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EUREKA

THE JOURNAL OF THE ARCHIMEDEANS

(The Cambridge University Mathematical Society: Junior Branch of the Mathematical Association)

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Editorial

THIS is our first issue for over three years which has not been edited by G. C. Shephard, of Queens'. We record here his great services to EUREKA and to the Archimedeans generally, and we hope to maintain the high standard he has set.

A glance through our back issues shows that on several occasions a hope is expressed that EUREKA might be published twice a year. The only result has been a slow precession towards October in the date of publication, and the impact of a considerable mass of contributions will be necessary to carry it forward to May. However, the next issue will appear as soon as we have sufficient material.

Contributions have only to be capable of entertaining the average undergraduate mathematician. Contributors are not limited in any way: we would even accept articles from Oxford. What more can we say?

The Archimedeans

THE year 1949–50 must certainly rank as one of the most successful in the life of the Archimedeans, particularly from the financial point of view (which would naturally concern a Scottish president). Although the committee consisted largely of B.A.s, it was able to conclude successfully most of the work which it undertook.

The six evening lectures were delivered by Professor J. B. S. Haldane, Professor J. E. Littlewood, Professor H. R. Hassé, Professor H. Davenport, Professor C. A. Coulson, and Mr. Brooke Crutchley. Of the other two evening meetings, one was the usual Problems Drive, and the other a Symposium on Telepathy. All these were well attended. Following our recent custom, the four tea-time lectures were given by research students. The social activities of the Society included a Christmas Party and a Dance. Both were very successful from every point of view. In addition there were several picnics. During the year an amplifier was built for the Society by Messrs. Haselgrove and Stringer. This was used at the Christmas Party and at the weekly meetings of the Music Group.

At long last a tie has been adopted for the Society; the design consists of Archimedean spirals with εὕρηκα between them. A new constitution was produced during the year; while following the spirit of the old one, it removed certain out-of-date features.

To the committee and all those who have in any way assisted with the above and the many other activities of the Society, I extend my thanks; to my successor and the new committee I wish every success.

J. H.

Greek Metamathematics

By N. A. Routledge

Achilles and the Tortoise—a Consideration

How often does the deep simplicity and insight of country folk confound the sophistries of the over-educated! It was round about 450 B.C. that a self-taught philosopher pointed out four paradoxes to the frequenters of the academies in the little Greek colony of Elea. Since then everyone with the slightest interest in philosophy or mathematics has been troubled by them and has developed highly ingenious theories for resolving them. But ingenuity has not been accompanied by conclusiveness: indeed the paradoxes "have probably occasioned more inconclusive disputation than any equal amount of disguised mathematics in history."

Three of Zeno's posers have been dealt with quite reasonably, but the other, concerning Achilles and the Tortoise, has never received a treatment that did not leave at the back of the mind a nasty feeling that the solver has been a little *too* clever. The argument is:

Achilles runs ten times as fast as the Tortoise.
He gives it a start of 100 yards.
When he has run this the Tortoise is 10 yards ahead.
When he has run this 10 yards the Tortoise is 1 yard ahead.
When he has run this 1 yard the Tortoise is 1/10 yard ahead.
Etc.

(*a*) Thus, according to this argument, Achilles never overtakes the Tortoise.

(*b*) Whereas, of course, we know that he does.

The usual way of treating this is to say that

$$100 + 10 + 1 + 1/10 + 1/100 + \ldots = 111\tfrac{1}{9}$$

and that Achilles overtakes the Tortoise after going $111\frac{1}{9}$ yards, but one cannot sum an infinite series in a finite length of time. This sounds, on a first hearing, as if it disposes of the contradiction, but the more one looks at it the less it seems to do so, and if one tries to rewrite Zeno's proof, one sees that the remarks have no bearing on the problem at all.

The complete solution introduces metamathematics—arguments about arguments. Metamathematics has been developed almost entirely in the last 50 years or so, and has yielded many startling and important results. The most comforting is the solution of Hilbert's *Entscheidungsproblem*: it has been demonstrated that no machine can deal with all mathematics. The most tiresome results concern unsolvability: one takes a logical system and shows that a certain statement in it cannot be shown to be true and cannot be shown to be false. Zeno's paradox in its correct form is precisely of this kind.

To resolve the paradox we merely alter the statement (*a*) to:

Thus, no argument of this kind, however long we continue it, will ever lead us to the conclusion that Achilles overtakes the Tortoise.

Does this now conflict with the statement (*b*)? Only if one argues:

Since our argument cannot show that Achilles overtakes the Tortoise, it must be true that he does not.

We can only assert this if we are sure that every statement can be demonstrated true or else shown false by such arguments. But this is not so, for these arguments allow us to say nothing concerning, for example, where the Tortoise is when Achilles has gone 112 yards.

The paradox has thus vanished.

We can introduce a formal logical system:

The symbols used will be R, (,) , ; , 0. I shall use the abbreviations:

1 for 00
2 for 000
3 for 0000
. . .
n for $n + 1$ zeros in a row.

I have just one axiom: $R(0 ; 1)$
and just one rule of proof:

From $R(n ; m)$ we may conclude $R(n + 1 ; m + 1)$.

Then one may easily see that $R(n ; 0)$ is not provable for any n.

