Eureka Editor
archim-eureka@srcf.net
The Archimedeans
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WA
United Kingdom

Published by The Archimedeans, the mathematics student society of the University of Cambridge

Thanks to the Betty & Gordon Moore Library, Cambridge

EUREKA

No. 31

1968

2/6

EUREKA

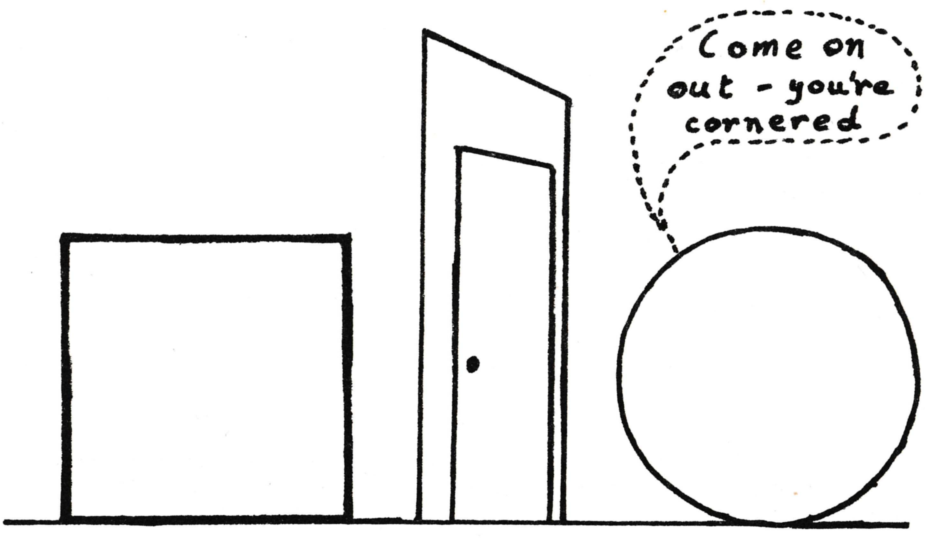
The Journal of the Archimedeans (Cambridge University Mathematical Society)

No. 31—October 1968

Editor: J. J. Barrett (Churchill)
Business Manager: C.D.Evans (Churchill)
Circulation Manager: C.R.Prior (Churchill)

CONTENTS

Correspondence for Eureka should be addressed to the Editor (or Business Manager or Circulation Manager as appropriate), Eureka, The Arts School, Bene't Street, Cambridge, England.



Come on out - you're cornered

Editorial

Why Cambridge?

What makes mathematicians choose to come to Cambridge in preference to the other forty odd universities in this country? Largely that along with 'the other place', it has the reputation of producing the best graduates. Because of this reputation Cambridge graduates get better jobs. Because it improves their employment prospects many more applicants want to get a B.A. (Cantab.) than there are places for them. The colleges can therefore pick the ablest of the able, who, irrespective of what they do in Cambridge, remain the most able and are therefore good at their jobs. Hence the reputation of Cambridge is maintained.

Exactly how this reputation grew up is not easy to explain, but there is little doubt that it does exist. Moreover, the viscious circle process just described would seem to prevent it ever dying out. But perpetual motion is not possible. There is an increasing number of school-leavers who, although they are bright enough to get into Cambridge, are more interested in the less conventional courses offered by the new universities. When they get into commerce and industry it may be that Cambridge graduates will feel the pinch. For motion to be maintained, an external force has to do work. The force is there, in the form of new ideas from the younger universities. But the Maths Faculty in Cambridge appears to be an immoveable object. It seems that while Cambridge Maths dons (and the same probably applies in many other faculties) are not altogether unaware of innovation elsewhere, it never occurs to them that Cambridge has anything to learn from other universities.

What can be learnt? To answer this let us first recall the two primary functions of a university: research and teaching. The mistake made in the Cambridge Maths course is to think that the teaching should be geared to the needs of those who will stay to do research. This inevitably neglects the vast majority who will leave with the B.A. degree. Other universities offer separate degree courses in Pure Mathematics, Applied Mathematics, Statistics and even Computer Science. But at Cambridge, time is spent almost exclusively on the two former while many people eventually take jobs involving the two latter. What is called for is a Maths course which would enable (but not compel) some specialisation at least as early as the second year. A graduate is after all supposed to be a specialist.

Another idea, which sounds almost unthinkable for Cambridge, but which would have advantages for both staff and students, is not to hold all examinations in the Easter term. For instance the Tripos could be divided into two parts, the first being taken after say four terms, and the second after nine terms. Apart from anything else, students would then be able to enjoy two of their three summers at university. It would also make it very easy to achieve the earlier specialisation already mentioned.

In June and July this year, art students up and down the country were demonstrating in no uncertain manner that they were not satisfied that their courses were appropriate to the work they would do subsequently. Mathematicians tend to be rather less exitable than artists, so there is little liklihood of a student 'sit-in' at the Arts School (the seats are too uncomfortable anyway!). What does seem to be likely is that before many years are out, no Maths students of any worth will be coming to Cambridge.

It is regretted that as from this issue it is necessary to charge 3s. 0d. for copies of Eureka sent by post, unless 11s. 0d. or more is sent in advance, in which case accounts will be debited 2s. 9d. per issue. These increases are due to the introduction of the 'Post Office Preferred' envelope sizes, into which Eureka will not fit, although it is an international paper size. Owing to the effect of sterling devaluation the dollar rates remain unaltered at 55c. per copy or 50c. per copy if \$2.00 or more is paid in advance. Anyone wishing to take out a postal subscription should write to the Circulation Manager at the above address, as should anyone interested in obtaining back numbers. Cheques and Postal Orders should be made payable to 'The Business Manager, Eureka' and crossed.

Those readers who receive their Eurekas through the Archimedeans will find a questionnaire enclosed, concerning the teaching of mathematics in Cambridge. We would be grateful if these could be returned to Mr. C. D. Evans, Churchill College by 9th November. We advise freshmen to wait until they have two or three weeks experience of the system before attempting to complete the questionnaire. All comments will be treated in strictest confidence. It is hoped that some analysis of the answers received will be published in next year's Eureka.

The editor wishes to thank all those who have contributed to the production of this magazine. He is particularly indebted to his predecessor, C. J. Myerscough, to C. D. Evans (Business Manager) and to C. R. Prior (Circulation Manager).

Logique mathématique et logistique militaire

par Paul et Naomi Laguerre

Dans le monde d'aujourd'hui on entend les cris souvents sonores et désespérés des étudiants contre les maux de la société, contre les inégalités, l'injustice et la guerre. Donc il convient de se souvenir que les mathématiciens échappent à la conscription. Ce fait est tout raisonnable, parce que la mathématique est une discipline libre et bienfaisante, ayante le seul but de poursuivre le bien de l'humanité. Mais parce que les mathématiciens des autres nationalités souvent ne le savent pas, il convient d'y donner une demonstration logique. Supposons, au contraire, qu'il serait possible qu'un mathématicien était conscrit. Depuis son service militaire il deviendrait ex-conscrit. Parce qu'on appelle mathématicien 'θx' (c'est-à-dire, tête-à-x) on aurait l'équation

$$\theta\text{x} = \text{ex-conscrit} \tag{1}$$

Divisions (1) par ex; nous obtenons

$$\text{Conscrit} = \theta/\text{e} \tag{2}$$

c'est-à-dire, conscrit = tête assurée, une absurdité bien évidente. Donc le théorème est établi.

A Question of Cubes

Set by C. V. Durell

Find <u>the</u> multiple of 3 which equals the sum of the cubes of its digits.

Mathematics and Evolution

by Cedric A. B. Smith

Professor of Biometry, University College, University of London.

The phrase 'applied mathematics' traditionally means only too often a rather dull kind of mechanics. Perhaps that is because the elementary laws of mechanics are simple and exact, so that it is easy to put them into mathematical form. But physics and chemistry are also largely mathematical. Living things are more complicated and capricious, and on the whole cannot be expected to obey absolutely precise laws. All the same in recent years there has been a great expansion of numerical and mathematical studies in biology and medicine, and the next few years will probably show similar advances in the social sciences. A quantitative approach is part of a scientific treatment, likely to lead to important discoveries. Looking back from the year 2100, if mankind still survives, the present state of the world, threatened by disease, hunger, poverty, violence and war, may seem incredibly ignorant and old-fashioned.

One of the best studied parts of mathematical biology is the theory of evolution, or more precisely, of genetic changes in populations. We can sometimes see evolution in action. During the last 150 years industrial smoke has blackened most tree trunks in eastern England. The moths who settle on these trunks are now mostly dark-colored, and so less easily seen by birds, whereas they used to be light. The birds eat more light moths than dark, and hence the proportion of light moths is lower in the next generation, still lower in the generation after that, and so on.

If we assumed most simply (but inaccurately) that light moths gave birth only to light moths, dark only to dark, and that the probability of survival of a dark moth was, say, 3 times that of a light one, then the effect would be that the ratio of the numbers of dark to light moths would be multiplied by 3 in each generation. Very soon all moths would become dark.

Such a simple type of evolution would have disadvantages. It would tend to reduce all members of a population to a single uniform 'best' type. If conditions changed, that type might no longer be able to survive, and the species would die out. Some diversity is desirable. Nature achieves this in various ways, especially by sex. In essence this means that each individual is a union of two halves, one derived from the father and one from the mother. Each part brings in its own 'genes', or units of heridity. For example, in man there are many different types of hemoglobin. For simplicity we consider here only two, and two corresponding genes, which we will here call G_1 and G_2. An individual ('G_1G_1') getting G_1 from his father, and G_1 from his mother, has normal or 'A' hemoglobin; one getting two G_2 genes has 'sickle-cell' or 'S' hemoglobin, while a G_1G_2 (i.e. G_1 from father, G_2 from mother) or G_2G_1 individual has a mixture of both. (The types G_1G_2 and G_2G_1 are physically identical, but it simplifies the algebra to distinguish them). G_2G_2 sickle-cell children are anemic, and die before the age of marriage. The G_1G_1 individuals (like most Europeans) are susceptible to malaria, as has been shown by direct experimentation. Hence both G_1G_1 and G_2G_2 are at a disadvantage in a malarial region. If G_1 is very much more common than G_2, the G_1G_1 individuals will be much more frequent than G_2G_2, and hence as a result of the selection many more G_1 genes will be eliminated than G_2, and the frequency of G_1 will decrease. In the same way, if G_1 is very rare, its relative frequency will increase. Thus selection tends to prevent either gene G_1 or G_2 from dying out completely, and hence all 3 types G_1G_1, G_1G_2, G_2G_2 will continue to exist. This theory appears to be completely verified in practice.

This argument can be put in a more precise algebraic form. The process of marriage and reproduction is a complicated one, but in practice it amounts very nearly to a random rearrangement of genes. If the gene G_1 is 4 times as frequent as G_2 in the adult population, then, as shown in Fig. 1, children will be of types G_1G_1, G_1G_2, G_2G_1,

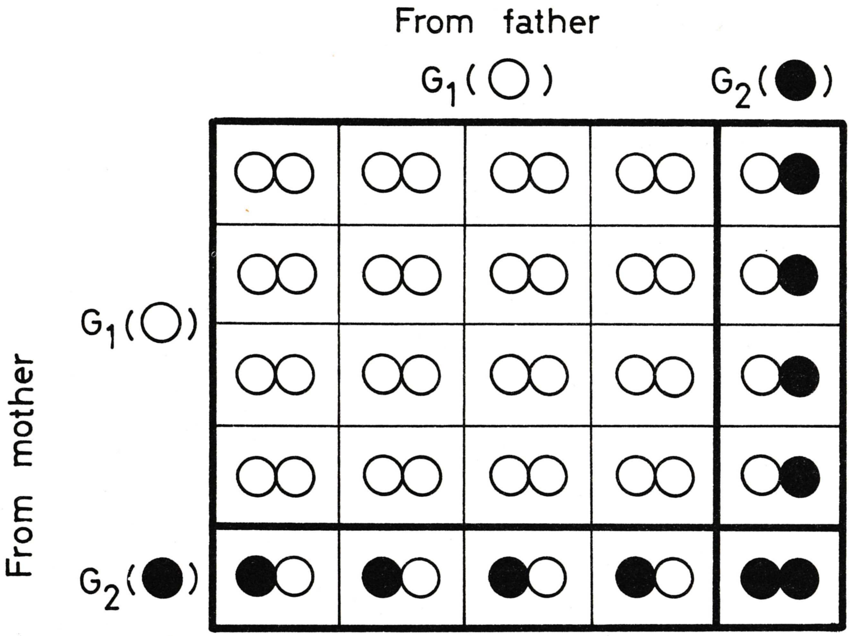


Fig. 1

G_2G_2, with frequencies in the ratio 16 : 4 : 4 : 1. More generally, if the genes G_1, G_2, G_3, ... have frequencies $p_1, p_2, p_3, \ldots$ then a proportion p_ip_j of children will have type G_iG_j (This fact was discovered by G. H. Hardy in 1908 at dinner in Trinity). Let $w_{ij} = w_{ji}$ be the probability of survival (and reproduction) of a G_iG_j child. Then the frequency of G_iG_j adults in the population in the next generation will be proportional to $w_{ij}p_ip_j$. The proportion p_i' of G_i genes in these adults and hence in the next generation of children is found by counting the G_i genes in these adults, i.e. it will be proportional to $\sum_j w_{ij}p_jp_i + \sum_j w_{ij}p_ip_j = 2\sum_j w_{ij}p_ip_j$. We can therefore write

$$p_i' = Kp_i\sum_j w_{ij}p_j, \tag{1}$$

where K is a constant of proportionality, fixed by the relation

$$\sum_i p_i' = 1, \tag{2}$$

so that

$$K = 1/\sum_{i,j} w_{ij}p_ip_j. \tag{3}$$

provided that (as we always suppose hereafter) $\sum_{i,j} w_{ij}p_ip_j \neq 0$.

