
Contributions	4
Postal Subscriptions and Back Numbers	4
The Theory of Games	5
A Theorem on Prime Powers	10
An Elevation Puzzle	11
The Problems Drive, 1956	12
The Mathematical Association	14
Euclid's Elements	15
Letter to the Editor	17
A Problem on Grills	18
Solutions to the Problems Drive	19
Solution to the Elevation Puzzle	19
Book Reviews	21

Contributions and other communications should be addressed to:

The Editor, "Eureka,"

Arts School,

Bene't Street,

Cambridge,

England.

Editorial

LAST year's President of the Archimedean, in his report published in this issue, writes of "the delays usually associated with *Eureka*." These delays have been even worse than usual this year, mainly as a result of the lack of suitable material at the time when we should normally be going to press. We have therefore to repeat, with ever increasing emphasis, the appeal for contributions which has so often appeared in these columns. To speak bluntly, the Archimedean cannot expect to have a Journal if they do not write for it. A feature of the present issue is the complete lack of poetical contributions, so we would remind you that, whilst we appreciate articles of serious mathematical content, contributions in a lighter vein are also acceptable.

Having made these comments it is only right that we should look through the pages and thank those who have helped to fill them. We are particularly grateful to Michael Hoskin who readily volunteered an article, to John Leech, who has done a great deal for *Eureka* in the past, to Terence Wall who has also written for us before, and to our reviewers and other authors. We must not forget the printers, nor our advertisers. The Editor, too, has a personal debt to the Business Manager, Ian Porteous, for his continual help and encouragement. Finally we thank you, the subscribers, for your support and for the appreciative letters which we receive from time to time from many diverse countries of the world.

The Archimedean

IT is difficult to write any report on the activities of the Archimedean without allowing it to become a mere list of lecturers and titles. It is not what is standard, but what has changed which is of importance. True, we had some interesting men with some very interesting things to say; these, with the Tea Meetings and the Christmas Party, formed the backbone of our syllabus.

One fresh departure made in the last year or so has been the introduction of discussions on various topics. In the Easter Term of 1955, Professor McCrea and Mr. Hoyle dealt with Cosmology, and in the same term of 1956 Professor Spencer Brown defended his views on Statistics against the animated criticism of Mr. Anscombe. These discussions were much appreciated, and we hope that there will be others like them.

Visits to Harwell and the National Physical Laboratory were arranged and were much enjoyed by all who went. Harwell comes dangerously near to entering the category of the first paragraph;

all to the good: the writer remembers a number of people whose ideas on a career were crystallised after one of these visits—some feeling inclined to take up residence at Harwell or some equivalent spot, others firmly declining to touch the place with a barge pole.

As retiring President, I wish to express my thanks to a Committee which was consistently reliable over its period of office, and to wish—belatedly perhaps, in view of the delays usually associated with *Eureka*—the new President and Committee, not only success in their routine responsibilities, but enterprise in undertaking further new departures.

JOHN L. MARTIN.

Contributions

No prize was awarded last year as no entry of sufficient merit was entered before the closing date. This year prizes will not be offered, but undergraduates and research students will be paid by book token for all articles, poetry, problems, etc., published in *Eureka* 20, at the rate of 10/- per printed page. Contributions should be sent to the Editor of *Eureka*, The Arts School, Bene't Street, Cambridge, as soon as possible and in any case not later than the end of July.

Postal Subscriptions and Back Numbers

FOR the benefit of persons not resident in Cambridge, we have a postal subscription service. You can enrol as a permanent subscriber and either pay for each issue on receipt or, by advancing 10s. or more, receive future issues as published at 25% discount, with notification when the credit has expired.

Copies of *Eureka* Numbers 11 to 16 (6d. each), Number 17 (1s.) and Number 18 (2s.) are still available. (Postage 2d. extra on each copy.) We should also like to buy back any old copies of Numbers 1 to 10 which are no longer required.

Cheques, postal orders, etc., should be made payable to "The Business Manager, *Eureka*," and addressed c/o The Arts School, Bene't Street, Cambridge.