If we interpret $R(n ; m)$ as meaning that:

> At the nth stage of Zeno's argument Achilles is $1000/10^m$ yards behind the Tortoise except when $m = 0$, when Achilles is level with the Tortoise,

we see that this system formalises Zeno's method of argument.

The really thrilling thing is to see how near to discovering metamathematics the Greeks were, and it is amusing to speculate what the trend of history would have been had they done so.

The First Year of the Edsac

By S. Gill

The story of the first electronic digital computing machines is a curious one. For four years after the first, the Eniac, was built, workers on both sides of the Atlantic were busily engaged in making plans for machines of all kinds, large and small. Many plans were scrapped in embryo, some were worked out in great detail and applied in imagination to various problems. Yet in four years, not one of these machines was completed: the Eniac stood alone.

The Eniac had proved the practicability of building a large electronic machine, but even as it was being assembled, the possibility was seen of making vast improvements in design. Soon the avalanche of blueprints began. Ideas streamed in plenty from the pens of mathematicians and engineers, who wallowed delightedly in this new field of ingenuity. So rapid was the stream, in fact, that it required some deliberation to collect together enough ideas to form a plan before they were swept away by better ones. This in fact was the main reason for the great hold-up. Some machines were even scrapped after building had begun, because they were already out of date.

There were other difficulties too. Some designers who had been relying on special components still being developed were forced to drop their plans or modify them, because the components did not come up to expectations. It was a disheartening period for the mathematicians. They tried to plan how they would use these machines when they came, but it was rather like finding one's way along a road in pitch darkness, to avoid getting lost in daylight. Some of us had haunting doubts whether the daylight would ever come: these machines all looked very clever on paper, but would they work? Would they really be useful?

The Edsac, though still not fully completed, has now been in use for nearly a year, and the Manchester machine is also in operation. The days of waiting are over and the active phase of electronic computing has begun. What has it taught us so far?

First and foremost, it has completely confirmed the practicability of the broad principles on which the use of the machine is based. Routines, sub-routines, and cycles of operation had all been foreseen years ago, but always with a certain element of doubt whether the whole scheme would prove to be manageable. It was an elaborate business, and there might well have been great difficulty in finding and correcting all the inevitable mistakes. The possibilities in the event of success seemed intriguing and unlimited. If complications had arisen, electronic computing might have turned out to be little more than an exhausting pastime for those who enjoy mathematical games.

In fact, the difficulties we have encountered at Cambridge are pretty much what had been expected—if anything, rather less serious than most people feared. We have been able to go a long way towards mastering the techniques required in this new field, further than we dared to hope a year ago. We have by trial and error accumulated a library of some 60 or 70 routines, with great flexibility in their use. By means of routines we have speeded up the job of finding mistakes in programmes.

It has become quite clear that a working library of routines is essential before full use can be made of a machine, and the preparation of this library is no small task: it is linked inseparably with the design of the machine itself. The whole project requires some years, and only after it is complete is it possible to finally assess the merits of the plan. Thus it seems that electronic computing machines are destined to undergo quite a lengthy period of evolution before designs become stabilized. At present many widely different designs have been proposed, but it is not yet possible to compare their performances.

In time the Edsac will become obsolete, but not, we hope, before it has played its full part as a leading pioneer machine, and seen many years of useful service.

(See D. J. Wheeler, "E.D.S.A.C.," EUREKA, No. 11, 23–25.)

■ ■ ■

Two Dissection Problems

(1) An "n-step" is a polygon with two sides of length n units, and $2n$ sides of unit length, all its angles being $\pi/2$ or $3\pi/2$. It is defined inductively thus:

(i) A 1-step is the unit square.

(ii) An $(n-1)$-step and an n-step may be fitted together to form a square of side n.

Given a piece of paper cut in the shape of an n-step ($n > 1$)show

(*a*) How it may be cut into four pieces which may be put together to form a square.

(*b*) That it is possible to solve the problem in three pieces if n has one particular value. Find this value.

(2) It is well known that $3^3 + 4^3 + 5^3 = 6^3$. Given cubes of sides 3, 4, and 5 units, show how to dissect them into a total of not more than ten pieces which can be put together to form a cube of side 6 units.

Solutions will be published in our next issue.

The Twelve Coin Problem

Related by BLANCHE DESCARTES as a moral story for the young.

Professor Felix Fiddlesticks
Is always up to foolish tricks.
His latest game is to collect
All fakes and duds he can detect,
And counterfeits. He says it is the truth
Forgers are not what they were in his youth.

One day, by high ambition lit,
He forged a perfect threepenny bit.
It was about as good a fake
As anyone could ever make.
His pride at this success he couldn't smother,
He rushed off home to show it to his mother.

"Oh, Mother, see what I can do,"
And from his pocket he withdrew,
Not, as he thought, one threepenny bit,
But twelve—alas! For all his wit,
The counterfeit he just could not locate
By sight, but knew it differed in its weight.

"Oh, clever Felix Fiddlesticks,"
His mother said, "you're in a fix.
The spurious threepence, can you state,
Is light, or is it overweight?"
"I can't remember." "Here's a balance, see,
Go find the counterfeit in weighings three."

■ ■ ■

SUPERVISOR'S COMPLAINT

I can always spot fallacies
In Anallacies.
But I'm right up a gum-tree
With any in Geumtree.

