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EUREKA

THE JOURNAL OF THE ARCHIMEDEANS

(The Cambridge University Mathematical Society: Junior Branch of the Mathematical Association)

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Editorial

ORIGINALLY it was hoped to publish this tenth number of EUREKA during the Michaelmas Term, but owing to many delays and difficulties, it has only recently been possible to send the manuscript to the printers. The response to our request for contributions has been very good, but more are required for our next issue. All articles will therefore be gratefully received.

Just one word of advice to the would-be contributor; glance at the London Mathematical Society booklet entitled, *Notes on the Preparation of Mathematical Papers* (No. 9350.c.110 in the University Library) before starting, as it will help you to avoid writing material in a form which is very difficult and costly to print.

Finally, this opportunity must be taken of expressing sincere thanks not only to the undergraduates who have contributed to this issue, and to the senior members of the University who have spared valuable time to write for us, but also to the Editorial Committee who have helped me with the difficult job of reading and checking the manuscripts.

The Archimedeans

DURING the Michaelmas Term there were two landmarks in the history of the Archimedeans—a tea-dance and a Christmas party. These proved huge successes and we intend to repeat them in future. The Society thanks Victor Hale and Chris Zeeman, the organisers, and also those who assisted with the decorations and other vital details.

Though we have come out of our shell socially, our other activities have not diminished. We are grateful to Michael Ash who allows us to use his splendid rooms in Trinity for the Music Group and Tea-time Meetings, and we apologise for any inconvenience caused to himself and his friends. The Play Reading Group has met, though it has not been as well supported as last year.

A. M. M.

Newton's Principia

By D. W. J. Cruickshank

Some kind whim recently made me acquire a copy of Cajori's revision of the translation of Newton's Philosophiae Naturalis Principia Mathematica. I say kind whim, because the study of the book has since been a continual source of pleasure. I don't quite know what I expected to find, perhaps a very theoretical work treating of the three Laws of Motion and the Law of Universal Gravitation in a rather dully abstract manner. Instead I found a book alive from the start, Newton's account of his own researches, written with all the enthusiasm of a discoverer, a happy intertwining of theory and (frequently ingenious) experiment.

In spite of the all-embracing title, the Principia is not an account of the whole of Mechanics or Physics; it contains hardly any Statics. It is primarily a research work, Newton's account of his investigations into the System of the World, but as in any research the author has come on side lines, which, while not directly bearing on the main end, are of great interest in themselves, so these are included as well. The heuristic process of discovery necessarily further upsets an even tempo of development; but, despite these factors, the book is a "treatise," very thoroughly covering its field, and as such is a remarkable tribute to the genius of its author.

The first few pages are devoted to a discussion of definitions, the three Laws and their corollaries. Newton also calls the Laws Axioms, for he takes them as given in the remainder of the Principia. After this introduction the Principia is divided into three books. From the point of view of the chief object, the first two books mathematically examine the consequences of various physical hypotheses, the first investigating the motion of bodies under various laws of attraction, and the second the motion of bodies in resisting media with differing laws of resistance. However, Newton occasionally sidetracks to deal with terrestrial physics or to describe experiments. The climax of the Principia is reached about 20 pages from the start of Book III. Newton sets out six phenomena, amounting pretty much to Kepler's Laws, and, the mathematics having been done in the first two books, he is then able very quickly to demonstrate the universality of gravitation. Thereafter he settles down happily to the discussion of all manner of details—aphelions and libration, nodes and tides. . . .