The Theory of Games

by

MICHAEL HOSKIN

THE theory of games was discussed for the first time by John von Neumann* in a lecture he gave in 1928 to the mathematical society at Gottingen. His lecture was followed by sixteen years of almost complete silence, but by no means complete inactivity, for in 1944 there appeared a volume of over six hundred pages entitled *Theory of Games and Economic Behaviour*. This work, which von Neumann wrote jointly with the economist Oscar Morgenstern, is already a classic; it contains an extensively developed mathematical theory, such as usually appears only after the publication of numerous research papers, and, more surprising still, there is a discussion of the application of the theory to economics. The publication of this book has been followed by a steady stream of literature, some of it strictly mathematical, some dealing with applications of the theory to economics and military tactics for example, and some of it set in a wider context, as in Professor Braithwaite's inaugural lecture and in a recent work of Père Dubarle. In fact, games theory is becoming a subject which one's vis-à-vis at dinner may introduce into the conversation, and the following notes have been written for those Archimedean who take their gamesmanship seriously.

A *game* is a set of rules within the framework of which the participants or *players* make their choices or *moves*. An actual instance of the rules being put into practice is spoken of as a *play*, so that a chess tournament, for example, consists of several plays of the same game. The characteristic feature of a game is that the moves made by any one player influence the outcome but do not completely determine it.

If a player decides in advance precisely the move he will make in every situation in which he may find himself, he is said to have adopted a particular *pure strategy*. The theory of games assumes that safety first is the consideration uppermost in everybody's mind, and on this assumption it examines the possible pure strategies which each player may adopt and helps him to make his choice.

An important section of the rules will determine the *information* about previous moves which a player has when making a move. In a game like chess the information is *perfect*, but in other games a player may have only a limited knowledge of the moves made by

* We regret to record the death of John von Neumann on February 8th, 1957, at the early age of 53.

the other players; and he may even be uncertain of his own previous moves, as for example when the term "player" is used to denote several persons who have joined forces. The possible situations in which a player may find himself can be divided into *information sets*, such that the player has enough information to avoid confusing a situation in one set with a situation in some other set, but insufficient information to distinguish between situations in the same set. Strategies must prescribe the same move for each of the situations of a given information set, and so a strategy is a function over the class of a player's information sets.

If the game does not involve chance elements, the outcome of each play is determined completely by the moves made by the players, so that once each player has picked his strategy the result may (in principle) be computed automatically. Even when there are chance elements, the play may be left to a computer equipped with random numbers. A game can be reduced in this way to its *normalised form*, in which player makes a single move (the choice of a strategy) without knowing the moves (or strategies) chosen by his opponents.

At the end of each play the players receive an amount, positive, negative or zero, called the *pay-off*. If there are no chance elements in the game, the pay-offs are functions of the strategies chosen. But if there are chance elements, the amounts which are important to the game theorists are the *expected pay-offs*, which again depend on the strategies chosen. When for all possible choices of strategies the sum of the pay-off functions is zero—that is, when on the average the sum of the payments made is zero—the game is said to be *zero-sum*. Non-zero-sum games are hard to theorise about. What, for example, is one to make of the (admittedly degenerate) game between A and B which consists of a single move by B, such that if B chooses the first alternative he receives £1 and A pays £1, while if B chooses the second alternative he receives £2 but A pays £5?

It is also hard to theorise about games which involve more than two players, as the possibility of temporary alliances and coalitions has to be taken into account. Consequently, much of the existing theory is concerned with *two-person zero-sum* games. Since in such a game B's loss is A's gain we can construct a pay-off table in the form of a matrix, the rows corresponding to A's strategies and the columns to B's, and the (i, j)-th element representing the pay-off to A if he chooses his i-th strategy and B chooses his j-th strategy. To decide what to do, A, who takes a gloomy view of his prospects, examines the matrix row by row. His object is to take out a gilt-edged policy, so he looks to see what is the minimum he can receive on each strategy. He then picks out the maximum of these minima, and he knows that by making the appropriate choice he will be sure

of receiving at least this amount. At the same time B, equally pessimistic, picks out the maximum in each column, and looks for the minimum of these maxima, for this is the smallest payment he can be sure of getting away with. If these two amounts are equal, then the game has one or more *saddle points* and A and B should (unless they have bribed the other's mathematical adviser) choose strategies which will lead to a saddle point. If they do not then they may receive a smaller pay-off than that indicated by the saddle point, which we call the *value* of the game. The following example has two saddle points, which are indicated by asterisks. A should choose his third strategy, but B may choose at random between his first and third. We notice that the element at each saddle point is the minimum in its row and maximum in its column.