Prime Numbers

By C. B. Haselgrove

The study of prime numbers has provided mathematicians with many fascinating and challenging problems ever since they were first discussed by the Greeks. Many people who dropped mathematics on leaving school find it difficult to believe that there are still many problems in mathematics which can be stated in terms which they can understand, but to which mathematicians have not, as yet, found an answer. But elementary number theory and, in particular, prime numbers do provide such problems. Further, in this subject, there have been many interesting and important developments recently. But before we describe these it is necessary to give a brief account of the history of the subject.

Euclid proved that there are infinitely many primes. It is of interest to ask how big the nth prime is, or what is the same thing, how many primes there are less than a given large number x. We shall denote this number by $\pi(x)$. It was conjectured that

$$\pi(x) \sim x/\log x$$

where the sign $\sim$ means that as $x \to \infty$ the ratio of the left side to the right side tends to unity. The problem was studied by several mathematicians in the nineteenth century, in particular Legendre, Gauss, Chebyshev and Riemann. But it was not until 1896 that this conjecture was proved independently by Hadamard and de la Vallée Poussin who used a method that had been introduced by Riemann. This method depended essentially on the theory of functions of a complex variable. The result is known as the Prime Number Theorem.

Most mathematicians felt that this situation was not satisfactory as, although the statement of the Prime Number Theorem involved only the most elementary ideas, it was not possible to prove it without using the very highbrow methods of function theory. This led to a question of general philosophical interest: "Are there problems of elementary mathematics which it is impossible to solve without using more difficult ideas?" This question was answered in the affirmative by Gödel, but even so there remained doubt whether it was possible to prove the Prime Number Theorem by elementary methods. According to a rumour that has been circulating in Cambridge recently, Professor Hardy is reputed to have said that he would throw all his books out of the window if an elementary proof was discovered, but on closer investigation this appears to be a gross exaggeration.

In 1948 Erdos and Selberg discovered an elementary proof of the Prime Number Theorem. It is true that the existence of this proof will mean that our methods of studying these problems will have to be revised, but there is still much that can be proved by the function theory methods which cannot yet be proved by the elementary methods.

The Prime Number Theorem is by no means the only problem in the theory of primes. Another problem which is older and simpler to state is Goldbach's Problem. In 1742 Goldbach conjectured, in a letter to Euler, that every number greater than 5 is representable as the sum of three prime numbers. Since 2 is the only even prime number this implies that every even number greater than 3 is representable as the sum of two prime numbers. Vinogradoff proved in 1937, using a method developed by Hardy and Littlewood, that every sufficiently large odd number is representable as the sum of three primes, but this leaves open the question of whether every even number is representable as the sum of two prime numbers. This problem appears to be extremely difficult. Another problem of apparently equal difficulty is the existence of an infinity of pairs of primes which differ by two, or, more generally, by any fixed even integer.

A further problem is the estimation of the minimum and maximum differences between consecutive primes. If p_n is the nth prime it is easily seen that $p_n \sim n \log n$ so that in some sense the average difference $p_{n+1} - p_n$ is $\log n$. It has been proved by Ingham that for all sufficiently large n it is less than $n^{\frac{5}{8}}$. Rankin has proved that it does become as small as $\frac{57}{59} \log n$, and that it becomes larger than

$$\tfrac{1}{4} \log n. \log \log n. \log \log \log \log n/(\log \log \log n)^2.$$

Noticing that $\log n$ increases more slowly than any positive power of n we see that the problem is far from being solved.

Now that the Prime Number Theorem has been proved it is of some interest to ask how accurately $x/\log x$ approximates to $\pi(x)$. It is found that the approximation is rather poor but that the function

$$\mathrm{li}\, x = \int_0^x \frac{dt}{\log t}$$

is a much closer approximation. It is conjectured that $\pi(x) - \mathrm{li}\, x$ is of order at most $x^{\frac{1}{2}} \log x$ but the best upper bound for the order up to now has been given by improving some results due to Tchudakoff and is $x.\exp(-(\log x)^{\frac{4}{7}-\epsilon})$. It has been proved by Littlewood that this function is sometimes as large as

$$x^{\frac{1}{2}} \log \log \log x/\log x$$

and that it takes each sign infinitely often. This last result has attracted considerable attention as for all x for which $\pi(x)$ has actually been evaluated, $\pi(x) - \mathrm{li}\, x$ is negative (that is for values of x up to 10^9). Skewes has proved that a change of sign occurs before

$$x = 10^{10^{10^{10^{29}}}}$$

and this number is said to be the biggest that has ever occurred naturally in mathematics.

There are many other problems to which the solutions are not known. For instance it is not known whether there are infinitely many primes of the form $n^2 + 1$. Even this seems easier than the famous problems of Fermat and Mersenne. Are there infinitely many primes of the forms $2^n + 1$ and $2^n - 1$? Recently Hua has considered the so-called Goldbach-Waring problem of the representation of numbers as the sums of powers of primes, and has made considerable progress.

It is never safe to make predictions, but I feel that it is likely that the very high rate of progress in this field will be continued. The subject has all the conditions favouring progress. There are, as we have seen, a large number of unsolved problems and there is a large group of mathematicians intensely interested in them. I hope that this article will encourage some more to take up the subject. But at the risk of putting some off I must insert a word of warning: the proofs of the theorems which I have mentioned are for the most part exceedingly long and complicated.