In the same way the frequencies p_i'' in the third generation are

$$p_i'' = K' p_i' \sum_j w_{ij} p_j' \tag{4}$$

and so on. What happens to the sequence of frequencies $p_i, p_i', p_i'', \ldots$? It can be shown that

$$\sum_{j,i} w_{ij} p_i' p_j' \geqslant \sum_{i,j} w_{ij} p_i p_j \tag{5}$$

with equality if and only if $p_i' = p_i$ for all i (i.e. at an equilibrium point). In other words, the probability of survival of a random child increases from one generation to the next. This suggests that the sequence will tend to a limit at a set of (p_i) making $\sum_{j,i} w_{ij} p_i p_j$ a (local) maximum; and such a set will be a stable equilibrium. This is not too difficult to demonstrate rigorously, using (5), Hence the solution depends in its general features essentially on the simple problem of finding the maximum points (in the region $0 \leqslant p_i \leqslant 1$, $\sum_i p_i = 1$) of the quadratic function $\sum_{i,j} w_{ij} p_i p_j$, $(w_{ij} \geqslant 0)$. Even this can lead to results which are not immediately obvious, at least not to me. For example, consider the case of 3 or more genes, $G_1, G_2, G_3, \ldots$ The w_{ij} can be chosen so that the following happens. There is an equilibrium in which G_1, G_2 are present, but G_3 is absent, i.e. $p_1 > 0$, $p_2 > 0$, $p_3 = 0$; and this equilibrium is stable at least as long as no G_3 genes are introduced (i.e. it is a maximum point on the line $p_3 = 0$). Similar remarks to G_2, G_3 with G_1 absent, and G_3, G_1 with G_2 absent. But there is no equilibrium, stable or unstable, with all three genes present.

The really difficult part of the theory is the proof of the apparently innocuous inequality (5). If we put

$$\sqrt{p_i} = v_i \geqslant 0, \text{ and } m_{ij} = m_{ji} = w_{ij} v_i v_j \geqslant 0 \tag{6}$$

then it becomes the matrix relation

$$(V^T M^n V)\,(V^T V)^{n-1} \geqslant (V^T M V)^n \tag{7}$$

for $n = 3$ (where T denotes matrix transposition). In fact, this is true for all positive integral n (under conditions 6), as was first independently discovered by H. Mulholland and P. A. M. Scheuer. But the proofs they gave were by no means trivial. A much simpler proof is due to J. F. C. Kingman. (Reference below). Can any reader improve on this? One can think of further generalisations, though some are of mathematical rather than genetical interest. What happens to (1) if $w_{ij} \neq w_{ji}$? If there are three genes G_1, G_2, G_3 and the w_{ij} take the values 1, c, c^2 according as $i-j = 0$, 1 or 2 (mod 3), then I and Roger Penrose have shown (in unpublished work) that the successive points (p_i), (p_i'), (p_i''), ... run cyclically round a cubic curve,

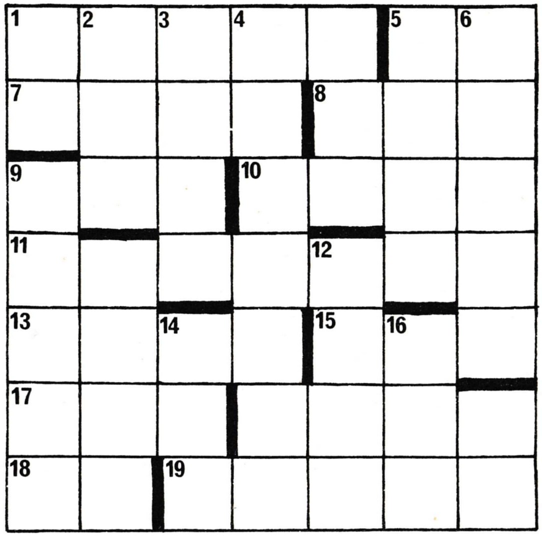
$$(c^2 + c + 1) \sum_{i,j} p_i^2 p_j - (c^2 + 1) \sum_i p_i^3 = a p_1 p_2 p_3; \tag{8}$$

but this is a very particular case. What happens if some w_{ij} are negative? Nobody knows. Many other problems are waiting to be solved.

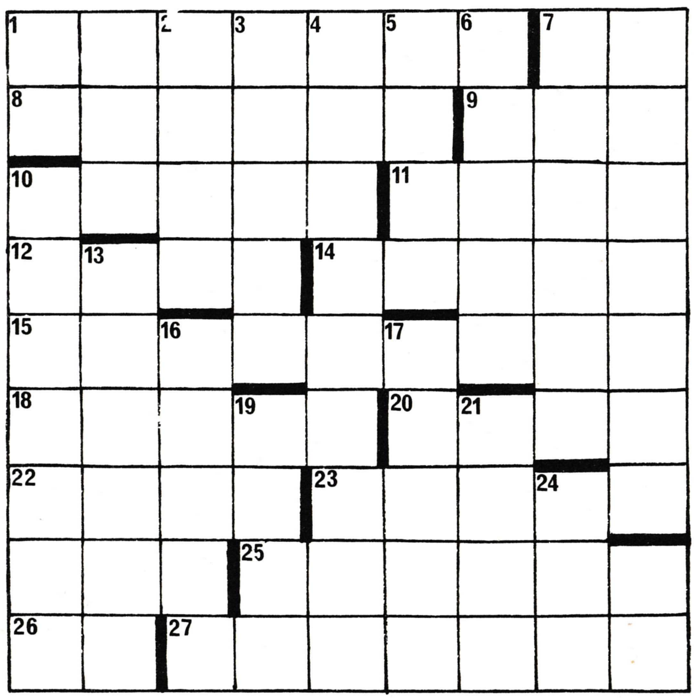
Reference:

J. F. Kingman. A matrix inequality. Quart. J. Math. 12, 1961, 78-80.

Two Crossnumber Puzzles



A



B

Square A

ACROSS

1: $2 \times$ (6 down)
7: $2 \times$ (8 across)
8: See 7 across
9: See 2 down
10: $2 \times$ (12 down); see 9 down
11: See 4 down
13: $2 \times$ (15 across)
15: $3 \times$ (14 down); see 13 across
17: Some other answer
18: Some other answer
19: Sum of three answers

DOWN

2: Multiple of 9 across
3: $2 \times$ (5 down)
4: $3 \times$ (11 across)
5: See 3 down
6: See 1 across
9: $2 \times$ (10 across)
12: See 10 across
14: See 15 across
16: Sum of two answers

Square B

ACROSS

1: $2 \times$ (13 down; some other answer
8: (7 down) + (24 down)
9: Sum of two answers
10: Multiple of 2 down
11: Multiple of some other answer
12: See 18 across
14: $2 \times$ (17 down)
15: See 4 down
18: $3 \times$ (12 across); see 3 down
20: See 16 down
22: See 21 down
23: $2 \times$ (19 down)
25: See 27 across
26: The number of even answers
27: $4 \times$ (25 across)

DOWN

2: See 10 across
3: $2 \times$ (18 across)
4: $5 \times$ (15 across)
5: See 6 down
6: Multiple of 5 down by a single-digited number
7: See 8 across
13: See 1 across
16: $4 \times$ (20 across)
17: See 14 across
19: See 23 across
21: $2 \times$ (22 across)
24: See 8 across

Solutions on page 33.

Edinburgh University Physical Society

are anxious to trace their Life Members, who are asked to contact the secretary, Ian Lacey, at Edinburgh University Physical Society, Department of Natural Philosophy, Drummond Street, Edinburgh, 8.

Fingers Out

The computer at an industrial firm we know bears this notice:

ACHTUNG ALLES LOOKENPEEPERS

Das computermachine ist nicht fur gefingerpoken und mittengrabben. Ist easy schnappen der springenwerk, blowenfusen und poppencorken mit spitssparken. Ist nicht fur gewerken bei das dummkopfen. Das rubbernecken sightseeren keepen hands in das pockets—relaxen und watch das blinkenlights.

Snowflake Tetrahedra

by P. T. Johnstone

A Snowflake Curve is the plane figure produced by the following process: take an equilateral triangle, add on three smaller triangles with their bases on the middle third of each side of the original triangle, then add still smaller triangles on the middle third of each side of the resulting figure, and so on. (See Fig. 1) The best known property of this figure is that its length diverges in the limit, although the area enclosed remains finite. I have given the name 'Snowflake Tetrahedron' to the result of a similar process in three dimensions, starting from a regular tetrahedron.

The obvious analogue of constructing equilateral triangles on each side is to place a tetrahedral cap on each triangular face, with one vertex at the mid-point of each side of the triangle; we then see that the number of faces is multiplied by 6 at each stage, the linear dimensions being halved. Applied once to a tetrahedron, the process gives us a stella octangula as the second term of the sequence. From now on we shall consider the effect of successive elaborations on a pair of faces ABC, ABD of the stella having a concave edge between them, after remarking that the surface of the stella divides into exactly 12 such pairs, and that the 'growths' of two pairs cannot interfere with one another. (I shall use the 'nth stage' to mean the process which takes us from the nth term to the (n + 1)th).

The first problem is that the two tetrahedra to be placed on Fig. 2 at the next stage have a common vertex at E; we must ensure that they do not overlap. Consideration of angles in the plane CED shows us that in fact they just touch, and therefore have a common edge EF (Fig. 3). Similarly, at the third stage, the tetrahedra to be placed on EFG, EFI, AEG and AEI will each share an edge with each of their two neighbours, and these common edges must meet in a point, enclosing an octahedral hole with one vertex at E. It turns out that these holes are a characteristic feature of the later terms, and it is worth considering separately what happens to one at each successive stage.

Since the angle between two adjacent faces of an octahedron is the same as that between ABC and ABD in Fig. 2, we see that the eight tetrahedra to be placed on the inside of the faces will share edges with their neighbours, and thus have a common vertex at the centre of the octahedron. They therefore divide it into eight smaller octahedra, one at each vertex of the original octahedron. We can thus divide the holes in the nth term into two types:

First-generation holes, occupying a volume which was not enclosed by the (n-1)th term.

Higher-generation holes, which were produced at the (n-1)th stage by a hole in the (n-1)th term dividing into six, as above.

We now define the following sequences:

F_n = no. of faces of nth term.

S_n = no. of holes in nth term.

u_n = no. of first-generation holes in nth term.

A_n = area of nth term.

E_n = exterior area (i.e. excluding holes) of nth term.

V_n = volume of nth term.

v_n = volume increment at (n-1)th stage = $V_n - V_{n-1}$.

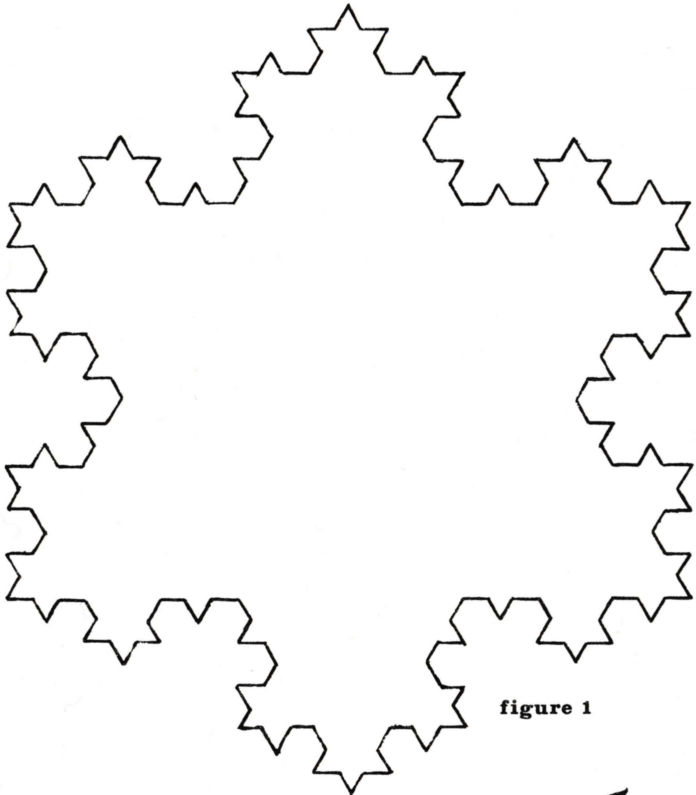


figure 1

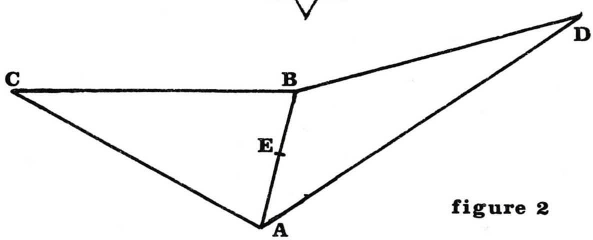


figure 2

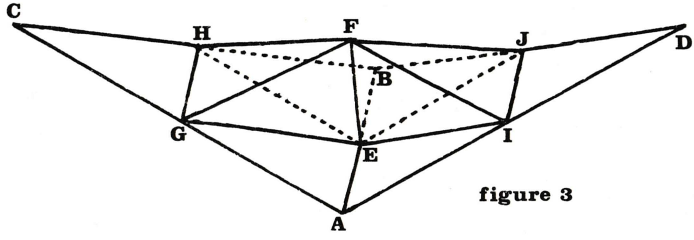


figure 3

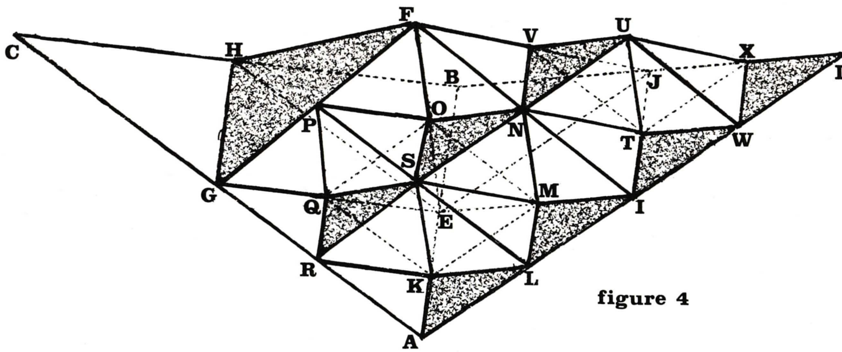


figure 4

Since each face, irrespective of position, gives rise to six new faces at the next stage, it is clear that $F_n = 4.6^{n-1}$ $(n \geqslant 1)$. We now proceed to discuss u_n and S_n; in Fig. 4 it is clear that first-generation holes will be produced at K, M, O, Q and T (one on each side), as well as at the corresponding points on the other side of the plane GFEJ. It is also easy to see that K, M, O and Q, are 'descended' from the point E which produced a hole at the last stage, and that any point which produces a hole at the (n-1)th stage will also produce four points like K, M, O, and Q ready to produce holes at the nth stage.