		B		
		5	7	5
Col. max.				
Row min.				
A	-1	-1	0	2
	-3	3	-3	4
	5	5*	7	5*

Now although von Neumann and Morgenstern have shown that the normal form of every two-person zero-sum game with perfect information has a saddle point, so that there are optimal strategies for playing games such as chess and draughts, the same is not always true of games without perfect information. For example, the matrix

		B	
		Head	Tail
A	Head	1	-1
	Tail	-1	1

associated with the game of matching pennies has no saddle point. In this case, by choosing a pure strategy, A cannot be sure of receiving more than -1 , and similarly B cannot be certain of paying less than 1 . But if we allow A and B to make their choices of strategy *at random*, selecting either strategy with (in this example) probability $\frac{1}{2}$, then from the resulting *mixed* strategies A and B may expect an average pay-off of zero, the value of the game.

This is an example of a general result. Suppose A and B form their mixed strategies by choosing their pure strategies with probabilities (x_i) and (y_j) respectively, and let us denote the resulting expected pay-off by $E(x, y)$. Then the fundamental "minimax"

theorem, proved by von Neumann in 1928, says that there exist optimal strategies with probabilities x^* , y^* such that

$$E(x, y^*) \leq E(x^*, y^*) \leq E(x^*, y).$$

In other words, A may expect at least $E(x^*, y^*)$ by choosing $x = x^*$, and B may expect to pay no more than $E(x^*, y^*)$ by choosing $y = y^*$; and these strategies are doubly safe, for if they are discovered by the opponent no harm results. So, the argument runs, A and B ought to content themselves with using their optimal strategies. But it is now time we looked at an example.

Lettice Hagelbarger, of Old Hall, is a friend of Jeremy ffoulkes-Fortescue, of Christhouse. Both are reading mathematics, but whereas she perseveres in attending lectures, he much prefers to sit in his rooms with a book. Now it so happens that he took her out to dinner the other night, but when the time of reckoning arrived he found that he had left his wallet at home, and Lettice had to rescue him with the loan of a pound. Just at present he would prefer not to have to repay this debt, and so he is for the time being avoiding Lettice's company. He is also a little jealous of Lettice's recent success in the Tripos and, though he would not admit it, he is secretly pleased when she has to miss a lecture.

Jeremy, on the morning in question, has to decide between staying in his rooms, going to coffee, and attending a lecture. Lettice likewise can attend the lecture or, since she is keen to have her pound back, search for Jeremy in his rooms or in the Espresso bar. Both have read articles on games theory, and as they always agree in estimating pay-offs we need only look at the situation from Jeremy's point of view. He begins constructing the matrix by considering his first choice, that of staying in his rooms. If Lettice calls, the pound goes but Lettice has missed a lecture and he can do a morning's work: reckon it at -3 . If Lettice goes to coffee, he does his work, keeps the pound, and she misses the lecture, say $+5$. If she goes to the lecture, both work but he keeps the pound: say $+3$.

Jeremy now considers going out to coffee. If she calls at his rooms, he keeps the pound and both miss work, say $+3$. If she finds him in the Espresso bar, both miss work, but the presence of mutual friends may enable him to escape with the pound: say -1 . If she goes to her lecture, he keeps the pound, but the balance of work goes against him: say $+1$ only.

Finally he contemplates the possibility of attending the lecture. If she calls for him, he keeps the pound, and she misses her lecture. The same is true if she tries the coffee bar, but then she need have no hesitation in calling on him later in the day, and so Jeremy rates the former at $+3$ and the latter at -1 . Finally, if both go to the lecture the pound is lost and the balance of work goes against him; say -7 .

The pay-off matrix is therefore:

		Lettice Hagelbarger		
		Calls	Coffee	Lecture
Jeremy ffoulkes- Fortescue	Rooms	-3	5	3
	Coffee	3	-1	1
	Lecture	3	-1	-7

The reader, confident that neither Jeremy nor Lettice would have stopped to solve this game had the matrix not been vulnerable to attack by gimmick, and having failed to find a saddle point, will notice that Jeremy may always choose coffee in preference to the lecture, since the figures in row 2 are greater than or equal to those in row 3. Jeremy can accordingly strike out row 3. We next observe that the elements of the third column are not less than the average of the elements of the first two columns, and so Lettice, who prefers as small pay-off as possible, can be relied on to distribute between her first and second choices any probability she may originally have thought of assigning to her third choice. So Jeremy can dispense with the third column, and he is left with the matrix:

		Lettice Hagelbarger	
		Calls	Coffee
Jeremy ffoulkes- Fortescue	Rooms	-3	5
	Coffee	3	-1

(We ought to remark that cancellation of rows and columns such as this may sometimes cause optimal strategies to be overlooked; for example, optimal strategies for the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

have the form $(1 - 2a, a, a)$ for both players, but if the matrix is reduced to

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

then the only optimal strategies to be found may be $(0, \frac{1}{2}, \frac{1}{2})$ for each player.)