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POSTAL SUBSCRIPTIONS AND BACK NUMBERS

For the benefit of persons not resident in Cambridge, we have a postal subscription service. Persons may enrol as permanent subscribers, and those who advance 10s. or more will receive future issues as published at 25 per cent. discount. This discount is not applicable to back numbers.

Copies of EUREKA Nos. 9 and 12 (1s. 6d. each, post free) and Nos. 10 and 11 (2s. each, post free) are still available. Cheques, postal orders, etc., should be made payable to "The Business Manager, EUREKA."

The Editor still requires copies of Nos. 1 to 7, and would be glad to hear from any reader willing to sell any of these. Photo-copies of these numbers may be obtained from the Science Museum Library, South Kensington, London, S.W.7, or through the Philosophical Library, The Arts School, Bene't Street, Cambridge.

The Problems Drive

THE following problems were among those set at the Archimedeans' 1950 Problems Drive. Competitors were allowed five minutes for each question. Solutions are given on page 20

(1) A dog-owner has 64 kennels, arranged in a square with 8 on each side. He has six quarrelsome dogs, who will bark for hours if two are too close together. In which kennels should he place them to get the best chance of a night's sleep? And how should they be rearranged if he gets a seventh dog, just as quarrelsome?

(2) Write down the next two numbers of the sequences:

(*a*) 3, 2, 1, 7, 4, 1, 1, 8, . . .

(*b*) 1, 15, 29, 12, 26, 12, 26, 9, . . .

Give reasons.

(3) Prove that any positive integer can be obtained by starting from the number 2 and performing a finite number of times (and in any order) the operations of cubing and taking the integral part of the square root. (For example, $1 = [\sqrt{2}]$, $4 = [\sqrt{[\sqrt{((2^3)^3)}]}]$.)

(4) The numbers 1, 2, ... n^2 are arranged to form an $n \times n$ magic square (that is, the numbers in each column, each row, and each of the long diagonals add up to the same number). Prove that, if n is odd, the determinant of the array is either zero or divisible by the sum of all the elements.

(5) Mr. Rookem hires out punts on the Cam. During the first few days of one month, he finds that the number of punts hired each day is the cube of the day of the month, and that the total number is the fourth power of the average charge, in shillings, per punt. What are his total takings over this period?

(6) Arrange four 4's, and any number of the ordinary mathematical symbols, to give as good an approximation to π as you can find. For example, $\sqrt{\sqrt{\left(\frac{4!!+4}{4!!}\right)^{4!!}}}$ is a very good approximation to e, and can clearly be modified to be as good as we please. ($\pi = 3{\cdot}14159265358979 32\ldots$; logarithms and trigonometrical functions may not be used.)

(7) The ground plan of "Four Gables" consists of five equal squares forming a cross; the roof, of two triangular prisms meeting in a pyramid whose sides are equilateral triangles. Sidney, the snail, lives half-way along the east side of the south gable; Emma, his girl-friend, half-way along the east side of the north gable. Sidney has a single crawling speed, which will take him straight up the roof to the nearest point of the ridge in one hour. In how short a time can he go from his home to Emma's?

Wari

By R. H. Macmillan

This is a game which has been played for generations by the Africans of the Gold Coast. It is played on a wooden board containing twelve declivities or cups arranged in two parallel rows of six. Initially four pebbles are placed in each cup, making 48 in all. It is quite possible to play without a board and use matchsticks for "men." The object is to have captured more men than one's opponent when the game ends.

Play is alternate; to make a move a player removes all the men from any one cup on his own side and then distributes them, one at a time and consecutively, into his own and his opponent's cups, proceeding anticlockwise and starting from the cup adjacent to that from which the men were first taken. If the last man to be placed makes the total in the cup into which it goes equal to either two or three, and that cup is on his opponent's side, then those two or three men are captured and removed from the board. If it is then found that the preceding cup (i.e. adjacent clockwise) contains two or three men, those also are captured, and so on. The maximum capture possible in a single move would thus be 18 (i.e. three in every one of his opponent's cups). If a pile of twelve or more men is distributed, then the cup from which they came originally is not filled on the second time round.

If A has no men left on his side after playing his move, B is obliged to leave him at least one man to move at his next turn, if he can. If this is not possible the game ends. The game may also end if A can capture all the pieces on B's side in a single move, thus leaving him without a move. Note that if A has captured fewer men than B, it is in A's interest to prolong the game, while B will try to terminate it. Obviously there is no point in continuing to play after either player has succeeded in capturing more than half the total number of men: this is the commonest ending.

The following considerations should govern the play. In the early stages it is more important to play for position than to make small captures. It is useful to get as many men as possible on one's own side, as this reduces the number one's opponent has and so limits his available moves. One should work for large captures (eight or ten men in a single move); a good way of effecting one is to collect a large pile of fifteen or more men in one cup.

A Diophantine Problem

By J. Leech

The writer was recently confronted with the following problem: Find two positive integers whose sum is a square, the sum of whose squares is a square, and the sum of whose cubes is a square, and which are not in the ratio 8 : 15. The following solution is typical of Diophantine problems of this kind, and may be of interest to readers. In the first instance we ignore the restriction that their ratio shall not be 8 : 15.