The points at T are descended from E in a different way; we can see that at each stage the number of 'replicas' of our original pair of faces ABC, ABD is doubled. (After stage 2 we had GHC, GHF and IJF, IJD. We now have NVF, NVU; WXU, WSD and two others.) And each such replica generates one new pair of hole-producing points at the next stage. The number of points like T over the whole figure in the (n-1)th term is $12.\,2^{n-4}$ (since there are 12 replicas of ABC, ABD in the stella octangula)and so the number of holes that they generate at the (n-1)th stage is $12.\,2^{n-3}$. $(n \geqslant 4)$

Since all first-generation holes are of type K or of type T, we have the recurrence relation

$$
\begin{aligned}
u_n &= 4u_{n-1} + 3.\,2^{n-1} \\
&= 4^2 u_{n-2} + 3.\,2^{n-1} + 3.\,2^{n-2}.\,2^2 \\
&= 4^{n-3}u_3 + 3(2^{n-1} + 2^n + \ldots + 2^{2n-5}) \\
&= 3.\,2^{n-1}(2^{n-3} - 1)/(2 - 1) \text{ since } u_3 = 0.\ (n \geqslant 3)
\end{aligned}
$$

Now

$$
\begin{aligned}
S_n &= 6S_{n-1} + u_n \\
&= u_n + 6u_{n-1} + 6^2u_{n-2} + \ldots + 6^{n-4}u_4 + 0 \\
&= 3(2^{n-1}.\,2^{n-3} + 6.\,2^{n-2}.\,2^{n-4} + \ldots + 6^{n-1}.\,2^3.\,2^1) \\
&\quad -3(2^{n-1} + 6.\,2^{n-2} + \ldots + 6^{n-1}.\,2^3) \\
&= 3.\,2^{2n-4}((3/2)^{n-3} - 1)/((3/2) - 1) - 3.\,2^{n-1}(3^{n-3} - 1)/(3 - 1) \\
&= 3.\,2^n.\,3^{n-3} - 3.\,2^{2n-3} - 3.\,2^{n-2}.\,3^{n-3} + 3.\,2^{n-2} \\
&= 3.\,2^{n-2}(3^{n-2} - 2^{n-1} + 1)\ (n \geqslant 2)
\end{aligned}
$$

We may check the first few terms of this sequence by actual enumeration; they are 0, 0, 24, 288, 2400... for n = 2, 3, 4, 5, 6...

Next we consider the volume and area properties of the sequence. Since linear dimensions decrease by a factor of 2 at each stage, the volume increment is $v_n = F_{n-1} . 8^{1-n} . V$ ($n \geqslant 2$), where V is the volume of the original tetrahedron. Thus $v_n = V . \frac{1}{2} . (\frac{3}{4})^{n-2}$. The total volume is given by

$$V_n = V(1 + \tfrac{1}{2} + \tfrac{1}{2} . \tfrac{3}{4} + \ldots + \tfrac{1}{2}(\tfrac{3}{4})^{n-2})$$

$$= V(1 + \tfrac{1}{2}((\tfrac{3}{4})^{n-1} - 1)/(\tfrac{3}{4} - 1))$$

$$= V(3 - 2(\tfrac{3}{4})^{n-1}). \quad (n \geqslant 1)$$

Thus the volume tends to a limit 3V.

The surface area $A_n = F_n . 4^{1-n} . A$ ($n \geqslant 1$), where A is the area of one face of the original tetrahedron. This clearly diverges; but if we consider only the external surface of the figure, we find

$$E_n = 4^{1-n} . A(F_n - 8S_n)$$

$$= 4^{1-n} . A(4 . 6^{n-1} - 4 . 6^{n-1} + 12 . 4^{n-1} - 12 . 2^{n-1})$$

$$= 12A(1 - (\tfrac{1}{2})^{n-1}) \quad (n \geqslant 2),$$

which tends to a finite limit 12A, or 3 times the area of the original tetrahedron. Furthermore, the total volume of the holes $\alpha\ 8^{-n}S_n$, which tends to zero, and so the figure obtained by 'filling in' the holes has a finite limiting volume 3V and a finite limiting area 12A. Our next problem is to identify this limiting figure.

First, we see from the similarity of the skew quadrilaterals CHFG, CBDA that C, F, D are collinear. The point U also lies on this line, and in fact points of the figure are ultimately dense on it. However, CD never contains interior points of an added tetrahedron, and so must lie in at least one face of the limiting figure. The twelve lines like CD are the edges of a cube (the circumscribed cube of the stella octangula).

Next, from the similarity of triangles AEF and AKS, we see that S lies in the plane ACD; and hence the intersection of the figure with this plane is a square mesh of lines (ALIWD, RSNU, GPF; ARGC, LSP, INF, WU) which grows twice as fine at each successive stage. Since this plane is part of one face of the cube, it can be seen that almost every point of the cube is ultimately inside the snowflake figure; and since no point of the figure ever lies outside it, we may say that the limiting snowflake tetrahedron is indeed this cube.

As a check, we may calculate the volume and area of the cube. Its volume is indeed 3V, but the area at $4\sqrt{3}$ A is too small by a factor of $\sqrt{3}$; this, however may be expected from the fact that every exterior face is inclined at an angle cos $(1/\sqrt{3})$ to the appropriate face of the cube. The fact that E_n varies at all is due to the 'edge effects' of faces like NVF and NVU, which have to be counted on two faces of the cube.

Regular Polytopes

by D. G. Hayes

A polytope is the 4-dimensional equivalent of a 3-dimensional polyhedron, i.e. it is a 4-dimensional body bounded by 3-dimensional polyhedral cells. There are six regular polytopes: the simplex, the tesseract, and four others which I shall refer to as the 16-cell, the 24-cell, the 120-cell and the 600-cell.

The simplex, tesseract and 16-cell are the 4-dim equivalents of the 3-dim tetrahedron, cube and octahedron. In any number of dimensions there are solids corresponding to these three polyhedra. In more than 4 dimensions, they are the only regular solids.

The vertices of the tesseract, 16-cell and 24-cell can easily be expressed in cartesian coordinate form.

The 8 points

$$(\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), (0, 0, \pm 2, 0), (0, 0, 0, \pm 2) \qquad \text{(i)}$$

are the vertices of a 16-cell. The 8 points

$$(\pm 1, \pm 1, \pm 1, \pm 1) \text{ with an even number of signs taken } + \qquad \text{(ii)}$$

form another 16-cell, and the 8 points

$$(\pm 1, \pm 1, \pm 1, \pm 1) \text{ with an odd number of signs taken } + \qquad \text{(iii)}$$

form another. If we take any two of the three sets of points (i), (ii) and (iii), the 16 points thus chosen are the vertices of a tesseract. The 24 points of (i), (ii) and (iii) are the vertices of a 24-cell.

The 120 vertices of the 600-cell can also easily be expressed in coordinate form. They are the points

$$(3 + \sqrt{5}, \sqrt{5} - 1, \sqrt{5} - 1, \sqrt{5} - 1)$$

$$(3 - \sqrt{5}, \sqrt{5} + 1, \sqrt{5} + 1, \sqrt{5} + 1)$$

$$(2\sqrt{5}, -2, 2, 2)$$

$$(4, 4, 0, 0)$$

together with the points similarly expressed but with

(i) the coordinates in a different order, or

(ii) an even number of the coordinates multiplied by -1, or both.

Every regular polytope, andindeed every polytope, has what is known as a dual polytope. The meaning of the word dual is best shown by an example. If we take any tesseract, then

(i) the centres of its cells are the vertices of a 16-cell,

(ii) the centres of its faces are the midpoints of the edges of a 16-cell,

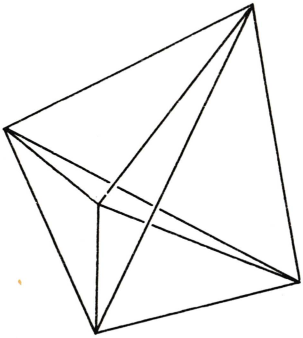


figure 1

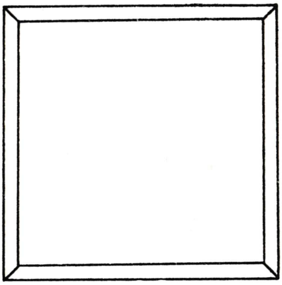


figure 2(a)

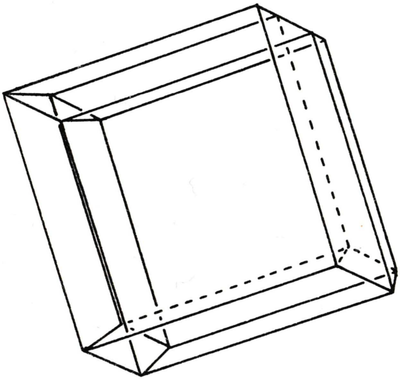


figure 2(b)

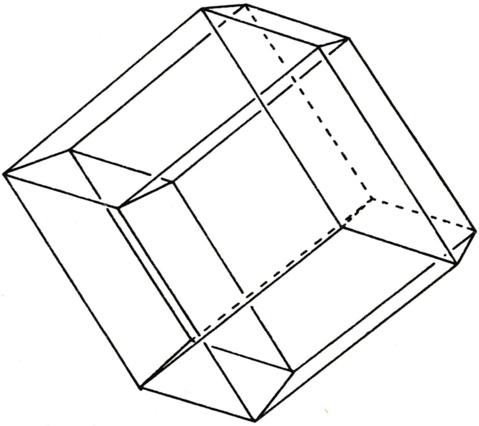


figure 2(c)

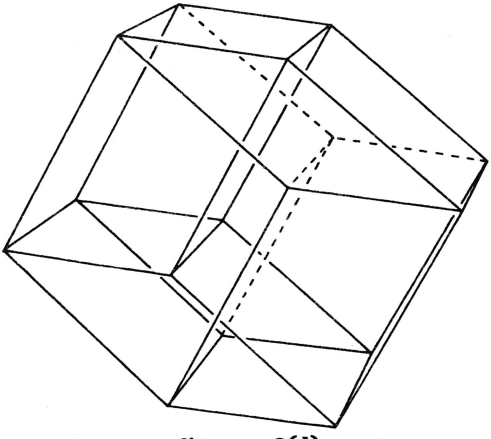


figure 2(d)

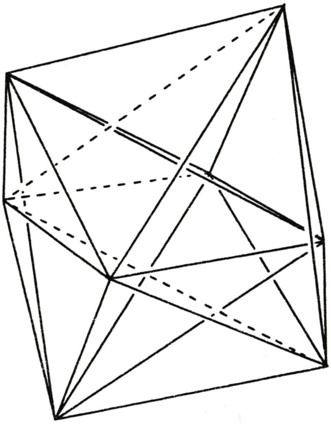


figure 3

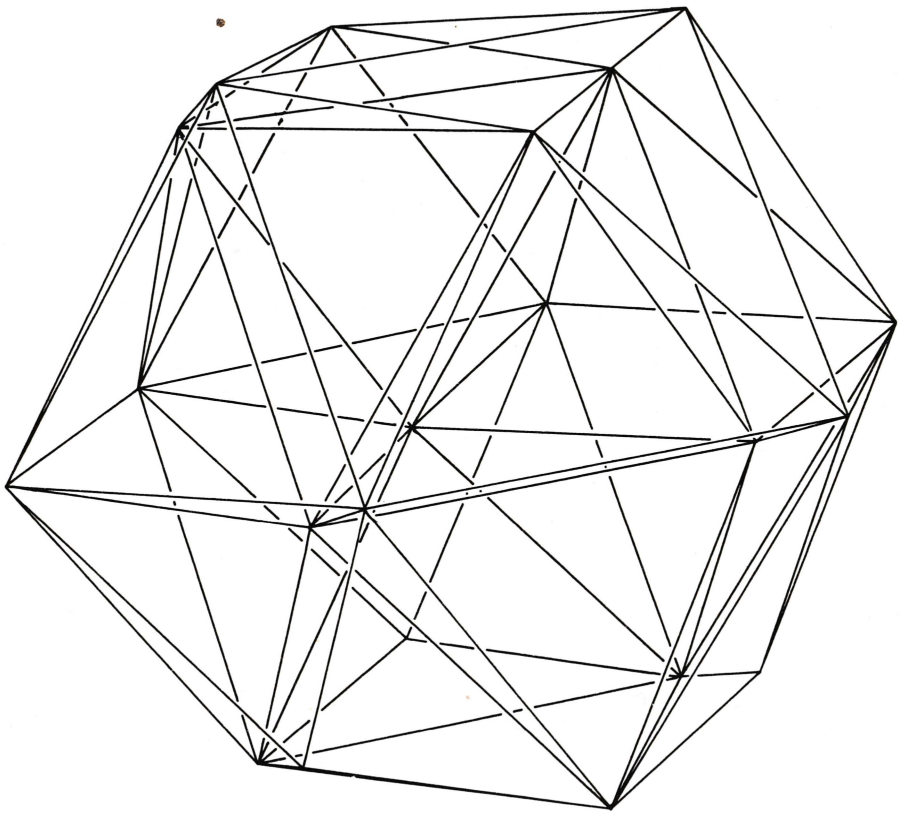


figure 4

(iii) the midpoints of its edges are the centres of the faces of a 16-cell,

(iv) its vertices are the centres of the cells of a 16-cell.

The 16-cell is said to be the dual of the tesseract, and vice versa.

Some of the properties of the regular polytopes are given in the following table.