It is now time for Jeremy to invoke the minimax theorem. He uses $x, 1 - x$ to denote his own probabilities for Rooms and Coffee respectively, and y and $1 - y$ for Lettice's probabilities for Calls and Coffee. Then

$$E(x, y) = -3xy + 5x(1 - y) + 3(1 - x)y - (1 - x)(1 - y) \\ = -12(x - \frac{1}{3})(y - \frac{1}{2}) + 1.$$

Jeremy's optimal strategy is to choose $x = \frac{1}{3}$, and Lettice's is to choose $y = \frac{1}{2}$. By doing this they can ensure an expected pay-off of 1 (the game is slightly unfair to Lettice). But to depart from the optimal strategy would be to run a risk of incurring an unnecessary loss. Returning to the original matrix, we see that an optimal strategy for Jeremy is $(\frac{1}{3}, \frac{2}{3}, 0)$ and for Lettice $(\frac{1}{2}, \frac{1}{2}, 0)$.

An easy exercise for the reader:

Find optimal strategies for A and B in the game whose matrix is

	B			
A	3	2	4	0
	3	4	2	4
	4	2	4	0
	0	4	0	8

Solution: $(0, 0, \frac{2}{3}, \frac{1}{3})$ for both A and B.

A Theorem on Prime Powers

by

C. T. C. WALL

RECENTLY it occurred to me to ask whether two prime powers can differ by unity. The brevity of the proof and the piquancy of the result encourage me to give them here.

Since one of the prime powers is even it must be a power of 2, so I write

$$p^n = 2^k \pm 1,$$

where p is an odd prime.

Suppose n is odd. Then in the two cases,

$$2^k = (p - 1)(p^{n-1} + \dots + p^2 + p + 1)$$

or
$$2^k = (p + 1)(p^{n-1} - \dots + p^2 - p + 1),$$

and the second factor in each expression contains an odd number of terms, and so is odd; it is clearly greater than 1 in each case, if $n > 1$. This is impossible.

Therefore, if $n > 1$, n is even, let us say $n = 2m$.

Then if $2^k = p^{2m} + 1$

write $p^m = 2x + 1$.

We have $2^k = (2x + 1)^2 + 1 = 4x^2 + 4x + 2$

and so is not divisible by 4. Thus $k = 1$, giving $n = 0$, which is trivial.

If $2^k = p^{2m} - 1 = 4x^2 + 4x$

we have $2^{k-2} = x(x + 1)$

and so $x = 1$, giving $p = 3$, $m = 1$, $k = 3$ and the solution

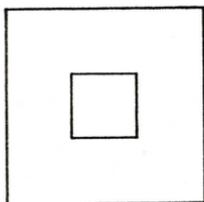
$$8 + 1 = 9,$$

which is the unique solution if we exclude Mersenne primes and Fermat primes, for which $n = 1$.

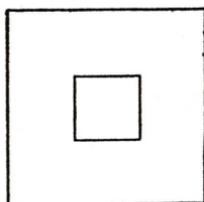
An Elevation Puzzle

by
ANTILOG

BELOW are shown the plan and front elevation of a solid figure. (All hidden edges in such drawings are represented by dotted lines.) The problem is to find what the side elevation looks like. There are no curved surfaces on the figure.



Plan



Front elevation

(For solution see page 19.)

The Problems Drive, 1956

Competitors work in pairs, and are allowed five minutes for each question. The winning pair have the privilege of setting the problems for the next year.

I.

1	2	3
4		
5		

Clues.

Across.

Down.