Suppose we have a solution a, b, that is, we have found a, b such that

$$\begin{aligned} a + b &= c^2 \\ a^2 + b^2 &= d^2 \\ a^3 + b^3 &= e^2. \end{aligned}$$

Then since $a + b$ divides $a^3 + b^3$, we have also

$$e^2 = a^3 + b^3 = (a + b)(a^2 - ab + b^2) = c^2f^2,$$

where

$$a^2 - ab + b^2 = f^2.$$

Further, if we have a solution x, y to

$$\begin{aligned} x^2 + y^2 &= z^2 \\ x^2 - xy + y^2 &= t^2, \end{aligned}$$

then

$$\begin{aligned} a &= x(x + y) \\ b &= y(x + y) \end{aligned}$$

is a solution of the problem, for

$$\begin{aligned} a + b &= (x + y)^2 \\ a^2 + b^2 &= (z(x + y))^2 \\ a^3 + b^3 &= (t(x + y)^2)^2. \end{aligned}$$

Also if $x + y$ has a repeated factor h^2, i.e. $x + y = h^2k$, then

$$\begin{aligned} a &= xk \\ b &= yk \end{aligned}$$

is a solution of the problem. All solutions can be derived in this manner, consequently it is sufficient to solve

$$\begin{aligned} x^2 + y^2 &= z^2 \\ x^2 - xy + y^2 &= t^2. \end{aligned}$$

The general solution in coprime integers of

$$x^2 + y^2 = z^2$$

is

$$\begin{aligned} x &= 2mn \\ y &= m^2 - n^2 \\ z &= m^2 + n^2 \end{aligned}$$

where m and n are arbitrary coprime integers, not both odd.

Inserting this in $\quad x^2 - xy + y^2 = t^2$

we obtain $\quad m^4 - 2m^3n + 2m^2n^2 + 2mn^3 + n^4 = t^2,$

which it is required to solve in integers; this is equivalent to solving

$$u^4 - 2u^3 + 2u^2 + 2u + 1 = v^2$$

rationally. Writing $v = u^2 - u + w$, this becomes

$$(2w - 1)u^2 - 2(w + 1)u + w^2 - 1 = 0,$$

i.e.

$$w^2 + 2u(u - 1)w - (u + 1)^2 = 0.$$

Now suppose we know values of u, w which satisfy this equation. The value of u gives rise to a quadratic in w of which one root is known to be rational, so therefore is the other which is, in general, different. This value of w gives a quadratic in u with, in general, two distinct rational roots, let us choose that which is different from the original value of u. This gives a quadratic in w, only one of whose roots coincides with the roots obtained above. By this means, further roots may be obtained *ad lib.* An obvious root is $w = -1$.

$w = -1$ gives $-3u^2 = 0$, $u = 0$ twice.
$u = 0$ gives $w^2 = 1$, $w = \pm 1$.
$w = 1$ gives $u^2 - 4u = 0$, $u = 0$ or 4.
$u = 4$ gives $w^2 + 24w - 25 = 0$, $w = 1$ or -25.
$w = -25$ gives $-51u^2 + 48u + 624 = 0$, $u = 4$ or $-52/17$.
$u = -52/17$ gives $17^2w^2 + 7176w - 35^2 = 0$, $w = -25$ or $(7/17)^2$.
$w = (7/17)^2$ gives $-17^2.191u^2 - 2.17^2.338u - 240.338 = 0$,
$u = -52/17$ or $-1560/3247$.

This is sufficient for the present problem.
$u = 0$ gives the trivial solution $a = 0$, $b = 1$.
$u = 4$ gives $m = 4$, $n = 1$, $x = 8$, $y = 15$, $a = 184$, $b = 345$, the solution which was specifically rejected.
$u = -52/17$ gives $m = 52$, $n = -17$, and we find a negative, b positive, contrary to the requirement of a solution in positive integers.
$u = -1560/3247$ gives $m = 1560$, $n = -3247$,

$$\begin{aligned} x &= -10130640, y = -8109409, \\ a &= 184783370001360, \\ b &= 147916017521041. \end{aligned}$$

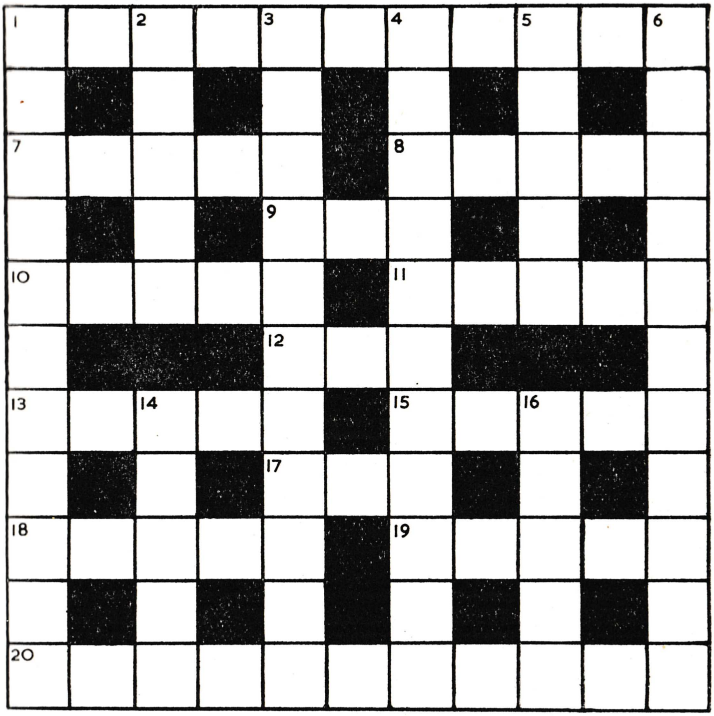
This is a solution of the problem.