I have mentioned Newton's skill as an experimenter. The very first experiment he describes gives a simple example of his ingenuity. The experiment was on the conservation of momentum, and was performed with bodies suspended on strings. The bodies, allowed to swing as pendulums, collide at their lowest points and from the heights of release and ascent, the velocities just before and just after the collision can be calculated. "To bring this experiment to an accurate agreement with the theory, we are to have due regard to the resistance of the air." Newton draws one body aside and lets the other swing freely. He marks the place, R, of release and the place V to which the body returns after one oscillation. The difference RV, he says, is due to the resistance of the air. If S, T, are points such that ST $= \frac{1}{4}$ RV and the interval ST is situated in the middle of RV, then a body released from S, above T, will have very nearly the same velocity at the bottom of its swing as a body moving *in vacuo* would have had if released from T. Newton then repeats this procedure for the positions to which the bodies rise. The remarkable feature of this experiment is not so much his allowing for air resistance as his manner of doing it. He does it in such a way that the allowance is made in his experimental set-up, his subsequent calculations remaining simple. The wiseacre of to-day performing the same experiment would quite likely put a formidable correction term into his calculations to achieve the same end.

Comets were a very bright feather in Newton's cap. The ability of his theory to deal with such eccentric paths, and to allow retrograde motions, was strong evidence of its correctness, while their motion was a sticky problem for the protagonists of Descartes' theory of vortices. Newton was not content merely to ignore the theory of vortices, he worked out its consequences and after discussing various resistance-velocity laws concludes triumphantly: "Let the philosophers then see how phenomena of the 3/2th power can be accounted for by vortices."

The theoretical discussions in the Principia are full of interesting matter, sometimes for the neat turns in the argument and sometimes from a historical point of view. He shows that the velocity of waves varies as the square root of the breadths in a very elementary manner, and a few pages later he enunciates the formula for the velocity of sound. Unfortunately, being unaware of the existence of two specific heats, he takes the wrong value for the elasticity of the air. To get the right result, he cheerfully "cooks" a couple of factors to deal with "crassitude" and "vapours of another spring"!

On the historical side there is much to note in the early forms of calculus, while Lemma I of Book I has a note not unfamiliar to the analysts of to-day, even if ideas on terminology have changed a little.

> "Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.

"If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference D; which is contrary to the supposition."

Newton had a complete mastery of the relative roles of observation and theory. The account of scientific method which he gives in the Principia has remained the guide of scientists ever since.

What more can be said? Only this, if you are seeking a book to keep you up all night, look no further.

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Note on a Diophantine Equation

By Diogenes

In the 1947 Problems Drive of the Archimedeans, the following problem was proposed:

To find unequal positive integers x, y, z such that

$$x^3 + y^3 = z^4.$$

Although there were some research students in Theory of Numbers among those who tried, not one person succeeded in solving it within the time, yet the solution is extremely simple. Take the equation

$$2^3 + 3^3 = 35$$

and multiply by 35^3, getting

$$70^3 + 105^3 = 35^4.$$

By the same method we can prove the following

Theorem: *If a, b are integers and l, m, n are integers greater than* 0 *such that* $(l, n) = 1$, $(m, n) = 1$, *then the equation*

$$ax^l + by^m = z^n \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

has an infinity of integer solutions.

Let p, q be arbitrary whole numbers, and put

$$ap^l + bq^m = r. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Then, since $(lm, n) = 1$, there are integers $h, k > 0$ such that $hlm - kn = -1$. On multiplying (2) by $r^{hlm} = r^{kn-1}$, we have

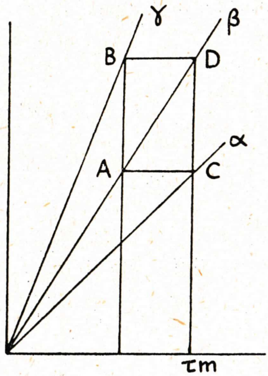
$$a(pr^{hm})^l + b(qr^{hl})^m = (r^k)^n.$$

Since there is an infinity of choice for p, q, (1) has clearly an infinity of solutions.

A Property of the Golden Number

By D. B. Sawyer

Rouse Ball, in his *Mathematical Recreations and Essays,* describes Wythoff's Game, a game involving two persons and two piles of counters. The solution of the game involves the sequence of number pairs (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), . . ., which are such that the r-th pair differ by r, and the smaller number of each pair is the smallest positive integer which has not yet appeared in the sequence. I give a demonstration, using geometrical ideas, of the fact that the r-th pair is $([r\tau], [r\tau^2])$, where τ is the golden number $(\sqrt{5} + 1)/2$, and the square brackets denote the integral part.