- | | |
|------------------|--------------------|
| 1. Prime number. | 1. Perfect cube. |
| 4. Prime number. | 2. Perfect square. |
| 5. Perfect cube. | 3. Perfect square. |

- A clock gains slightly. At midnight December 31/January 1 last it showed the correct time. At 1.5 a.m. on January 1 the hands of the clock were exactly coincident, pointing to slightly after 1.5. At what time on what date will the clock next show the correct time, assuming that it continues to gain at the same rate?
- Two new straight roads, both the same length, less than 20 miles, now connect Hayford with Beeford and Hayford with Seaford. Formerly it was necessary to travel due North for 16 miles and then turn due East (for an integral number of miles) to reach Beeford. The journey from Hayford to Seaford entailed going 19 miles due North and then due West for an integral number of miles. How far apart are Beeford and Seaford, using only the roads mentioned?
- In the following addition sum the letters represent digits in the scale of 4.

$$\begin{array}{r}
 \text{EAT} \\
 \text{TEA} \\
 \text{EAST} \\
 \hline
 1300 \\
 \hline
 \text{TASTE} \\
 \hline
 \end{array}$$

5. In an election there were three candidates, A, B and C. At the previous election only A and B stood. The electorate of 100 all voted both times. B gained the same number of votes in both elections. This time A had twice as many votes as C. The numbers of votes gained by A and C from B between elections were the same. 36 of the electorate voted for A on both occasions. The number of people who voted for B both times is one more than A's vote at this election.

Find B's majority over A at this election.

6. Give the next member of each of the following sequences:—

(a) 3, 4, 6, 8, 12, 14, 18, . . .

(b) 1, 3, 4, 13, 53, . . .

(c) 1, 1, 3, 6, 15, 36, 91, . . .

7. Show that the sequence of numbers defined by

$$[k + \sqrt{k} + \frac{1}{2}], \quad k = 1, 2, 3, \dots$$

i.e.

$$2, 3, 5, 6, 7, 8, \dots$$

includes all prime numbers. $[]$ denotes "integral part of" and 1 is not regarded as a prime number.

8. Find an integral non-trivial solution of the equation:

$$x^2 + y^2 = 10z^2,$$

where

$$x \neq y \neq z.$$

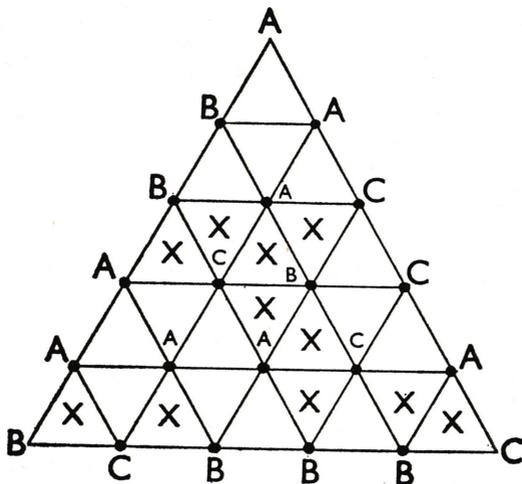
9. In a set at tennis, A beat B 6-3. 5 games went against service. Who served first?

10. On a circular railway line, trains travel non-stop in both directions (apart from a short stay at the terminus). Those starting in an easterly direction take two hours and those starting in a westerly direction take three hours. Two trains start simultaneously, one in each direction, every fifteen minutes. How many trains travelling in the opposite direction can be seen by a passenger in each train, starting to count after leaving the station?

11. Construct a square circumscribed to a given quadrilateral, i.e., such that each side of the square passes through a vertex of the quadrilateral.

12. A triangle ABC is divided into smaller triangles by drawing n lines parallel to the sides, as in the figure ($n = 4$). The points of the figure are then lettered as follows: A, B, C

retain their lettering; points on BC may be lettered B or C, but not A, and similarly for CA and AB. Interior points may be lettered A, B or C indiscriminately. Prove that the number of small triangles with three different letters at their vertices is odd (11 in the figure).



(For solutions see page 19.)

Mathematical Association

President: PROFESSOR G. F. J. TEMPLE, C.B.E., F.R.S.

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 21s. per annum: to encourage students, and those who have recently completed their training, the rules of the Association provide for junior membership for two years at an annual subscription of 10s. 6d. Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published four times a year and deals with mathematical topics of general interest.