Polytope	Dual	Shape of cell	Number of cells	Number of faces	Number of edges	Number of vertices
simplex	simplex	tetrahedron	5	10	10	5
tesseract	16-cell	cube	8	24	32	16
16-cell	tesseract	tetrahedron	16	32	24	8
24-cell	24-cell	octohedron	24	96	96	24
120-cell	600-cell	dodecahedron	120	720	1200	600
600-cell	120-cell	tetrahedron	600	1200	720	120

For any polytope we have the relation

$$F + V = E + C$$

where C, F, E, V are the numbers of cells, faces, edges and vertices of the polytope respectively. Compare $F + V = E + 2$ for polyhedra, and $V = E$ for polygons.

An n-dimensional tesselation is a way of filling n-space with n-dim solids. It may be regarded as the boundary of an infinite (n + 1)-dim solid. The dual of an n-dim tesselation is defined in the same way as the dual of an (n + 1)-dim solid. Thus the dual of a given 4-dim tesselation has for example one vertex at the centre of each polytope of the given tesselation. Except when $n = 2$ or $n = 4$, the only regular n-dim tesselation is the n-dim equivalent of the 3-dim tesselation of cubes. In 4 dimensions, if a regular polytope is to be able to form a regular tesselation, the angle between the 3-spaces of two adjacent cells of the polytope must be of the form $2\pi/n$. n will be the number of polytopes meeting at each face of the tesselation. There are three polytopes which satisfy this requirement. They are the tesseract, (for which the angle is $\pi/2$), the 16-cell ($2\pi/3$) and 24-cell ($2\pi/3$). We thus have 3 regular 4-dim tesselations. The tesselation of tesseracts is its own dual; the other two are each other's duals.

Two-dimensional representations of the simplex, tesseract, 16-cell and 24-cell are shown in figs 1, 2, 3, 4. The tesseract is shown being rotated about the plane containing the points (0, 0, 0, 0), (1, 4, 6, 3), (3, 5, 2, 7). The two-dimensional representations are made to look like two-dimensional pictures of three-dimensional representations of the 4-dimensional polytopes. In all but one of the pictures, the edges at the 'back' of the polytope are shown dotted. The 'back' of the polytope means the back with respect to projection from 4-space into 3 space. In the picture of the 24-cell (fig. 4) the edges at the back have been ommited altogether.

The pictures were obtained with the aid of the computer Titan in the following way. First the coordinates of the vertices of a polytope symmetrically placed with respect to the coordinate axes were found. Then a suitable rotation matrix was found. The matrix was applied to the vertices of the polytope, to give the coordinates of the vertices of the polytope after it had been rotated. For any vertex (x, y, z, w) of the polytope the corresponding point of the 2-dim picture is $(x/(1 - hz), y/(1 - hz))$. The purpose of the factor $1/(1 - hz)$ is to make the picture look like a 3-dim representation of the polytope, viewed from the point $(0, 0, 1/h)$. h was given the value 0. 12 and the circumradius of the polytope was equal to 1.

The Archimedeans

The Archimedeans have had another successful year, with good attendance at the evening meetings. The talk by Mr. S. R. Deards on 'Graph Theory and Electrical Networks', Professor Coulson's address on 'The Relation between Pure and Applied Mathematics' and a Brains Trust (reported elsewhere in this magazine) with a panel of members of the teaching staff were particularly notable. Amongst the Tea Meetings, special mention should be made of Mr. R. M. Damerell's talk on 'Patchwork Quilts', after which the four colour problem remains unsolved, and 'Extra-Marginal Activities' delivered by Mr. H. L. Montgomery (title suggested by Dr. J. Conway), which was followed by a demonstration of infinitely intelligent reasoning.

The Problems Drive was again a success, although the Invariants were unable to attend, owing to the lack of transport facilities between Oxford and Cambridge. The computer Group has had an extremely active year, and has even greater plans for the future. The Puzzles and Games Ring is thriving once more, and the Bridge and Music Groups have continued to hold regular meetings. The Tiddlywinks Match against the Dampers, and the Punt Party were both very successful, although the latter only just escaped destruction by the weather. Many of the visits organised had to be cancelled through lack of support, and those that took place were poorly attended. The bookshop has continued to provide a service to undergraduate mathematicians, and has had another good year.

This year's evening meetings will include an address from Sir Richard Woolley, Astronomer Royal, who will talk about 'The Box Orbits of Stars in the Galaxy', on 29th November. We hope also to have a Careers Meeting on the lines of those held in previous years, and arrange a visit to Oxford during the Michaelmas Term. The Problems Drive will again be held in February; the Invariants will be challenged to take part in it. A Tiddlywinks Match against the Dampers and a Punt Party to Grantchester will be arranged as usual. Visits are being arranged for next year, and if there is enough support there may be trips to some London theatres.

It is hoped that all members of the Society will find something in the coming year's programme, but all suggestions as to possible changes in future years will be welcomed. These should be made either directly to the Secretary, or through the suggestions book kept in the Arts School.

P. E. SMITH, Secretary.

Some Problems

Set by Dr. H. T. Croft

(1) In this question the bodies are to be thought of as solid and hence nonoverlapping; also 'touch' will here mean 'having a common surface of strictly positive area'. Guess configurations which give the maximum values of the following (you are not expected to prove that they are maximum):

(a) the number of disjoint translates of a given cube C which touch C.

(b) the number of disjoint cubes, each touching all the others.

(c) the number of disjoint tetrahedra (not necessarily the same shape), each touching all the others.

(2) $[a_i, b_i]$, $i = 1, \ldots\ldots, n$ are a finite set of intervals on the real axis, some overlapping others. The total length covered (less than the sum of the lengths of the intervals, of course) is L. Show that one can extract a subset of the set, which are disjoint and cover a length at least $\frac{1}{2}$L.

(3) C is a closed, convex, differentiable curve. Show that there exists at least one nondegenerate triangle inscribed in C, such that a particle can describe it periodically, bouncing elastically at each vertex (i.e. each time it hits C).

(4) (a) S is the set of configurations of 5 points P_i ($i = 1, \ldots, 5$) in an equilateral triangle of side a. Find $\max_S \min_{i, j; i \neq j} (P_iP_j)$.

(b) 5 discs of unit radius are placed non-overlapping in an equilateral triangle. How large must it be?

(5) Prove that for $n = 5$ and $n \geqslant 7$, a regular n-gon cannot be inscribed in the Cartesian square lattice, with each vertex on an integral point.

(6) A convex 3-dimensional body possesses an internal point G with the property that every section through G is circular. Prove that it is a sphere.

(7) (a) (Quickie) Guess how many sections of a regular tetrahedron have the centroid of the tetrahedron as centroid of the section.

(b) Restricting consideration to triangular sections, but dropping the assumption that the tetrahedron is regular, prove that no other sections have the property of (a).

Solutions on page 32.

The Brains Trust on Mathematics Teaching, March 1968

A Brains Trust on Mathematics Teaching in Cambridge, organised jointly by the S.R.C. Committee on Mathematics Teaching and the Archimedeans, was held in the Arts School on Friday, 8th March 1968. The questions were answered by a panel consisting of Dr. J. C. Polkinghorne, Dr. J. H. Williamson, Dr. G. A. Reid and Dr. J. P. Dougherty, and the discussion was chaired by Mr. C. J. Myerscough, the chairman of the S.R.C. Committee.

Question 1 was: 'Do the panel think it would be desirable for undergraduates to have the possibility of a certain amount of specialisation in Part IB?'

Dr. Polkinghorne thought that the Tripos ought to present a series of graded options; he could see no objections to this suggestion. Dr. Williamson felt that either half of the present IB course was probably unsatisfactory for those whose main interests were on the other side; his own idea was to have a first year comprising largely mathematical methods courses, a second year in which one could delve deeper into Pure or Applied Maths, or both, and a third year similar to the present one. Dr. Reid agreed that the Applied he had done in Part IB was not much use to him now, but felt that the present system was desirable on the grounds of a general education. Dr. Dougherty also felt that two years of non-specialised work were about right, and suggested that it would be unwise to speculate on this subject while school curricula were still under extensive discussion.

Supplementary comments from the floor included a complaint (with which Dr. Williamson agreed) that the Pure and Applied Departments did not collaborate enough in planning the Part I courses. Dr. Williamson pointed out that the degree structure (which allows 3 years for one's B.A., and 3 more for a Ph.D.) virtually dictated the level at which specialisation should start; if 8 years overall were available for a Ph.D., it would be better to have no specialisation at all in the B.A. courses. However, the Cambridge timetable is generally taken as desirable by other universities.

Mr. Myerscough felt that the whole Tripos course was geared more to the needs of those who intended to do research, than to the needs of those who left on taking their B.A.; it was agreed that the latter probably did not need all the pure or all the applied they learned in Part IB. To a suggestion that the course would have to be changed sooner or later, Dr. Polkinghorne replied that a great many changes had been made in the last 10 years, and he hoped that the present course would last for some time yet. Dr. Reid said that the desirability of this particular change was still not proven, at least as far as he was concerned.

Question 2 was: 'Should lecturers distribute detailed summaries of their courses?'

Dr. Reid, who had tried this last term, recalled the incident during his own undergraduate days when Dr. Taunt had distributed notes for a course of 9 a.m. lectures, and one drowsy student had fallen off a bench halfway through a lecture. If duplicated notes had a fault, it was that they discouraged people from using books. Dr. Polkinghorne found the word 'summaries' of the question misleading; he himself felt that students should read books—they are excessively spoon-fed at the moment—and he had distributed detailed literature about which books to use before his courses. He felt that lecturers should choose the most suitable book and adopt its notation and approach, rather than invent their own. Dr. Dougherty likes to be able to gesture at the key results he has written on the board, but he had nevertheless decided to experiment with duplicated notes next year. He thought that spontaneity might be lost if students sat through the lectures simply ticking off the appropriate paragraphs of their duplicated notes.

One supplementary question asked whether there was room for a general course on how to study, at the beginning of Part IA; Dr. Williamson said that this was the job of Directors of Studies. It was suggested that it might be better to distribute notes after the lectures rather than before, and that notes might be produced on a semi-commercial basis. Dr. Reid thought that, if it was definitely desirable to use, say, 1000 stencils a term on duplicated notes, we should go ahead and do it regardless of cost. It was suggested that lists of key theorems, definitions, etc. might be an acceptable halfway stage.

Question 3 was : 'Would it not be beneficial if some or all of the Tripos exams. were replaced by less formal tests of ability?'

Dr. Williamson said that the present 3-hour exams. with unseen questions were a standard pattern, and produced fairly satisfactory results. Dr. Polkinghorne pointed out that there were considerable difficulties when large numbers of people were involved, and when lecturers were not the same people as examiners. Dr. Dougherty said that experiments had been made in Part III, but candidates tended to prefer the standard potboilers rather than the imaginative questions. Dr. Reid said that anyone who could memorise nine lecture courses must be a good mathematician, and that, although the questions in Part III were mostly bookwork, this was not as bad as it sounded.

A questioner complained that Part III students had little opportunity for writing essays before the exams. themselves, but Dr. Reid said that essays were judged on their mathematical content rather than their literary style. The panel were agreed that the present exams. did very well in sorting out the firsts, seconds and thirds; Dr. Williamson said that outside organisations attached less importance to exams. than we tended to think, although Mr. Myerscough thought that certain dividing lines were very important in industry.

On the question of whether it was necessary to have one exam. every year, Dr. Polkinghorne said that he disapproved of the Oxford system which concentrated most of the worry into a single exam. at the end of the course. It was claimed that in Cambridge this worry occurs once a year, but Dr. Dougherty professed himself unable to understand the mentality of those who worry about exams. Dr. Williamson mentioned the American system where each course is examined separately, but did not think it particularly advantageous.

Question 4 was: 'The present IA course on computing and numerical analysis seems inadequate even as an introduction. Could it not be extended?'

Dr. Dougherty said that numerical analysis was a perfectly good subject, and he would not object to having more of it; but he felt that courses on programming were out of place in the Tripos—they had the same relation to mathematics as the art of glass-blowing did to physics. There was no further discussion of this question.

Question 5 was: 'Could not the lighting and ventilation of the rooms in the Arts School be improved?'

Dr. Polkinghorne had not been aware of this problem, and he pointed out that it was useless to discuss expensive alterations in view of our impending move. But he was willing to consider any inexpensive changes which the S.R.C. Committee could suggest. Mr. Myerscough said that roller blackboards would be an improvement on the present type. Dr. Reid said that the Arts School compared very favourably with the D.P.M.M.S.

The discussion was then thrown open, and a number of points were raised. These included a suggestion that the pure courses in Part III could be more systematically arranged, and that the lecture list should state clearly how many units each course was worth; the panel agreed, but felt that the number of units required for Part III at the moment was not excessive.

A questioner asked why the optional IA physics courses displaced applied rather than pure courses. Dr. Polkinghorne pointed out that these courses were really only for those intending to change to Natural Sciences after a year, and that the scientists liked people with a good pure mathematical grounding. Dr. Dougherty thought that the whole idea of a physics option was obsolete, in view of recent improvements in the Part IA applied syllabus.

There was some discussion of the new Part II supervisions; Mr. Myerscough said that the replies to the S.R.C. questionnaire had not been analysed yet, but most people seemed to be satisfied with the new system. Dr. Polkinghorne said that the change was welcome to supervisors, who had less repetitious work to do, and that the system aimed at producing homogeneous classes of 7 to 8 people. It had been found that follow-up supervisions with two people were often unproductive. Mr. Myerscough said he preferred supervising the higher-ability groups, but Dr. Polkinghorne disagreed.

It was said that some people had difficulty in finding supervisors for the IA Probability and IB Statistics courses; Dr. Williamson said that the Statistical Laboratory had offered to help with this, but that the IA course counted as pure. A suggestion that pure and applied questions be mixed within each paper in Parts IA and IB was welcomed by Dr. Reid as a simple way of introducing options without damaging the present course structure. Dr. Polkinghorne recalled the rumours that Mr. Swinnerton-Dyer had done all but one of the pure questions and none of the applied when he took the Tripos.

The speakers were then thanked by Mr. A. C. Norman, the President of the Archimedeans, and Mr. Myerscough declared the meeting closed.

P. T. JOHNSTONE

Problems Drive 1968

Set by A. K. Manning & B. J. Scorer

(1) A cylindrical hole is bored through a solid sphere. The generators of the cylinder are 6" long. What volume of the solid sphere is left?