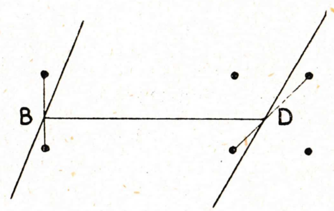


Diagram 1 (left)
Diagram 2 (above)

In the first place, τ is a root of $x^2 - x - 1 = 0$, so that we have $\tau^2 = \tau + 1$. Thus $r\tau^2 = r\tau + r$ and so $[r\tau]$ and $[r\tau^2]$ differ by r. It remains to show that the numbers $[r\tau]$, $r = 1, 2, \ldots,$ and $[s\tau^2]$, $s = 1, 2, \ldots$, include each positive integer once and once only.

Consider an integral lattice in the (x, y) plane and let α, β, γ be the lines $y = x$, $y = \tau x$, $y = \tau^2 x$. Let the line $x = m$, where m is a positive integer, cut β, γ at A, B, and let $x = \tau m$ cut α, β at C, D. Then A, B, C, D have coordinates $(m, \tau m)$, $(m, \tau m + m)$ $(\tau m, \tau m)$, $(\tau m\ \tau m + m)$ respectively, and form a rectangle. (See diagram 1.)

Since τ is irrational, $\tau m + m = n + \theta$ where $n = [\tau^2 m]$ and $0 < \theta < 1$. Thus D is $(n - m + \theta, n + \theta)$ and so is on the diagonal of the unit square $(n - m, n)$, $(n - m + 1, n + 1)$. Since $\tau > 1$ we see from diagram 2 that $(n - m)\ \tau < n$, $(n - m + 1)\ \tau > n + 1$, and so $[r\tau] \neq n$ for any integer r. In other words, if $[s\tau^2] = n$ for some s, $[r\tau] \neq n$ for any integer r.

Conversely, it is clear that if $[r\tau] \neq n$ for any r, the line $y = \tau x$ must cross a diagonal such as $(n - m, n)$, $(n - m + 1, n + 1)$ for some m, and so, reconstructing the rectangle, we have $[m\tau^2] = n$. Thus the numbers $[r\tau]$, $r = 1, 2, \ldots$, and $[s\tau^2]$, $s = 1, 2, \ldots$, include each positive integer once and once only.

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The Cambridge Mathematical Laboratory

By R. V. Baron

The phrase "mathematical laboratory" once invoked in me a picture of bearded professors busily engaged in tossing coins, playing roulette, cutting and dealing cards, and calculating the soundest methods of winning football-pools. The Cambridge Mathematical Laboratory is more prosaic, but its contents are fascinating in their complexity.

The Laboratory includes several computing rooms, where the Brunsviga (hand) and Marchant (electric) general-purpose calculating machines save research students many hours of tedious calculation. In adjoining rooms are the National machine (used for interpolation and construction of tables of functions) and the Mallock machine, invaluable for solving simultaneous linear equations. Far more spectacular is the large differential analyser (with its Meccano baby brother) which gives graphical solutions of differential equations. This machine was employed on war work from 1940 to 1946. The Laboratory has recently acquired a Hollerith punched-card sorter and tabulator (with the accompanying card punches and verifiers) for the convenient and speedy treatment of large amounts of numerical data. Its action is most impressive.

The latest and most remarkable activity of the Laboratory is the building of an electronic high-speed general-purpose calculator, EDSAC. This Electronic Delay Sequence Automatic Calculator with its 2,000 valves, will be considerably more flexible than the ENIAC with 18,000 valves. All the necessary electronic research has been carred out at the Laboratory under its Director, Mr. M. V. Wilkes, and includes the design of an ingenious "memory" employing supersonic vibrations travelling down mercury tubes. This machine, which is now being completed, will work in the binary scale, but will deliver the results of its calculations to any desired accuracy by means of teleprinters working in the decimal scale. It will perform 20,000 operations (e.g. addition or multiplication) per minute, so that complex calculations, once considered too onerous to be possible, will now be comparatively easy.