Euclid's Elements

SOME years ago, well before the present American reprint was issued, I had the good fortune to be able to buy Sir Thomas Heath's edition of the Elements of Euclid. Prior to then, I had only seen old school editions of Euclid, and had thought of it as a rather dull presentation of elementary geometry with which our fathers were cursed at school but which had now been superseded and forgotten. Forgotten, perhaps. Superseded, never! I had to read Heath's translation and notes to realise just how much of the dullness was the work of later editors rewriting for school use. Because Euclid is not a school book. It was written for mature men, of a race whose maturity was exceptional. It was taught at universities over two thousand years after it was written, and was only taught in lower schools since the early part of the nineteenth century. Its abstract aloofness is not appropriate to schools, and it has now quite rightly been superseded in school use. As it is no longer taught in universities, no one now learns Euclid. More is the pity: I often wonder whether the Tripos would not be better for a section on the History and Philosophy of Mathematics, comparable to that in the Natural Sciences Tripos.* For no mathematician's education can be called complete without some such study, and Euclid's Elements must without doubt be by far the most important single work of mathematics ever written.

It will be in order to mention a few of the more remarkable features of Euclid's work. His Fifth Postulate—the Parallel Axiom—asserts “that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.” In introducing this as an axiom, Euclid avoided a *petitio principii* of which Aristotle had complained in the treatment of previous writers. (Unfortunately we know very little of the work of previous writers—it was so completely displaced by Euclid that it didn't survive.) Euclid no doubt found that he had to make some such postulate, as he could not prove the assertion. For over two thousand years, mathematicians tried to prove the postulate—some of them still do—but only during the last century did Riemann exhibit that it could not be proved, that Euclid was correct in making it an axiom.

Or again, we may look at Books Seven to Nine of the Elements, which aren't even geometry but are what we now call Theory of

* It might have saved a recent Tripos lecturer having to apologise for a difficult geometrical result—that a cube can be inscribed in a dodecahedron—instead of quoting it as a “well-known theorem of Euclid”!

Numbers. These books culminate in a method for constructing Perfect Numbers (numbers which are the sum of their divisors, such as $28 = 1 + 2 + 4 + 7 + 14$). He is handicapped by the lack of a good working notation, but having in the previous proposition summed a geometric progression, he proves that if we sum a series in double proportion (geometric progression with ratio 2) beginning with unity and going on until the sum is prime, and multiply this sum by the last term added in, we obtain a perfect number. In modern notation, $(1 + 2 + 2^2 + \dots + 2^{n-1})2^{n-1} = (2^n - 1)2^{n-1}$ is perfect whenever the first term is prime. He does not know whether the perfect numbers are finite or infinite in number, but he has shown that there may be infinitely many as there are certainly infinitely many primes (Book Nine, Prop. 20—the well-known proof by multiplying together any finite set and adding 1, thereby constructing a number which is divisible by, or is itself, a prime not in the finite set). Whether there are infinitely many perfect numbers is to this day unknown. Whether there are any perfect numbers not given by Euclid's construction is unknown. All that is known is that Euclid's construction gives all the even perfect numbers; the existence of odd ones is a notorious unsolved problem.

A study of Book Five is rewarding as this book records the work of Eudoxus on the theory of real numbers. The Greeks—probably Pythagoras—had discovered that not all ratios of lengths of lines in their geometrical figures were ratios of integers; for instance, the side and diagonal of a square were not in the ratio of two integers. The arithmetic of integers was therefore insufficient to deal with all their requirements and a more general theory was required. Eudoxus based it on the definition of equal ratios which would today be stated thus: Two ratios $x:y$ and $z:t$ are equal whenever, for all pairs of integers m, n , the same symbol of equality or inequality obtains between nx and my as between nz and mt . In other words, two ratios are equal whenever they make exactly the same partition of the rational numbers m/n into those greater than, equal to, or less than, the given ratio. This corresponds exactly to Dedekind's definition of real numbers.

Euclid, not surprisingly perhaps, is not an infallible oracle. Even his first proposition will not hold water in the light of modern ideas of rigour. He requires to construct an equilateral triangle on a given line AB, and proceeds by drawing circles with centre A radius AB and centre B radius BA, and then joining A and B to a point of intersection of these circles. But it needs some sort of an axiom of continuity to prove that the circles intersect, and he has no such axiom. Or again, consider his fourth proposition, that if two triangles have two sides of one equal to the respective sides of the

other, and the included angles equal, then the third sides will also be equal. His proof is to pick one of the triangles up and place it on the other; the third sides then coincide and so are equal. But, an objector may ask, why can't the length of the third side change *en route*? It would if the triangles were made up of geodesics on a rugger ball! The assumption that it does not is in fact the result to be proved. Modern practice is to take this result as an axiom, it being acknowledged that Euclid's proof is worthless. In defence of Euclid, it appears that he didn't like the method of proof, as elsewhere he refuses to use it, preferring other, often longer, proofs.