(2) Solve the following multiplication sum where A, B, C, D, E, F are distinct digits in the scale of ten.

$$\begin{array}{r} \text{A B C D E F} \\ \text{E} \\ \hline \text{F A B C D E} \\ \hline \end{array}$$

(3) 1, 18, 3, 8, 9, 13, ...add 3 terms

3, 11, 17, 21, 27, ...add 2 terms

301, 477, 602 . . . . . .add 2 terms

0, 2, 5, 9, 12, 2, 1, ...add 2 terms

(4) 42 million men, each six feet tall, stand at intervals of one yard around the equator. A light elastic string of modulus 7 lbs. wt. lies taut at their feet. If they all pick it up and put it on their heads, what force does it exert on each of them?

(5) An explorer sets out from his hut and makes the traditional journey one mile south, then one mile east, and then one mile north to return to his hut. What can be said about the position of the hut?

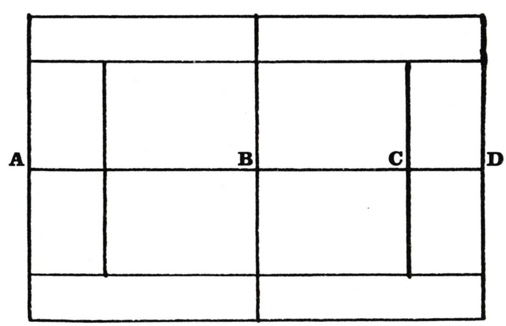
(6) Cylindrical tins of baked beans are to be packed in square cardboard boxes. The tins have circular cross-section of unit diameter, and the boxes have sides of integral length and height equal to that of the tins. They may be packed in either a square or triangular pattern. For large boxes the triangular pattern is most efficient. What is the largest box for which the square pattern is more efficient?

(7) Solve in non-zero integers m, n, and r:-

$$n^{m+n} + m^m(m + n)^m = (2m + n)^n$$

$$r^3 + (r + 2)^3 + (r + 4)^3 = r^4 + 2$$

figure 1



(8) A certain tennis player serves from the point A to the right hand court. The ball may be considered as travelling in a straight line leaving his racket at a height three times that of the net. It is aimed at a point of the line BD, all points being equally likely. BC = (2/3)BD. (Figure 1). What is the probability of his scoring a double fault?

(9) Selwyn College second boat is 30 ft. long. Each oar is 12 ft. long and is pivoted three feet from the end at which each oarsman pulls with a force 50 lbs. wt. On one outing no. 3 cannot row. Assuming that the cox uses the rudder only while the oars are in the water and that the oars are approximately perpendicular to the boat and that the boat pivots about its centre, at what angle to the line of the boat should the cox hold the rudder in order to steer a straight course? The normal force on the rudder when at angle θ is ($40\sin\theta$) lbs. wt.

Answers on page 33.

Impossible Objects

by D. R. Woodall

It is not easy to produce an exact definition of an 'impossible object', since, being 'impossible', it cannot exist as an object, but only as a drawing! The defining property of such a drawing is that it appears at first sight to represent a three-dimensional object, but that a closer examination reveals features which are incompatible with this interpretation. An 'impossible object' could perhaps be defined to be what such a drawing would represent, if it could. Such drawings, much beloved of wags and practical jokers, have considerable psychological interest, compared with which their mathematical construction is deceptively simple.

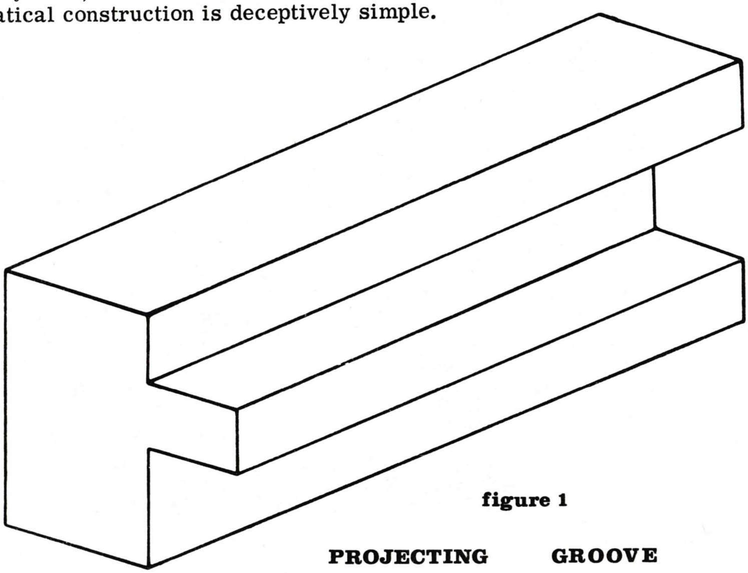


figure 1

PROJECTING GROOVE

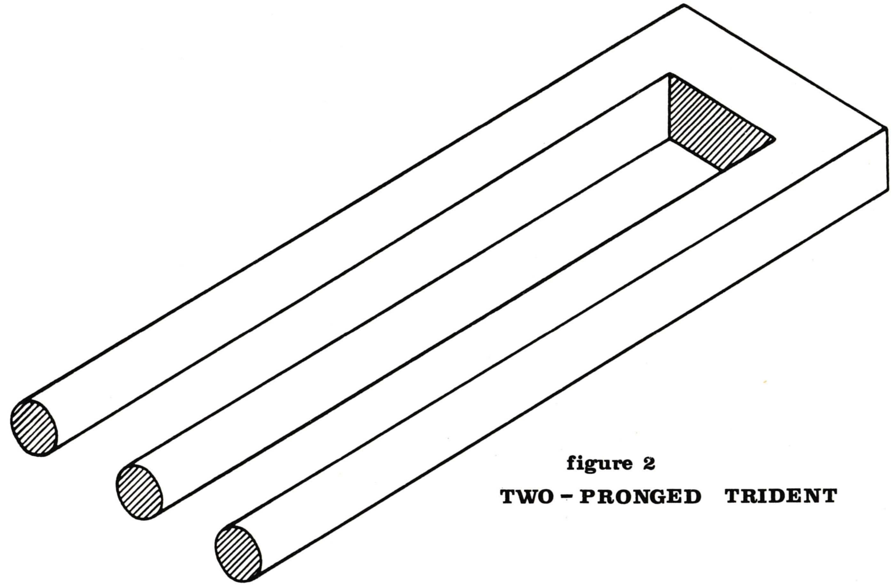


figure 2

TWO - PRONGED TRIDENT

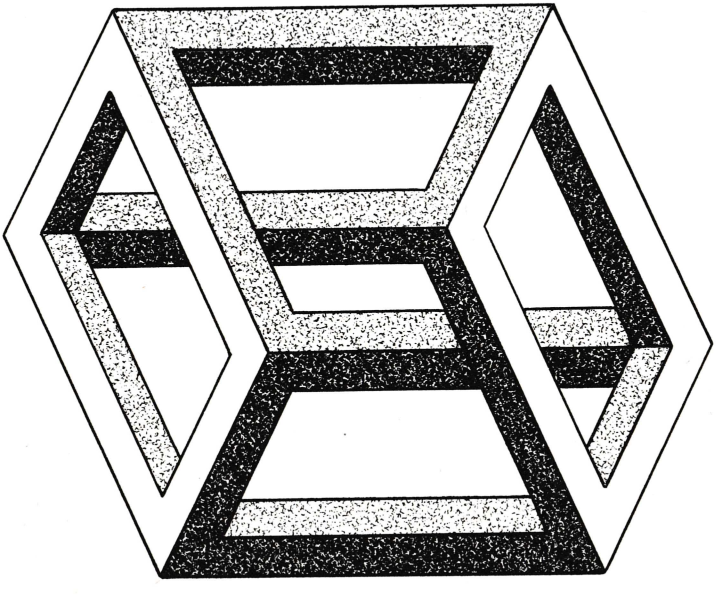


figure 3

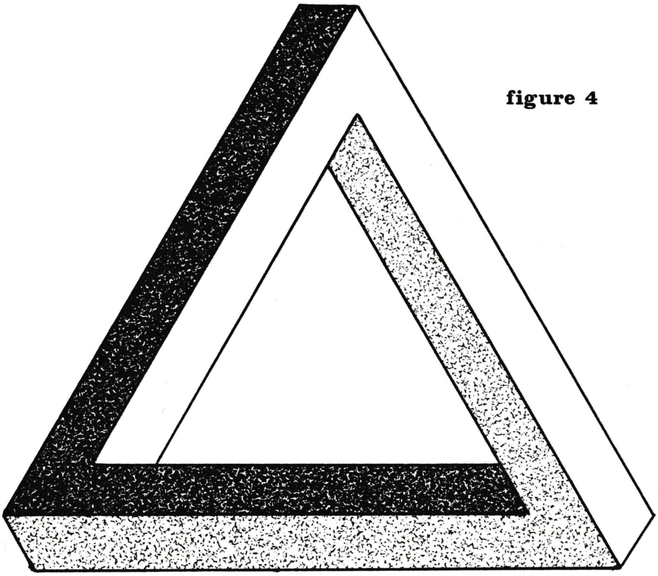


figure 4

The five accompanying figures are relatively simple drawings of this nature. In each case it is possible to draw a line somewhere through the middle of the picture in such a way that each half of the picture makes sense, as a two-dimensional representation of part of a three-dimensional object, but the whole picture, formed by putting the two halves together, does not make sense. The result of this, as pointed out by Penrose and Penrose [2], is that as one's eye moves round the picture there comes a point at which the picture demands a sudden change in interpretation. More complicated examples may include several 'sensible' components put together in a non-sensical way, and when one tries to take these in, more frequent changes in interpretation are required.

These drawings seem to fall naturally into three groups or orders. There is no sharp dividing line between these, and a complicated figure may include components of more than one order. Nonetheless this grouping provides a useful classification.

Drawings of the lowest order are constructed in the obvious way. A drawing is made of some three-dimensional object, and the drawing is cut across the middle. The lines which cross the cut are then continued and completed in a way which is incompatible with the original drawing. Sets of parallel straight lines are particularly useful in this respect, and Figures 1 and 2 show two possible applications of them. Needless to say, there are many more. More complicated drawings of this nature may be constructed by making a drawing of some object, and then altering the drawing of some of the corners, or of crossing-points of struts, so that a part which was originally behind appears to be in front. However, this more complicated process does not yield anything basically new, and the simplest examples are generally the most effective. Indeed, the 'two-pronged trident' has such an appeal that it is perhaps with some regret that we must classify it with the lowest group. But the blatant way in which a prong seems to grow out of nothing, which gives the picture its appeal, itself ensures that the fallacy is very easily spotted. The intriguing property of 'impossible objects' of the highest order, as of logical paradoxes of the highest order, is not just that they are meaningless, but that they are meaningless despite the facts that they appear at first sight to be the reverse, and that it is not easy to see where the fallacy lies. For example, there would be no possibility of shading either of Figures 1 and 2 in the consistent way in which the later figures are shaded—the fallacies are too blatent to allow it.

At first sight Figure 3, due to Escher [1], might be thought to be of the lowest order, as it is obtained from a picture of a cube by altering some of the corners and crossing points. But the alteration has been carried out systematically so as to exploit the age-old 'reversing cube' illusion, and this picture belongs to an altogether more subtle order of 'impossible object'. A rule for constructing a picture of this type might run as follows. Make a rough outline sketch of some simple object. In general there will appear to be more than one possible interpretation of the sketch, there being at least one part of the picture which can be imagined either as rising out of the paper or as receding into it. Now fill in the picture in detail, but drawing the corners and crossing-points in such a way that one part of the picture imposes one interpretation on the ambiguous portion, while the remainder of the picture imposes the other interpretation. It will now be possible, if the drawing has been filled in with sufficient skill, to shade it consistently, as Figure 3 is shaded.

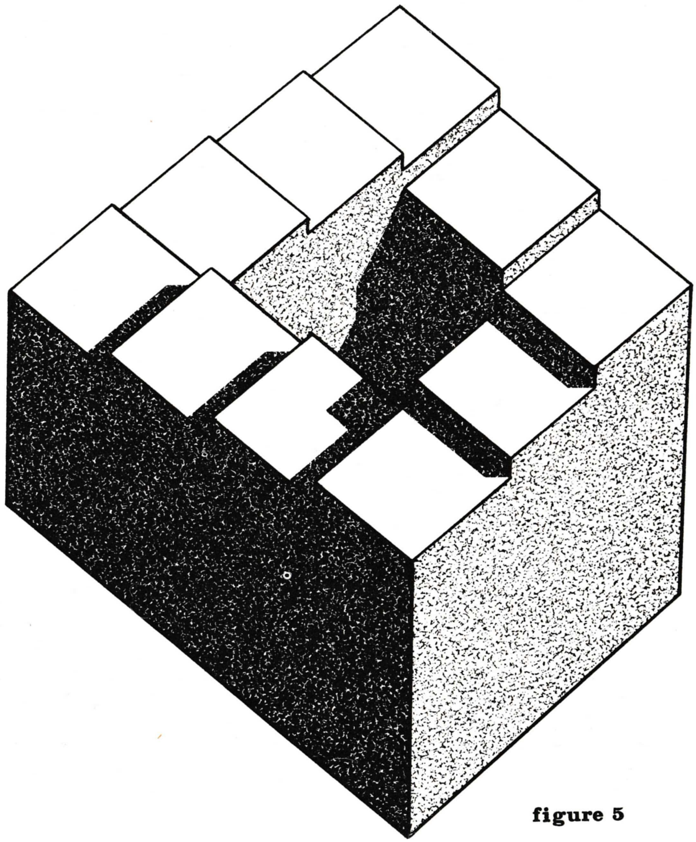


figure 5

Figures 4 and 5, due to Penrose and Penrose [2], are of a higher order still. They are formed by taking an object and drawing it from such an angle that two distinct parts of it appear to join up in a continuous manner. For example, for Figure 4 one could have taken three mutually perpendicular joined struts, AB, BC and CD, say, and for Figure 5 a staircase round a square well with four steps in each of the middle two flights, three steps in the top flight, and only two steps in the bottom flight. (The picture in [2] on which Figure 5 is based is an actual photograph obtained in this way). If the manner in which the ends join up is really continuous, it will be possible to shade the resultant figure consistently, and to draw it in perspective (as Figure 4 is, in the Penroses' article). Moreover, it will be impossible to tell where the join was, as in general there will be many possible points at which the join could have been.