Other obvious solutions in u, w are $u = -1$, $w = 0$; $u = -\frac{1}{2}$, $w = \frac{1}{2}$; $u = 1$, $w = \pm 2$; none of these gives rise to solutions fundamentally different from those above. There may be other solutions giving values of u which are ratios of small integers which would give smaller integers a, b than those above.

Readers caring to compute $a + b$, $a^2 + b^2$, $a^3 + b^3$ and so verify that each is a perfect square are welcome, but we expect the above reasoning to convince most readers without further computation.

The problem cannot be extended by adding the requirement that $a^4 + b^4$ be square also, for it is well known that the sum of two non-zero fourth powers is never square.

Puzzle



ACROSS

1. 720.
7. Tenth power of a root of 10 across.
8. Twice the greater zero of 3 down.
9. Smallest number with more than thirty divisors.
10. Equation with no real roots.
11. Product of 2 down and cube of diameter of 20 across.
12. One of the values of s satisfying 6 down.
13. Cube, if t satisfies 6 down.
15. See 2 down.
17. Beauty's opposite number.
18. Beauty's opposite.
19. Difference of zeros of this is twice the greatest root of 1 down.
20. Circle, centre $(5, -\frac{1}{2})$.

DOWN

1. The other roots of this are -1 and $-\frac{1}{3}$.
2. Cube of 15 across $\times$ $\operatorname{cosec}^3 t$.
3. See 5 down and 8 across.
4. Cube.
5. Modulus of the lesser zero of 3 down.
6. Simultaneous quadratic.
14. If Cambridge, this is mathematical.
16. Are they even smaller than epsilons?

Permitted symbols:—Digits 0, . . ., 9; letters $a, \ldots, z$; indexed letters $a^2, \ldots, z^2, a^3, \ldots, z^3$ (each indexed letter is treated as a single symbol); signs $+, -, \times, \div, =$. Polynomials are in either ascending or descending order. $0, +, \times, =$, and $\div$ are not used as initial symbols.

The solution will appear in our next issue.

Telepathy Experiment

A MEETING of the Archimedeans was held on 8th March, 1950, to discuss and experiment in telepathy. The speakers, Mr. E. D. M. Dean and Mr. C. B. Haselgrove, gave brief accounts of the psychological and mathematical aspects of the subject. An experiment involving the audience was then performed.

A pack of 25 "Zener" cards, containing 5 sets of 5 different symbols, was dealt face upwards on to a desk in such a position that they could not be seen by the audience, at a rate of about one card every $1\frac{1}{2}$ seconds. Each member of the audience, who could hear the impact of the cards on the desk, was asked to write down his guess of the symbol as each card was turned up. The experiment was performed four times with a new set of cards each time.

It can be proved mathematically that the expected number of correct guesses is 5 with a standard deviation not greater than 2·1 whatever method of guessing the subject may employ. It is only by telepathy or clairvoyance (or of course by fraud) that a different average can be obtained.

It has not been possible to analyse the results in full up to the present, but a preliminary check on the results of about 60 of the persons present has been made. It was decided to count also the number of precognitive guesses $1\frac{1}{2}$ and 3 seconds ahead, in view of the results of Professor J. B. Rhine. The average numbers of guesses in these three cases were

$$4{\cdot}64 \qquad 4{\cdot}74 \qquad 4{\cdot}35$$

which differ from the chance expectations by

$$-0{\cdot}36 \qquad -0{\cdot}06 \qquad -0{\cdot}25.$$

The standard deviation in each case is 0·13. The negative values may possibly be explained by errors in checking the results. It was found that some people when checking these results failed to notice correct guesses although they did not mistake incorrect for correct guesses. These deviations seem to be inconsistent with chance, although the significance is not great. It is hoped that further checking will elucidate the matter, but the labour involved is considerable. The results are open to the inspection of anyone wishing to see them, who should apply to Mr. C. B. Haselgrove, King's College.

The experiment may be criticized on several grounds. Perhaps the most important is that any correlation between the guesses of the members of the audience would invalidate the probability theory. Such a correlation might easily arise from associations with the symbols on the cards in the mind of "the average member of the audience." But we cannot go into these matters here owing to lack of space.

C. B. H.

Book Reviews

Outline of the History of Mathematics. By R. C. ARCHIBALD. (American Mathematical Monthly, Vol. 56, No. 1 (January, 1949), Part II.) \$1.

"An attempt is here made to give indications of the development of mathematics before the nineteenth century, and to refer briefly to some developments of the nineteenth and twentieth centuries in connection with topics usually discussed at undergraduate colleges." So says the introduction to this *Outline*. Being condensed into 50 pages without serious omission, the text is unavoidably rather heavy and less readable than the old style histories of mathematics, though this is not a serious fault in a work which is to be regarded primarily as one of reference. The text is very fully documented; the literature list and notes occupy as much space as the text itself, and they form an admirable guide to the immense body of the literature on the history of mathematics. This alone will make this *Outline* indispensable to students of the history of mathematics and of considerable value to all mathematicians.