But these reservations are insignificant blemishes on what must stand for all time as the most outstanding monument in the whole literature of mathematics. To quote Heath's preface, "No mathematician worthy of the name can afford not to know Euclid," and though the current price of Heath's edition may mean that some can't afford to buy it, let them borrow it and read it. Then perhaps they will afford to buy it; certainly they will benefit by having read it.

J. LEECH.

Letter to the Editor

DEAR SIR,—The article in the last number of *Eureka* by Mr. Roberts on the bias of dice was very welcome, being to my knowledge the first to treat this subject quantitatively. It is unfortunate that his experiments were not sufficient to provide any significant test of his theory, but we have two sets of experiments carried out by Prof. Weldon at the turn of the century of much greater magnitude which will allow a critical test.

We introduce two assumptions. We first assume that any bias due to the geometry of the dice will cancel out over the number of dice (twelve and eighteen) used by Prof. Weldon. Mr. Roberts' data on the six he examined shows that this is reasonable. Secondly we assume that the displacement of the centre of gravity due to the pits in the faces was the same for Prof. Weldon's dice as for those used by Mr. Roberts, and we take, using his notation, $(a + b) = 15.1 \times 10^{-3}$, $(a + b + c) = 17.25 \times 10^{-3}$.

We adopt the following notation:

$P(p|H)$ is the probability of the proposition p on the hypothesis H .
 H_0 denotes the hypothesis that the dice were unbiased.

H_1 denotes the hypothesis that the dice were biased in accordance with Mr. Roberts' formula.

P is the probability that a given value of χ^2 is reached or exceeded in random sampling.

The first of Prof. Weldon's experiments is quoted by Pearson in the paper in which he discovered and developed the χ^2 test (I have been unable to find any writings by Prof. Weldon himself on this subject).

Twelve dice were thrown 26,306 times, and the frequency of the total number of 5's and 6's in each throw was tabulated. Using $P(5 \text{ or } 6|H_0) = \frac{1}{3}$ we find after some grouping that $\chi^2 = 35.4$ on 9 d.f., giving $P = 0.000,05$, so that the chance that the dice were unbiased is about 1 in 20,000. However, using $P(5 \text{ or } 6|H_1) = 0.337,258$, we get $\chi^2 = 8.5$ on 9 d.f., and P is approximately $\frac{1}{2}$. In other words, while if the dice were not biased we should expect the observed deviation from the theoretical values about once in twenty thousand times, if Mr. Roberts' values are accepted the observed deviation should occur on every other occasion.

The second experiment, quoted by Edgeworth, is more unsatisfactory on several points, although the evidence it provides is little less conclusive than the first. Three sets of six dice were thrown 4,096 times, and the frequencies of the total number of 4's, 5's and 6's in the first two sets and in the first and third were plotted in a 13×13 table. This, while giving a smaller number of throws than in the previous experiment, required the calculation of about seventy frequencies and residuals. Taking $P(4, 5 \text{ or } 6|H_0) = \frac{1}{2}$, and again after some grouping, we obtain $\chi^2 = 67.5$ on 36 d.f., giving $P = 0.001,2$, and taking now $P(4, 5 \text{ or } 6|H_1) = 0.504,483$ we obtain $\chi^2 = 49.3$ on 36 d.f., leading to $P = 0.067$.

Thus we see that in the two experiments taken together there is overwhelming evidence that the dice were biased, and that the use of the corrections suggested by Mr. Roberts to allow for the displacement of the centre of gravity of the dice due to pits in their faces is completely adequate to explain the observed results.

Yours faithfully,

J. R. PROBERT-JONES.

Trinity College,
Cambridge.

REFERENCES

- Edgeworth, R. Y., *Encyclopedia Britannica*, 11th Ed. (1911), Vol. 22, p. 393.
 Pearson, K., *Phil. Mag.*, **50** (1900), p. 167.
 Roberts, J. D., *Eureka* 18 (November, 1955), p. 8.