I must leave the reader to decide the following question. Is it possible to construct three-dimensional objects which are representations of 'impossible objects' in four dimensions? If so, what do such objects look like, and is it possible to classify them in any way analogous to the above?

References.

1. The Graphic Work of M. C. Escher, Oldbourne Book Co. Ltd. (1961).
2. L. S. Penrose and R. Penrose, Impossible objects: a special type of visual illusion, British Journal of Psychology **49** (1958), 31-33.

The Cambridge SRC Maths Teaching Committee

This body was set up at the beginning of the Lent Term by a number of mathematicians who considered it worthwhile to improve the already fairly good communications between teachers and taught in the Faculty of Mathematics. An early policy decision was that we should not attempt wide-ranging surveys of the whole student mathematician body, but should rather concentrate on specific points of concern.

Accordingly, we chose two topics for surveys in the Lent term in conjunction with the Faculty Board, notably Dr. Polkinghorne, whose advice and suggestions have been a great help to us. One was of reaction to the change-over from small supervisions to classes for Part II that has taken place over the last year in many colleges. The other was aimed at finding the precise reasons why so many give up maths after Part IA or IB. Reports on the above surveys are now available, price 1s. each, from N. Maclaren, Corpus Christi College, Cambridge.

At the end of the Lent term we held a most successful 'Brains Trust', which is described in greater detail elsewhere in this magazine. Activities during the Easter term have been restricted, but it is hoped that next year there will be enough support for work to continue. So if you have ideas on what should be done to improve the teaching here, and care enough to want to have them implemented, contact Nick Maclaren, the new chairman, at Corpus.

C. J. MYERSCOUGH, Outgoing Chairman.

La coloration des cartes

par Blanche Descartes et Rose Descartes

La conjecture célebre des quatre couleurs, qu'on puisse colorer tous les pays d'une carte de geographie quelconque de telle manière que deux pays contigus aient toujours des couleurs différentes, est une hypothèse fascinatrice pour tous les mathématiciens. Néanmoins tout le monde la trouve bien difficile, parce qu'il s'agit de la topologie planaire, c'est-à-dire d'un espace de 2 dimensions. Nous montrerons ci-dessous qu'on peut la réduire à une hypothèse unidimensionelle. Par conséquent, la démonstration ou la réfutation de la conjecture devient chose tout simple, et nous la laissons au lecteur.

Choisissons 4 couleurs, par example azure, blanc, couleur de chocolat, et rose (A, B, C, R). Nous appelons une suite finie c(0), c(1), . . . , c(n) des quatre couleurs une suite cartésienne si elle satisfait aux conditions suivantes: (i) deux couleurs adjacentes sont toujours différentes, c'est-à-dire, $c(r) \neq c(r+1)$; (ii) la suite $c'(s) = c(2s)$, $(s = 0, 1, \ldots)$ est aussi cartésienne. Par example, la suite ABCARABAC est cartésienne, parce que deux couleurs adjacentes ne sont jamais égales; et si l'on efface les éléments alternatifs, on obtient une autre suite $c' =$ ACRBC possêdante la même propriété; une soustraction des éléments alternatifs de c' nous donne $c'' =$ ARC, ayante la même propriété; et ainsi de suite. Soit

$$0 \leqslant i_0 < i_1 < \ldots < i_m \leqslant n$$

une suite croissante arbitraire d'entiers non-négatifs.

La conjecture cartésienne dit: Etant donnés les numéros n, i_r, il existe une suite cartésienne possédante la propriété que la suite partielle $d(s) = c(i_s)$ est aussi cartésienne. Par example, si $n = 8$, $u_0 = 0$, $u_1 = 1$, $u_2 = 4$, la suite ci-dessus $c(0) = A$, $c(1) = B$, $c(2) = C$, $c(3) = A$, $c(4) + R, \ldots$, est cartésienne, et la suite partielle $c(0) = A$, $c(1) = B$, $c(4) = R$ est aussi cartésienne.

Nous disons que les deux conjectures sont équivalentes l'une à l'autre. Pour la démontrer, prenons un exemple (fig. 1) d'une carte de geographie. Nous voulons démontrer d'abord que la vérité de la conjecture cartésienne implique l'existence d'une coloration de la fig. 1 en 4 couleurs. Les lettres F, G, H, I, correspondent aux pays de la figure, et la lettre E à l'extérieur. Il nous convient d'employer le réseau dual (fig. 2). Chaque point, au 'noeud' de ce réseau correspond à un pays de la carte de la fig. 1, si 2 pays E, F sont contigus, on joint les noeuds E, F de la fig. 2 par une seule courbe ('arête') EF. Donc il faut colorier les noeuds du réseau, avec la condition que les deux noeuds de chaque arête aient des couleurs distinctes.

Selon sa définition, les pays de la carte de la fig. 2 ont au moins 3 côtés. On ajoute des autres arêtes (FH, FI de la fig. 3) pour rendre tous les pays triangulaires. Chaque coloration éventuelle de la fig. 3 nous donnera tout de suite une coloration de la fig. 2, qui en est une partie. Il serait possible qu'il y avait sur une figure telle que la fig. 3 un triangle 'non-élémentaire', c'est-à-dire que ni l'intérieur ni l'extérier sont vides. Cependant, en ce cas-ci on pourrait colorier indépendemment l'intérieur et l'extérieur. Donc il suffit de considérer seulement les cas sans triangle non-élémentaire, comme dans la fig. 3. Un théorème de Hassler Whitney (Ann. Math. (2), **32**, 1931, 378-390) nous assure l'existence d'un circuit EFGHIE qui traverse chaque noeud une seule fois. Nous dessinons la fig. 3 à nouveau, dans la fig. 4, de telle manière que la suite EFGHI du circuit (sauf la dernière arête IE) devient une ligne horizontale. L'extérieur du circuit se trouve au-dessus de la ligne, et l'intérieur endessous.

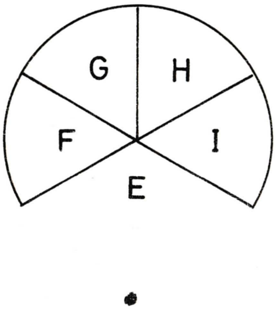


Fig. 1

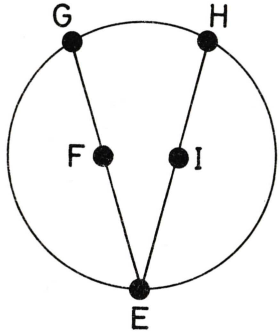


Fig. 2

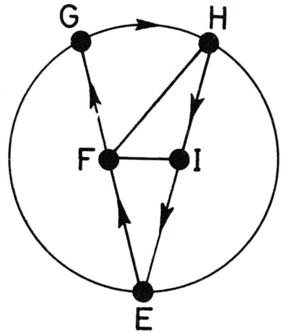


Fig. 3

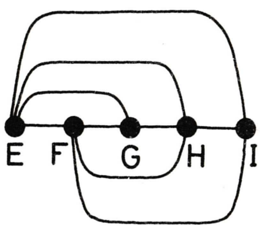


Fig. 4

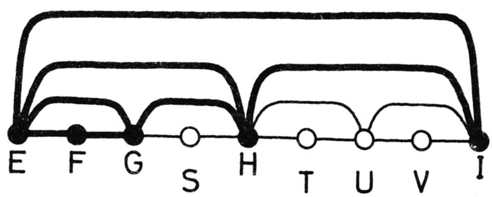


Fig. 5

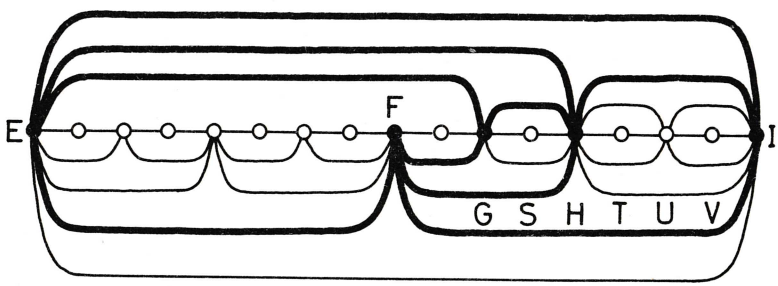


Fig. 6

Considerons d'abord la ligne EFGHI et les arêtes supérieures. Dans la fig. 5 nous y ajoutons des autres noeuds (S, T, U, V) et arêtes (lignes minces) pour la rendre symétrique. Parce qu'il y a un triangle EGH, on y ajoute un triangle HUI, posé symétriquement par rapport au noeud H. Parce qu'il y a triangle EFG, on construit également des nouveaux triangles GSH, HTU, UVI, (et ainsi de suite).

Avec chaque coloration des noeuds de la fig. 5, deux noeuds adjacents de la suite EFGSHTUVI reçoivent des couleurs distinctes, parce qu'ils sont liés par une arête. Si l'on en soustrait les noeuds F, S, T, V, la suite partielle EGHUI que reste aura la même propriété, et ainsi de suite. On voit ainsi que chaque coloration des noeuds de la fig. 5, par example ABCARABAC, nous donne une suite cartésienne, et vice versa.

Réintroduisons ensuite la partie de la fig. 4 inférieur à la ligne horizontale. Faisons la même construction qu'auparavant sur cette partie, avec des nouveaux noeuds et arêtes, au nécessaire. Nous obtiendrons ainsi la fig. 6, dont la fig. 5 sera la partie supérieure. Chaque coloration des noeuds sera une suite cartésienne, puisque la partie inférieure de la figure sera maintenant symétrique, en même façon que dans la fig. 5; de plus, la coloration des noeuds E, F, G, S, H, T, U, V, I, nous donnera en même temps une suite cartésienne, pour la même raison. Or, la conjecture cartésienne affirme précisément l'existence d'une telle suite, et donc d'une coloration de la fig. 4 contenue dans la fig. 6 (lignes épaisses), et donc de la fig. 2.

Nous avons démontré que la conjecture cartésienne implique la conjecture des 4 couleurs. Parce que la conjecture cartésienne, appliquée a la fig. 6, affirme simplement l'existence d'une coloration, la converse est tout de suite évidente.

Solutions to Problems

A Toroidal Twister (Eureka 30 (1967) page 5)

(This solution is not unique)

0	0	1	0	0	0	1	0	0
0	1	2	2	1	2	2	1	0
1	1	2	1	1	1	2	1	1
1	2	0	0	2	0	0	2	1
2	2	2	0	2	0	2	2	2
2	0	1	1	0	1	1	0	2
0	2	0	1	2	1	0	2	0
2	1	0	2	1	2	0	1	2
1	0	1	2	0	2	1	0	1

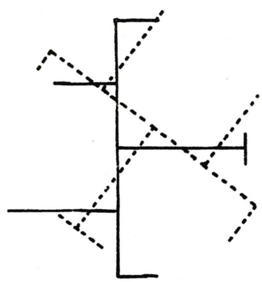


figure 1

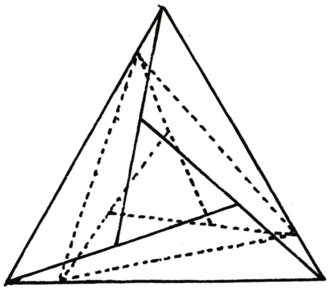


figure 2

Some Problems (Page 17)

(1) (a) 14; let say 4 touch each face $|x| = 1$; 2 each face $|y| = 1$; 1 each face $|z| = 1$.

(b) 6; let them all touch a common plane. Then fig. 1 shows how: the heavy lines denote cubes one side, dotted those the other.

(c) 8; similar to (b): see fig. 2.

I have proved (a), (b), do indeed give the maximum (submitted to J.L.M.S.). For (c) it is unknown whether 8 is the 'right' answer. Baston (Some Properties of Polyhedra in Euclidean Space) has shown that more than 9 is impossible. Fig. 2 is due to Bagemill (AMM. **63** (1956) 328-9).

(2) We may assume: (i) no interval encloses another; (ii) the length covered consists of just one stretch $[A, B]$—otherwise add up. Select a subset of $[a_i, b_i]$, renumbering thus: take $a_1 = A$; then $a_2 =$ that a that is $\leqslant b_1$ and with the greatest b_2; $a_3 =$ that a $\leqslant b_2$ and with the greatest b_3; etc. We get an interlacing subset that totally covers A, B with

$$a_1 < a_2 \leqslant b_1 < a_3 \leqslant b_2 < a_4 \ldots \leqslant b_{r-1} < b_r.$$

Then pick either all the 'even' such intervals or all the 'odd' ones.

(3) Consider the set of all triangles inscribed in C; take one with the longest total length. It is clear: (i) such a bound is attained (by compactness); (ii) all the vertices are distinct (or else the length can be increased); (iii) it has the 'bouncing' property at each vertex (or else the length could be increased by small movements of that vertex along the curve).

(4) (a) $\frac{1}{2}$a. Clearly attainable; and there is nothing better, for if the triangle be cut into 4 similar triangles $\frac{1}{4}$ the size in the obvious way, then (by counting) some pair of points must lie in some one such smaller triangle, and are not more distant than $\frac{1}{2}$a.

(b) Side $4 + 2\sqrt{3}$. Their centres fall in a smaller equilateral triangle, sides one unit from the big one. Now apply (a) to that.

(5) If it exists, there is a smallest (non-zero) such one (for its sides must be of the shape $\sqrt{(p^2 + q^2)}$, p, q integral). Let it be $P_1P_2 \ldots P_n$. Draw P_1P_1' parallel and equal to P_2P_3, and similarly all the way round. Then $P_1'P_2' \ldots P_n'$ is regular, smaller but non-zero.

(b) Take 2 such intersecting circles C_1, C_2 as we may since G is inside. They determine a sphere S. Let X be a point of the surface not on S, if possible. Take a plane P through G and X but not through the common point of C_1, C_2. Since GX pierces the interior of each of circles C_1, C_2, plane P determines a circle C_3 that intersects with each of C_1, C_2, and in distinct points; that is, C_3 has 4 distinct points in common with S, and hence lies on it.