J. L.

Methods of Mathematical Physics. By H. JEFFREYS, M.A., D.Sc., F.R.S., and B. S. JEFFREYS, M.A., Ph.D. (Cambridge University Press, second edition, 1950.) £4 4s.

This book, as its title implies, is an account of the methods used in the mathematical description of physical phenomena. The classification of the contents is therefore mathematical: each chapter or series of chapters deals with one mathematical idea, fully illustrated by examples from every branch of mathematical physics. Throughout the book the authors emphasize the essential interdependence of mathematics and physics: physical phenomena are understood in terms of abstract mathematical concepts, but these concepts are suggested by physical experience. The authors also point out that many mathematical concepts and methods sometimes regarded as inadequate or excessively abstract are very close to what is required by physics: examples of this are continuity, approximations, and the justification of the use of operational methods in solving partial differential equations for continuous systems.

The book starts with a series of chapters on such basic concepts as the real variable, tensors, matrices, and multiple integrals; the ideas are then related at once to various physical topics, problems of all degrees of difficulty being considered. It is refreshing to find tensors discussed before matrices, and treated as a natural extension of vectors; and to find the Stieltjes integral introduced as a simple generalization of the Riemann integral. This section of the book ends with two chapters on operational methods (which should be well understood before reading later parts of the book), and a useful chapter on numerical methods.

The next section of the book begins with the introduction of the complex variable, and several chapters contain mainly pure mathematics; those who are not used to operational methods should read these chapters carefully as such methods are used frequently later in the book. Elementary applications of complex variable theory are made in the chapters on Fourier series and factorial functions; more difficult ones are discussed in the chapters on linear differential equations and asymptotic expansions—important topics discussed in these chapters are Bessel functions, the Airy integral, and group velocity.

The third and last section of the book is an introduction to the subject of partial differential equations. All the most familiar equations are discussed in relation to various boundary conditions and the solutions of the resulting ordinary differential equations found. Some familiar functions, such as Bessel functions, are found from first principles. The authors are probably wise to introduce the hypergeometric function after several particular differential equations have been solved; the power of this more general method is shown by the simple deduction of Laguerre, Hermite, and Weber functions. Many particular physical problems are discussed during the development of solutions.

The book is remarkable for its wide scope and for the fact that the authors never deviate from their purpose of discussing methods, while satisfying the demands of both pure mathematics and physics. The reader must not expect a treatise on the logic of physics or on atomic structure—any special subject must be studied elsewhere. The authors are sympathetic towards the reader, explaining difficult points lucidly and pointing out common errors. The arrangement of the book is very good; it is inevitable that the discussion of one or two examples, such as the spinning top, should be split up. For Tripos purposes, a thorough knowledge of the bookwork and of the simpler examples from chapters 1-8, 10-19, and 24 should satisfy any set of Part II examiners. At the end of each chapter there is a well-chosen set of examples for the reader.

This is a well-balanced book: a good investment for those reading for the Tripos Part II or Part III (applied), and a more than useful reference book for research students in theoretical physics.

J. S. R. C.

The following books have also been received, and reviews will appear in our next issue:

Probability and the Weighing of Evidence. By I. J. Good. (Charles Griffin.)

An Introduction to the Theory of Statistics. (14th edition.) By G. U. Yule and M. G. Kendall. (Charles Griffin.)

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A Paradox

An astronomer has announced recently that he has proved that space is curved and that the universe as we know it consists of the surface of a hypersphere, the radius of which is increasing steadily at the velocity of light. He remarks that as this implies that the circumference of the universe is increasing at a rate $2\pi c$ where c is the velocity of light, we shall never be able to see all the way round the universe. But a colleague, propounding a rival theory, states that the above theory cannot be right as it is clear that light would actually travel along an equiangular spiral and so we should be able to see as many times round the universe as our telescopes allow.

The Revision of the Tripos

"I do not wish to reform the Tripos, but to destroy it," said Hardy in 1926. The more moderate will welcome changes which remove the gaps that have been a perennial grumble in recent years, and few will have any immediate complaints. Some of us enjoyed Differential Geometry, some Astronomy (even Lowndean Professors of Geometry and Astronomy can hardly have enjoyed both), but it was hard to feel that they had a real place in the Part II syllabus. On the other hand, everyone who has taken Part III recently has had to learn for himself at least one, perhaps two, of the three subjects (Abstract Algebra, Random Variables, and Thermodynamics) on which new courses are to be given.

In the "Advanced" courses, by nature more flexible, the immediate changes are less striking. Several moves in the last few years, and two new courses now announced, make the "Introductory" lectures far more useful as an introduction to modern mathematics than they were not long ago. The division of "Advanced" courses into two groups is welcome in so far as it provides guidance in planning one's Part III syllabus, but it will be unfortunate if it discourages those not sure of another chance from attending (merely "for amusement") some courses "intended primarily for graduates."

G. H. T.

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The Mathematical Association

President: Prof. H. R. Hassé, D.Sc.

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 21s. per annum: to encourage students, and those who have recently completed their training, the rules of the Association provide for junior membership for a limited period at an annual subscription of 10s. 6d. Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published four times a year and deals with mathematical topics of general interest.