The Mathematical Laboratory does not forget its link with the past. Amidst all the modern machinery there is a portion of the earliest calculating engine, made by Charles Babbage, the Lucasian Professor, in 1840.

An Alphabet

A for Analysis, first on the list
Of subjects whose purpose is usually missed.

B is for Body, an object most frigid
Which even in heat waves stays perfectly rigid.

C is for Conic: oh! common of curves,
It crops up so often it gets on your nerves.

D is for ∇^2, for div and for det,
And several others we try to forget.

E for ϵ that's greater than nought.
This magical symbol will save us much thought.

F is for Field; not where buttercups grow,
But where magnets and charges bring currents in tow.

G is for Gravity, clear to us all,
Or what else would happen to Newton's old "ball"?

H is for Hydromechanics, a study
Of sources and streams—not the kind that are muddy!

I for Infinity, mythical place
Where circles and parallel lines show a face.

J for Jacobian, a pleasant device
For making the nastiest integral nice.

K is for Kepler, who left us some laws
Of planet'ry motion, effect but not cause.

L stands for so many things, that, in doubt,
I've chosen the Limit that's often about.

M is for Matrix, a mighty array—
If we didn't leave blanks we'd be writing all day.

N is for Normal, a misleading word,
For a "non-normal" normal's not even absurd!

O is for ORBIT; we'll readily trace
The path of a body that's moving in space.

P is for PARTICLE having no size;
It's wonderful what it can do when it tries.

Q is for QUADRIC, the Conic's big brother;
What's true for the one may be true for the other.

R is for RANK; but the Major is out,
For here it's the Minors we're worried about.

S is for SIGN that's so often mislaid,
Explaining mistakes that should never be made.

T is for TRIP.: how I wish that implied
A journey by car or a char-à-banc ride.

U for UNIQUENESS, important, I'm sure,
But the proofs of the theorems are rather too pure.

V is for VECTOR: all lecturers say
That the sum is the same if you take it each way.

W must obviously stand for a WAVE;
The problem arises: "How does it behave?"

x, *y* and *z*, from their own point of view,
Are complaining: "We have far too much work to do;
It seems that for axes we're much better than
All the others; they use us whenever they can;
Though mathematicians may do as they like,
Beware! We may yet go on strike!"

"PLUTO."

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BACK NUMBERS

A FEW copies of the last two issues of EUREKA are still available; No. 8 at 1s. and No. 9 at 1s. 6d. post free. Cheques, postal orders, etc., should be made payable to "The Treasurer, The Archimedeans."

Brown University, Providence, Rhode Island, U.S.A., require copies of No. 1 and No. 2 for their library. Any reader who is willing to sell these should communicate with the Editor.

Some Ratio Theorems

By G. C. Shephard

Descriptive Geometry is the study of method of depicting points of [3]* on a flat sheet of paper in such a manner that geometrical constructions can be carried out in the plane corresponding to constructions in space, and various results deduced graphically from them.

In four dimensions, though it is comparatively simple to imagine figures, it is necessary to use some form of descriptive geometry if exact constructions are to be carried out. I propose the following simple method, showing in particular how some theorems on incidence in [4] lead to interesting theorems on proportion in [2]. Throughout we shall be working in euclidean space.

Let π_1 and π_2 be two perpendicular planes in [4], i.e. every line in π_1 is perpendicular to every line in π_2. A point P in [4] is determined by P_1, P_2 its orthogonal projections on π_1 and π_2 respectively. In this manner we have established a (1, 1) correspondence between the points of [4] and the pairs of points on the two planes. Throughout, a letter without a suffix will denote a point of [4] whilst the letter with suffix 1 or 2 will denote the projections of the point on π_1 or π_2.

A line is determined by any pair of points A, B on it. The projections of all the points of the line AB lie on the lines A_1B_1 in π_1 and A_2B_2 in π_2. However, it is not sufficient merely to specify the projection lines in π_1 and π_2 to determine the line uniquely, since all the points of a plane in [4] have projections on these two lines. It is also necessary to specify on the two lines the projections A_1B_1, A_2B_2 of two points A, B of the line in [4].