(7) (a) 7; 4 equilateral triangular and 3 square sections.

(b) Let ABCD be the tetrahedron, centroid G; B'C'D', B"C"D" the planes cutting off corner A, one parallel to BCD, the other not but having the desired property. Then B'G, B"G are partial medians of triangles B'C'D', B"C"D". Hence if B'M', B"M" are the complete medians, B'G/B'M' = B"G/B"M" = 2/3. So by similar triangles in the plane B' B"G, we see that if ANY point on edge AB be joined to G and produced to cut plane ACD, then G divides it in the ratio 2 :1. But this is ridiculous, for G lies midway between the midpoints of AB and CD (being c.g. of 4 equal masses at the vertices).

A Question of Cubes (page 4)

153.

Crossnumbers (page 8)

5	5	6	4	8	3	2
1	6	9	4	8	4	7
1	4	1	5	5	5	8
1	4	8	5	2	9	2
1	4	2	8	7	1	4
1	7	3	7	7	7	6
6	1	8	6	9	3	1

A

B

1	8	8	2	7	8	4	3	1
3	0	9	3	0	2	9	0	8
2	6	9	9	7	4	4	8	8
3	9	9	9	2	9	9	9	2
1	4	1	4	5	1	4	3	7
1	1	9	9	7	4	8	8	8
4	3	5	8	1	9	7	3	4
8	9	5	6	8	9	1	6	1
2	2	2	7	5	6	6	4	4

Problems Drive (page 22)

(1) 36π

(2) 142857 or 153846 (compare 1/7 and 2/13).

(3) 5, 4, 5. (A, R, C, H, I, M, E, D, E, . . .)
29, 33. (100—primes)
699, 778. ($1,000 \times \log_{10}n$)
0, 0. (Number of moons of planets starting with Pluto)

(4) $(2\pi^2/63) \times 10^{-12}$ lbs. wt.

(5) Either at the North pole or at any point distant $1 + 1/2\pi n$ miles from the South pole.

(6) 7×7.

(7) $m = 2, n = 3; r = 7$.

(8) 121/144.

(9) 15°.

The winners were K. Loveys (St. Johns) and R. G. Newcombe (Trinity) who scored 65%.

Mathematical Association

22 Bloomsbury Square, London, W.C. 1

President: Professor C. A. Coulson, M.A., Ph.D., D.Sc., F.R.S.

The Mathematical Association, which was founded in 1871 as the Association for the Improvement of Geometrical Teaching, aims not only at the promotion of its original object but at bringing within its purview all branches of elementary mathematics. The subscription is 2gns. per annum; for students and those who have recently completed their training junior membership is available at 10s. 6d.

The Mathematical Gazette is the journal of the Association. Published four times a year, it deals with mathematical topics of general interest. The present Editor is Dr. E. A. Maxwell.

Book Reviews

It is regretted that, owing to shortage of space, a number of reviews have been held over until our next issue.

ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE. By D. R. Curtiss. (Dover) 17s.

This book provides a wide coverage of complex variable theory. Topics dealt with include the origin and application of the theory, real functions of real variable, complex functions with derivatives, applications in geometry and physics, integrals of analytic functions, infinite series, singularities of single-valued analytic functions, and analytic continuation. It would serve as a good introductory reading for the Analysis III course but does not contain enough meat for the mathematical specialist. References to more detailed works are given after each chapter.

C. BAMFORD
M. PEMBERTON

THE MUSIC OF THE SPHERES. Vols. I and II. By G. Murchie (Dover) 19s each.

The Universe is an unknown symphony of which we slowly glimpse tunes and harmonies. Using this thread to write his work, Mr. Murchie examines our conception of matter and energy. He analyses first the macrocosm—planets, stars, galaxies, cosmology—and then the microcosm—matter, atoms, waves, radiation, relativity. Each topic is historically dissected and then integrated into the prevalent theme of the harmonious unity.

The information-content is modern and the author's prose is not hindered by a limited vocabulary. For mathematicians, the books provide a stimulating and eminently readable exposition of current scientific thought and knowledge. R. T. EDDLESTON

DIFFERENTIAL AND INTEGRAL CALCULUS. By F. Erwe (Oliver & Boyd) 57s. 6d.

A substantial volume intended to provide an undergraduate course for students aspiring to postgraduate work in analysis. The subject matter is basically that of the first year courses in Cambridge. There are a large number of digressions from the main development, but many of the proofs in the book are surprisingly sketchy.

I cannot imagine that any but first year undergraduates who already have a lifetime of analysis as their ambition will gain any real benefit from this book. However a library copy would provide an excellent source for further details outside the scope of the Tripos. C. D. EVANS

SPHERICAL HARMONICS. By T. M. MacRobert. (Pergamon) £5.

Professor MacRobert begins with the theory of Fourier Series and their applications to problems in heat conduction and vibrations of strings. After a detailed discussion of hypergeometric and Legendre functions, he develops the theory of gravitational and electrostatic potentials. The latter half of the book concerns the theory of eccentric spheres, ellipsoids of revolution and Maxwell's theory of spherical harmonics. The book concludes with chapters on Bessel Functions and asymptotic expansion.

There are a large number of intricate problems and, whilst not being particularly readable, it would be a worthy addition to any college library. G. R. FARREN

AN INTRODUCTION TO A MATHEMATICAL TREATMENT OF ECONOMICS. By G. C. Archibald and R. G. Lipsey. (Wiedenfelt & Nicholson) 50s.

The text fills a gap in the literature for the first university course in Economics, by explaining clearly and easily the mathematical tools needed for elementary economic analysis. It then applies the tools to problems in the theory of production, the Phillips curve and other macro problems. The work is set out simply with plenty of clear and helpful diagrams.

The text might also be useful to a second class of reader—those who are studying for the Maths Tripos and who wish to see whether economic theory and analysis have anything to offer as a future specialisation for the Mathematician. The applications described are simple, but they do give a view of the wide range of mathematical usage in economics. However, to see the depth and intensity of maths in any particular field, more advanced texts should be recommended. J. FILOCHOWSKI

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE, VOL I. By A. I. Markushevich (translated by R. A. Silverman) (Prentice-Hall).

This, the first of three volumes of a free translation of the original Russian tome, contains most of the material relevant to Part I (both halves) of the Tripos. The style is readable and the treatment thorough, though somewhat marred by rather too many careless printing errors, mostly wrong letters in expressions but some misnumbered cross-references and the astonishing definition: 'The sets in Σ are said to be pairwise disjoint if $E \cap F = 0$ for any pair of disjoint sets $E \in \Sigma$, $F \in \Sigma$.

Chapter 1 introduces analytic real functions and defines complex numbers as the field generated by the reals and the extra element $i = \sqrt{-1}$; in Chapter 2 this is shown to be consistent with the other usual definitions. Sets, convergence and continuity are introduced in Chapter 3, and connectedness in Chapter 4. The concept of infinity is dealt with at some length in Chapter 5, and there follows a chapter on differentiation and three chapters on the standard mappings, with a lengthy section on Lobachevskian geometry. The last seven chapters deal with integration, uniform convergence and power series.

There are plenty of exercises (no answers, but hints to, and some comments on, solutions) and a comprehensive index and bibliography.

The early chapters have a somewhat unrigorous appearance as the author, or perhaps the translator, seems to have been in some doubt as to how much to assume from real variable theory, but, for anyone familiar with this, the book provides a sound, if (at 450 pages) rather lengthy, introduction to its subject. J. S. GRANT

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE, VOLS II AND III. By A. I. Markushevich (Prentice-Hall)

These two volumes of Markushevich's book give a comprehensive treatment of Analytic Function Theory based on the firm foundation given in vol. 1. The first two chapters of vol II deal with Laurent series and the Calculus of Residues at about the same level as taught in the Analysis III course. Also included in this volume are interesting chapters on the use of complex potentials in fluid dynamics and the inverse and implicit function theorems (of analysis IV fame!) for functions of several complex variables. Volume III considers some more advanced aspects of the theory such as elliptic functions, Riemann surfaces and ramifications of conformal mappings. The volume ends with the elaboration of one of the most remarkable theorems in the subject, Picard's great theorem—in any deleted neighbourhood of an isolated essential singularity, any analytic function takes every finite value with one possible exception.

Throughout both these volumes the translator has maintained a high standard in his exposition of the subject. The book is a free translation from the Russian and many fascinating examples have been added to the English edition. Whilst this book is unlikely to be purchased by many undergraduates, it should certainly be purchased by every college library since many people profit by browsing through this book for an hour or so on a dull Cambridge day. P. O. GERSHON

LECTURES ON MODERN ALGEBRA. By P. Dubreil and M. L. Dubreil-Jacotin. (Oliver and Boyd) 105s.

An extremely comprehensive exposition of most topics of algebra covered in a degree course but the treatment is dry and concise, and tends to be overburdened with definitions resulting in a fault common to algebra books: over dependence on previous chapters.

A knowledge of set theory and mappings is assumed, but otherwise the book is self-contained and progresses from the crudest algebraic structure, a set with internal composition law, through semigroups, groups, rings, fields, modules and vector spaces in a delightful way, neatly punctuated by interesting chapters on lattices and the equivalence of Zorn's Axiom, Zermelo's Axiom and the Axiom of Choice. However for any particular topic the treatment is hardiy adequate and more specialist texts are to be preferred; the book is probably most valuable for insight into algebraic technique and collation of ideas. D. R. GREY

DIALOGUES ON FUNDAMENTAL QUESTIONS OF SCIENCE AND PHILOSOPHY By A. Pfeiffer (Pergamon) 25s.

Professor Pfeiffer's book is a document of his generation. Pfeiffer, now a scientist at an East German university, lived through the tremendous scientific revolution of relativity and quantum mechanics, and the social upheaval of pre- and post-war East Germany. His book, the first part on 'Science and the Theory of Cognition', the second on 'The Ethical Problem, Nature and Culture', strongly reflects these scientific and political influences. It presents a far reaching dialogue between Pfeiffer as a young man with his inherited and morally impeccable standards, and Pfeiffer after his concepts have been changed out of all recognition by his experience of life. This contrast is what makes the book so interesting, although I often found myself much more sympathetic towards the views of Pfeiffer as a young man, and continually exasperated by the ease with which his opinions were quashed by the older Pfeiffer. However, bearing in mind Pfeiffer's background, this set of ten dialogues forms a very readable and stimulating account of the philosophy of an eminent scientist and communist.

S. N. H. MACKIE

PROBABILITY. By J.R. Gray. (Oliver and Boyd). 17s. 6d. paperback, 27s. 6d. hardback.

Probably the most striking feature of this book is the wide range of examples within the text and exercises (totalling over 100) at the end of each chapter. In the first few chapters, the author deals with axioms, basic results, discrete and continuous random variables. He goes on to the use of recurrence relations and differential equations in problems, and finally Markov chains and queues. The exposition is clear throughout and the book would undoubtedly prove to be very useful if one had time to work through all the examples. S. G. NEWTON

FOUNDATION OF EUCLIDEAN AND NON-EUCLIDEAN GEOMETRIES. By L. Redei. (Pergamon.) £7.

By the end of the last century three different types of geometry had been presented as possible answers to the question 'What is space like and how is it constructed?'. The first was due to Euclid, and of the other two, one was constructed by Bolyai and Lobachevski, and the other by Riemann. In the terminology of Felix Klein, these are now referred to as the parabolic, hyperbolic and elliptic geometries respectively.

Of the methods of defining the basis of these geometries that due to Klein—through a projective extension of space—is of considerable importance, but as yet has not received adequate coverage in the literature. It is the purpose of Professor Redei's book to remedy this deficiency.

The book is confined mainly to the foundation of geometry by developing the group of motions and the proof of consistency. But sections are included dealing with an introduction to measurements of segments and angles—according to the principles of Cayley—Klein—and to some notions of trigonometry. Chapters I to V deal with the development of projective geometry.

The author assumes a knowledge of set-theoretical ideas, and an aquaintance with some well-known concepts of algebra and analysis and the methods of analytic geometry is also presumed.

Throughout the book, Professor Redei has tried—and in most cases succeeded—to use the simplest possible methods for the proofs of the statements. An interesting innovation is that when dealing with space, plane co-ordinates have been introduced first, and point co-ordinates only in the further development of the subject. The resulting exposition has far greater clarity than most other books on the same topic.

The book cannot be recommended for any one of the many courses comprising the Mathematical Tripos, but it should prove an invaluable source of reference for graduates and other mathematicians of an equivalent level whose research work involves study of geometrical topics. The price is rather high for most students, but every college library should certainly purchase a copy. C. R. PRIOR

CARTESIAN TENSORS. By Nils O. Myklestad. (Van Nostrand). 37s.

This book is intended for engineers rather than for mathematicians. It may, however, be found useful as an introduction to vector and tensor algebra by students before coming up to university. It is unsuitable for the mathematics course as it is too slow and contains too little material to make it worthy of buying.

The main virtue of the book is its extreme clarity and I also liked its development of vector algebra from the tensor point of view (It pre-supposes no previous knowledge of vector algebra). Its main fault is the development of two-dimensional and three-dimensional theory separately, hence producing a totally unnecessary distinction and involving considerable repetition. The worked examples, too, tend to obscure the mathematical principles involved by their tedious arithemetic. I. H. ROSE

AN INTRODUCTION TO FLUID DYNAMICS. By G. K. Batchelor. (Cambridge University Press.) 75s.

This book, if only by its length, reminds us how rapidly the knowledge of hydro-dynamics has increased in recent years. The change in emphasis from the study of the motion of ideal to that of real fluids can be appreciated by comparing it with its two best known English predecessors, Lamb's 'Hydrodynamics' and Goldstein's 'Modern Developments in Fluid Dynamics'. Many of the elegant but unrealistic theorems which adorn the pages of Lamb are not to be found in Batchelor. Goldstein's book could, so far as its mathematical methods are concerned, almost be regarded as an extension of Lamb's, designed to make the text useable in treating aeronautical problems; indeed, though Goldstein's name is rightly associated with the book, it was Lamb who started it though he did not live to see it published. In Batchelor's treatment the physical properties of fluids, so far as they are relevant to continuum mechanics, are introduced at the very beginning. The simplicity of the analysis of flow problems when the viscosity is small enough to be neglected is not brought out till about half way through the book. Mathematical arguments are expressed in vector form, but the ratio of the number of words to the number of symbols is greater than is usual in mathematical text books and the reader is constantly made aware of the physical meaning of the mathematical formulae by diagrams showing the concordance between them and experimental results. A fine collection of flow photographs reproduced from many recent articles should help students to avoid wandering into unrealistic bypaths.