SOLUTION TO TWELVE COIN PROBLEM

F set the coins out in a row
And chalked on each a letter, so,
To form the words: "F AM NOT LICKED"
(An idea in his brain had clicked).
A bold man must he be who thinks he licks
Our wonderful Professor Fiddlesticks.

And now his mother he'll enjoin:

	Coins put on left hand side	*Coins put on right hand side*
1st weighing	"MA, DO	LIKE
2nd weighing	ME TO	FIND
3rd weighing	FAKE	COIN".

By weighing thus, he can detect
The spurious coin by its effect;
And more than that, with confidence he'll state
Whether the dud is light or over-weight.

For instance, should the dud be L
And heavy, here's the way to tell:
First weighing, down the right must come;
The others, equilibrium.
Each coin can thus be tested—or perhaps
F left the dud behind, a frequent lapse.

Such cases number twenty-five,
And by F's scheme we so contrive
No two agree in their effect,
As is with pen and patience checked:
And so the dud is found. Be as it may
It only goes to show CRIME DOES NOT PAY.

(For further information on the subject, see Smith, C. A. B., *Math. Gazette*, **31** (1947), p. 31; Hammersley, J. M., *Proc. Cambridge Phil. Soc.*, **46** (1950), p. 226).

■ ■ ■

SOLUTIONS TO PROBLEMS DRIVE

(1) Referring to the kennels by the Cartesian co-ordinates of their centres, a possible solution is that the six dogs should be placed in (0,1) (0,7) (3,4) (4,0) (6,7) (7,3), giving a least separation of $\sqrt{17}$ units, and the seven in (0,0) (0,4) (2,7) (4,0) (4,4) (7,2) (7,7), with a least separation of $\sqrt{13}$ units.

(2) (*a*) 5, 2. (Odd terms from decimal expression of π, even from that of e.) (*b*) 23, 7. (These are days of the month at fortnightly intervals from 1st January.)

(3) In fact, it can be done with all the "cubing" first, and all the "integral-part-of-square-rooting" afterwards: in this case, we will clearly get the same result by taking the integral part only once, at the end. Hence the result of m cubings, followed by n of the other operations, is

$$\left[2^{3^m/2^n}\right] = \left[2^{2^{ma-n}}\right]$$

where $a = \log_2 3$. Clearly a is irrational; hence $ma - n$ can be made arbitrarily near any real number by an appropriate choice of m and n: in particular, we can ensure that $N \leqslant 2^{2^{ma-n}} < N + 1$ for any positive integer N, which can thus be obtained as required.

(4) Add all the rows to the last, then all the columns to the last. The required factor, $\frac{1}{2}n^2(n^2 + 1)$, can now be removed, $\frac{1}{2}n(n^2 + 1)$ from the last row, and n from the last column (since $n^2 + 1$ is even).

(5) £388 16s. (Takings for n days are $\{\frac{1}{2}n(n + 1)\}^2$ shillings. This is a fourth power, so $\frac{1}{2}n(n + 1)$ is square: only relevant case is $n = 8$).

(6) The best result we have found is

$$\sqrt{\sqrt{\{4 \times 4! + \sqrt{\sqrt{4}}/\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{(4!)}}}}}}}}}\}}} = 3{\cdot}14159862196 \ldots$$

(7) 3 hours. (Unfold the roof surface.)

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Solutions to Problems in Eureka No. 12

CROSS NUMBER PUZZLE

Across:—1. 331776. 5. 3481. 7. 484. 8. 400. 9. 441. 11. 125. 12. 4624. 13. 239121. *Down:*—1. 314432. 2. 144. 3. 784. 4. 690561. 5. 3844. 6. 1024. 10. 169. 11. 121.

CROSSWORD PUZZLE

Across:—1. Whip-p(upp)y. 4. Fly-paper. 10. Expansion by rows. 11. Cal-amity. 12. †Albeit. 14. †Evil. 15. Ancho(vy)-rite. 18. P.E.R.P.-ended. 19. Aver(roes). 22. Orchid. 23. C-arb-olic. 25. Trig-on-o-metrical. 26. Canis-ter. 27. M-yo-sis.

Down:—1. Wren(tou)ches. 2. Impulsive action. 3. Pa-na-ma. 5. Link. 6. Pa-y-cle-rk. 7. †Projective locus. 8. Resite(ppa). 9. Listen-in-g. 13. Sh-o-em-aker. 16. Epsilons. 17. Trickles(s). 20. Coptic. 21. †Uberty. 24. Dome(stic).

(† = anagram.)

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TWO PROBLEMS ON FRAMEWORKS

(1) Take three arbitrary points and their reflections in a line in general position. Insert rods joining each of the six points to all of the others except its own reflection. This gives a framework of twelve rods, each belonging to just two 3-cycles, which can be shown to be deformable (see G. T. Bennett, "Deformable Octahedra," *Proc. London Math. Soc.* (2), **10** (1912), 309-343).

(2) Take two sets of five points each, insert rods joining each point of one set to every point of the other, and remove one of these rods. The resulting framework clearly contains no 3-cycles: it has 10 joints and 24 rods, and hence $3.10 - 24 = 6$ degrees of freedom, and so will be non-deformable for general positions of the ten points. It is easily proved that any non-deformable framework with fewer rods must contain a 3-cycle.