A Problem on Grills

A determinant is called a grill if each element is either 1 or 0. Prove that the value of an $n \times n$ grill cannot exceed

$$2\left[\frac{1}{4}(n+1)\right]^{\frac{1}{2}(n+1)}.$$

Construct grills which have this value when $n+1$ is a power of 2.

The solution to this problem will be published in our next issue.

G. H.

Solutions to the Problems Drive

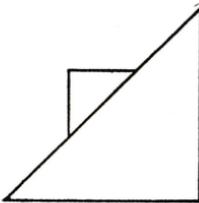
¹ 5	² 4	³ 1
⁴ 1	4	9
⁵ 2	1	6

1.

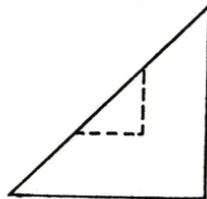
2. Noon on March 12. (Note that this problems drive took place in Leap Year.)
3. 18 miles.
4. $A = 0, T = 1, E = 2, S = 3.$
5. 5.
6. (a) 20 (Primes + 1).
(b) $690 (u_r + 1 = u_r u_{r-1} + 1).$
(c) 231 ($\frac{1}{2}(u_r + 1)u_r$, where u_r is Fibonacci's sequence 1, 1, 2, 3, 5, 8, 13, 21 . . .
7. The sequence omits perfect squares only.
8. 13, 9, 5.
9. A.
10. 20.
11. If ABCD is the quadrilateral, draw AE perpendicular to BD and equal to it in length. Then CE is one side of the required square, which may easily be completed by drawing perpendiculars.
12. Number each line segment in the figure 0 or 1, according as the letters at its ends are the same or different. The three sides of a small triangle lettered ABC then sum to 3; other triangles give 0 or 2. The sum along each edge of the large triangle must be odd. Sum over all the small triangles in the figure; interior segments are counted twice, so the answer is odd. Hence there must be an odd number of triangles numbered 3.

Solution to the Elevation Puzzle

(See page 11)



or



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Book Reviews

Integral Functions. M. L. CARTWRIGHT. (Cambridge Mathematical Tracts.) 18s.

The tract gives an account of certain aspects of the theory of integral functions with particular reference to those functions $f(z)$ which are of finite order. The Phragmén-Lindelöf indicator function $h_r(\theta)$ is used extensively as a means of examining the behaviour of $f(z)$. Its fundamental properties are established early and there follow theorems concerned with the minimum modulus and distribution of zeros of $f(z)$. Further applications are made later in connection with the Pólya theory of functions of exponential type.

Much of the theory is developed in full generality by introducing proximate orders in the definition of the function $h_r(\theta)$ and establishing general theorems for functions regular in an angle. In this way the importance of the function $h_r(\theta)$ is emphasised and its extreme properties are well illustrated by a number of special functions which are considered in the text.

Although lack of space has prevented the author delving too deeply into the theory of exceptional values, a chapter concerned mainly with lines of Julia is included. Existence theorems are proved and a brief survey is given of the corresponding theorems associated with directions of Borel.

Essentially the tract presents a classical approach to the theory of integral functions; primarily for the advanced reader, it covers techniques which the author and others have found so fruitful in this field.

C. N. L.

Theory of Games as a Tool for the Moral Philosopher. R. B. BRAITHWAITE. (Cambridge University Press.) 6s.

This inaugural lecture by the Knightbridge Professor of Moral Philosophy at Cambridge deals with the problem of how two people with possibly conflicting interests should behave so that those interests should be met as far as possible. Their most appropriate courses of action are not in every case clear. The answer is obvious if the situation is wholly *co-operative*; in a wholly *competitive* situation, von Neumann's Theory of Games provides a complete solution. For these cases there is nothing further to say. It is the intermediate case—that in which the competition is only partial—which is considered in this book.

Professor Braithwaite's procedure is to analyse the partly competitive situation into two parts, one wholly competitive and one wholly co-operative; the solution of each part is known by existing theory, and we are thus led to an answer to the given problem. However, there is a great deal of scope allowed in the details of such a procedure, and the author has made his method unique by certain simple assumptions of linearity. A remarkable parallel procedure is discovered in projective geometry; in particular, the geometry of the parabola provides a useful representation of the situation. Most of the presentation is consequently in geometrical terms and gains in lucidity thereby.

On the whole this is an interesting and lightheartedly written examination of a rather intriguing problem, one which the reviewer enjoyed reading.

J. L. M.

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