Some branches of hydrodynamics have been omitted in order to keep the book from being too bulky for convenient handling, effects of compressibility and magnetohydrodynamics which have been treated in several recent text books, for example. The omission of all but a very slight mention of stability is perhaps surprising but again this has been the subject of the recent text book of Chandra Sekhar.

Some of the problems included have not appeared in text books before. The excellent

accounts of jets and cavities and of rotating fluids will be welcomed by research students as well as by the less advanced. Effects of turbulence could not be excluded altogether from a work which is expected to be useful to Engineers, but there is no discussion of the phenomenon itself.

In spite of these omissions, which Professor Batchelor hopes perhaps to treat in a subsequent volume, the book gives a clear account of the more successful mathematical discussions of the flow of real fluids. It also contains a useful list of references and a few good exercises one of which, on p. 398, is worthy of some thought.

G. I. T.

HOMOLOGY THEORY by P. J. Hilton & S. Wylie. (C.U.P.) 22s. 6d. paperback.

This book was first published in 1960 in hardback form. Subtitled 'An Introduction to Algebraic Topology', a more accurate description, it presents a comprehensive account of the standard techniques of simplicial theory, homology theory and elementary homotopy theory. It starts at a very elementary level, assuming very little knowledge of analytic topology or abstract algebra and is very thorough in its treatment of simplicial homology. There follow useful chapters on elementary homological algebra and covering spaces. The second part of the book is devoted to singular theory, obstruction theory and cup products, ending with an introduction to spectral sequences and the homology of modules.

The book is well-written but falls short of the ideal on two counts. Firstly the notation is many places is confusingly complex, making it difficult to use for reference purposes. There are also some rather unpleasant 'washing-machine' proofs. Secondly it would be easier to read and understand if the authors had totally adopted the language of categories. They refer obliquely to naturality and only introduce the definitions of category and functor in an appendix to the penultimate chapter.

It is a pity that there is no treatment of Poincare duality but this is a minor drawback. The book is a two-star text text for Part III courses on the subject and at its paperback price chapters 1, 2, 3, and 6 make good reading for the Algebraic Topology course in Part II.

A. G. TRISTRAM

INTRODUCTION TO THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS. By M. G. Smith. (Van Nostrand) 25s. paperback.

This book is intended as a text-book at the senior honours undergraduate level (e.g. Part II) and owes a great deal to the more formidable and advanced 'standard works' on the subject which are mentioned in the bibliography. For a full understanding of the material presented, it will be necessary to have one or two supplementary texts on hand unless results from complex analysis and linear algebra are to be taken on trust and such topics as Fourier expansions, Legendre polynomials and the various Bessel functions (to name but a few) are known intimately.

In a largely traditional subject there is little room for innovation at this level, but the author manages to cover the ground-work very thoroughly. His derivation, in the first chapter, of most of the partial differential equations met in Applied Maths by the undergraduate is impressive on account of its brevity; the chapter introducing Generalised Functions makes the best of a difficult job in a short space; extension of the theory to n independent variables and equations of order greater than two is rightly given a mention but only a brief one.

It is unfortunate that one of the best features of the book, its organisation and unity, only becomes apparent on reading it as a whole and may be lost to a reader who only dips into it for specific topics as I tend to.

B. M. THOMPSON

THE CLASSICAL MOMENT PROBLEM. By N. I. Akhiezer. (Oliver and Boyd) 70s.

In the language of probability theory, one can describe moment problems as being concerned with such questions as: 'Given the expected value of X^k ($k = 0, 1, 2, \dots$) what can we say about the distribution of X?". In fact, the title of this volume is slightly misleading in that it deals with a number of different (though related) problems, not all of which can fairly be described as classical. Other standard works on this subject possess a certain quality of impenetrability, and Professor Akhiezer has managed to retain much of this great tradition; this is certainly not the sort of book which the casual enquirer can pick up in order to find out whether such-and-such a theorem is true under such-and-such conditions. On the other hand it is beautifully produced and reasonably (though not wholly) free of misprints, and the determined reader will learn a great deal from its 253 densely-packed pages. P. M. LEE

A COURSE OF HIGHER MATHEMATICS. Vol. III. Part I 'Linear Algebra'. By V. I. SMIRNOV. (Pergamon). 63/-

This book gives an account of two branches of modern mathematics—linear algebra and group theory.

The subjects are given a definite slant for the physicist (or applied mathematician) and I would not recommend this book to any Cambridge trained mathematician. There is a detailed treatment of determinants in the first chapter and the notion of a matrix is introduced from an 'array of numbers' type of approach.

There is no formal definition of a vector space and those considered are over the real or complex rather than the general field.

In chapter two linear maps and their matrix representation are introduced; also discussed are characteristic equations; quadratic and Hermitian forms; infinite dimensional space (Hilbert Space).

In chapter three there are a few examples of groups (e.g. Lorentz group, permutations) followed by a brief account of basic group theory. The rest of the chapter deals with linear representations of groups and the elements of the theory of continuous groups.

B. P. McGUIRK

SMP ADVANCED MATHEMATICS. BOOK 2. (C.U.P.) 25s.

This is the second of the projected four volume treatment of the new School Mathematics Project 'A' level course. It is described in the introduction as the 'core' of the advanced course, but this somewhat misleads, as this volume has only one chapter on the important topic of mechanics. The topics covered are: some introductory statistics, and a quite exhaustive treatment of probability; techniques of differentiation and integration; 'further' work on vectors and trigonometry; a long chapter on the quadratic function; work on matrices as applied to systems of (2×2 and 3×3) linear simultaneous equations and the introduction to mechanics, which closes the volume. These topics apart, some shorter topics are touched upon (eg Σ notation).

It seems a shame that some of the topics which really do provide a background to quite a lot of work at this level are still not covered in this volume—it would be useful to have complex numbers dealt with properly and not in two pages at the end of a chapter; it would be an improvement too, if some practical examples were offered for solution when calculus techniques are discussed; since no mechanics has been done by this point however, there is little scope for introducing problems on such subject matter.

Generally, the subject matter which is included is unnecessarily long-winded in verbal presentation, and even in the strictly 'mathematical' parts there are long, and frequently unclear, explanations. The desire to write a book which the student may read without direct supervision seems to have resulted in a style which is too conversational. The verbal style often obsures the argument. This is particularly true in the sections on mechanics and probability.

Two particular criticisms seem particularly important—first the whole treatment of probability seems wrong to me. The importance of notation in mathematics cannot be overstressed, but here many of the examples are little more than exercises in the use of notation which I find unnecessarily heavy (what is wrong, if $p(\underline{A}) = n(A)/n(\xi)$, with defining $p(\underline{A}') = n(A')/n(\xi)$ we surely don't need to introduce the $p(\sim\underline{A})$ at all). Later the introduction of the more subtle ideas such as the distinction between 'independence' and 'pairwise independence' seem likely to obscure some of the basic ideas. The same is true of the examples. In an effort to produce something more interesting than the usual drawing of cards and throwing of dice, the examples have become necessarily verbose as explanations have to be incorporated into the questions themselves. All of this results in the chapter appearing very difficult, while the material as actually presented is, of course, very straightforward indeed.

Secondly the chapter on the solution of equations by matrix methods has suffered considerable amendment from the draft of this text, and in my estimation, the omissions here seriously detract from the final text. Originally, there were several paragraphs giving a more rigourous treatment of determinants and, although these were difficult, I feel they should have appeared in the final text—even if marked as 'optional extras'. The omission means that the section on inverse matrices suffers badly; the row operation method for the inverse remains but the useful algorithm ($(\text{Det } A)^{-1}$ Adj A_{ji} for matrix (a_{ij})) is lost. It seems a pity that this should be lost-after all, that is the way we calculate the inverse in the 3×3 case and the SMP syllabus aims at 'practical' mathematics.

Finally, I am sorry to say that, after the text has been through several draft editions, it still contains carelessly set exercises—for example Pg. 657 Ex 5 overlooks the obvious exception $n = 1$, even if one assumes that the question is talking about integral n. In the same example, I am highly suspicious of the sort of question which asks 'what general result does this suggest?' Here the only sensible answer is 'none'—how can the results of an enormous number of particlar, definite integrals suggest anything at all about the result of a general, indefinite one?'

All in all, a useful addition to the Common Room shelf—but I would not be happy to give this to a sixth former to use without guidance, nor would I like to teach much material in it without a second more pithy and slightly more rigorous companion text.

M. A. LEWIS

A MATHEMATICIAN'S APOLOGY. By G. H. Hardy. Foreword by C. P. Snow. (C.U.P.) 15s.

MATHEMATICS FOR THE MILLION. By Lancelot Hogben. (Allen & Unwin) 40s.

A 'classic' book is one which can be reread at intervals with profit each time. Hardy's book certainly satisfies this criterion. Like many other students of mathematics, the reviewer first read it shortly after his arrival in Cambridge. But he has come to appreciate far more the significance of certain sections of this most honestly written work only later.

Lord Snow's foreword to this work is taken from his book 'Variety of Men'. Several other people discussed there—for example, H. G. Wells, Einstein, and Lloyd George—have much in common with Hardy. Born near the beginning of the long period of

European peace and economic expansion between 1870 and 1914, they grew up to believe in the solution of problems by rational thought, leading to the steady progress of civilisation. The impact of the Second Thirty Years' War was quite shattering to this generation. Hardy knew he was one of the luckier ones: he could conceive of no evil use for his work. Though Bernard Shaw once said of Einstein that 'There is no blood on his hands', one reason why the greatest of all theoretical physicists produced no work in the last twenty years of his life may have been his mental torment at what seemed to him an inexorable march towards self-extermination in which his work had played a small but undeniable part. Thus it is hardly surprising that Hardy's views on the application of his subject seem alarming and possibly one-sided to most people born in the twentieth century—and particularly since 1945; we have had to be tougher minded.

Such a person is, of course Professor Hogben. He makes it clear in the introductions to all his works that he regards them as Primers for the Age of Plenty. Dismissing academic mathematics as an anachronism, he sounds the clarion call for social revolution, to attain his view of a solid, Presbyterian, clean-living, egalitarian, and probably rather dull society. There was undoubtedly far more reason for such polemics before the war than there is now—the first few issues of Eureka are full of such writing.

But this is not the main reason why one cannot be very enthusiastic about this reissue. The subject matter, though revised, is now out of date, the most serious ommission being the lack of any discussion of the advent of computers and the techniques needed to exploit them fully—a basically mathematical approach to the organisation of production, distribution and exchange. Here, surely, is a field for Professor Hogben's expository genius—we need a text good enough to make it compulsory reading for every politician, industrialist, and trade union official in the country.

C. J. MYERSCOUGH

ORDINARY DIFFERENTIAL AND DIFFERENCE EQUATIONS. By Frank Chorlton. (Van Nostrand) **21s.**paperback

To quote from the preface 'this book aims at a systematic and practical treatment of ordinary differential and difference equations. Its contents are sufficient to cater fully for the needs of undergraduate scientists and engineers of all kinds and it is hoped that the work will furnish a suitable introduction to the subjects for first year mathematical students and for sixth form scholarship pupils'. The author has certainly gone to a lot of trouble to provide a clear and logical exposition of the subjects, one of the best features being the large number of examples drawn from many disciplines which are worked through in the text. To supplement these worked examples there are many more for the student to attempt, solutions being provided at the end.

The author makes a point of introducing the calculus of the D-operator very early in the development, later bringing in the finite calculus of E and Δ with emphasis on their similar nature. He has included some numerical and graphical techniques for completeness and devotes a chapter to the Laplace Transform. The book is well set out and very readable, for this the author and publisher are to be commended.

D. C. JOYCE

STUDIES IN REAL AND COMPLEX ANALYSIS edited by I. I. Hirschman, Jr. (MAA Studies in Mathematics, vol. 3).

This book is a collection of seven essays on various topics in modern analysis, most of which give very full descriptions of their respective subjects in remarkably short space. The essays on harmonic analysis and the Laplace transform are particularly useful, whilst those on Toeplitz matrices, several complex variables and nonlinear mappings between Banach spaces are perhaps of rather more specialist interest. The 'introduction' to the Lebesgue-Stieltjes integral is by the Daniell approach which views the Lebesgue integral as the continuous extension of the functional 'Riemann integral' on a quotient space of the space of continuous functions of compact support to the completion of that space in a suitable norm. The article on semigroups is very short and fails to go as deeply into its subject as the other essays.

The level of all these essays is such as to make them accessible to a Part II student, and the book can be thoroughly recommended for reading before starting Part III analysis.

P. G. DIXON

ELEMENTS AND FORMULAE OF SPECIAL RELATIVITY. By E. A. Guggenheim. (Pergamon) 21s.

This book covers relativistic kinematics, mechanics and electromagnetic fields. Concise proofs of all formulae are given and it is a useful reference book though not particularly good value.

G. R. FARREN

QUANTUM MECHANICS. By R. A. Newing & J. Cunningham. (Oliver & Boyd) 17s. 6d. (Paperback), 27s. 6d. (Hardback)

Quantum mechanics is here developed concisely in terms of the theory of vector spaces. While the mathematics of the subject is concentrated upon throughout, the physical significance of the results obtained is by no means neglected.

After a short introduction, the authors set up the necessary theory of vector spaces and linear operators. They then base their postulates on this theory, going on to discuss the Schrodinger coordinate representation of a system and several applications thereof—potential barriers, hydrogen atom, etc. The remainder of the book covers scattering theory, angular momentum, electron spin, perturbation theory, and the Dirac electron.

This book is a useful introduction to quantum theory, although the complete newcomer to the subject may desire rather more motivation for the postulates than is given.

Generally though it maintains the high standard of the Oliver & Boyd series.

J. J. BARRETT

Produced by LINEOPRINT at The Gresham Press, Unwin Brothers Limited, Woking, Surrey, England