The following theorems immediately present themselves:

T1. *If* L_1, L_2 *are chosen on* A_1B_1, A_2B_2 *respectively so that*

$$A_1L_1 : L_1B_1 = A_2L_2 : L_2B_2, \quad \ldots \quad \ldots \quad \text{(i)}$$

then L_1, L_2 *are the projections of a point* L *on* AB.

Conversely, any point L *on* AB *has projections* L_1, L_2 *satisfying* (i).

T2. *The condition for the lines* AB, CD *to intersect* (*i.e. that the four points* A, B, C, D *are coplanar*) *is that if in* π_1 A_1B_1, C_1D_1 *meet in* L_1, *and* L_2 *is determined similarly in* π_2, *then*

$$\begin{aligned} A_1L_1 : L_1B_1 &= A_2L_2 : L_2B_2 \\ \textit{and} \qquad C_1L_1 : L_1D_1 &= C_2L_2 : L_2D_2 \end{aligned}$$

(for this is the condition that L lies on both AB and on CD).

Since A, B, C, D being coplanar implies also that AC meets BD, and AD meets BC, we can deduce the following theorem:

* [n] denotes space of n dimensions.

T3. *Given two sets of four points* A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2 *in a plane,* A_1B_1 *meets* C_1D_1 *in* L_1, A_1C_1 *meets* B_1D_1 *in* M_1, A_1D_1 *meets* B_1C_1 *in* N_1, *and* L_2, M_2, N_2 *are similarly determined, then*

$$A_1L_1 : L_1B_1 = A_2L_2 : L_2B_2$$

and

$$C_1L_1 : L_1D_1 = C_2L_2 : L_2D_2$$

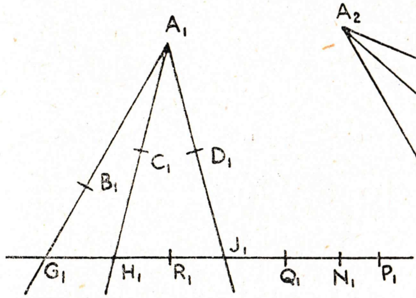
imply

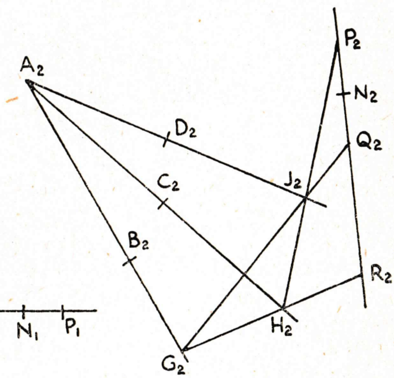
$$A_1M_1 : M_1C_1 = A_2M_2 : M_2C_2$$
$$B_1M_1 : M_1D_1 = B_2M_2 : M_2D_2$$
$$A_1N_1 : N_1D_1 = A_2N_2 : N_2D_2$$

and

$$B_1N_1 : N_1C_1 = B_2N_2 : N_2C_2.$$

As another illustration of this method of using incidence in [4] to deduce theorems on proportion in the plane, we shall consider the well-known property that if l, m, n are three general lines in [4] then one and only one transversal to them can be drawn. (The [3] (l, m) meets n in a point N. If L, M are determined similarly, then L, M, N lie in three solids, and therefore are collinear. This line is the unique transversal.)





This leads to the following theorem:

T4. *If* $A_r, B_r, C_r, D_r, E_r, F_r$, $(r = 1, 2)$ *are two sets of six points in a plane, then it is possible in one and only one way to draw two lines* l_r *meeting* A_rB_r *in* P_r, C_rD_r *in* Q_r, *and* E_rF_r *in* R_r, $(r = 1, 2)$ *satisfying the conditions*

$$A_1P_1 : P_1B_1 = A_2P_2 : P_2B_2$$
$$C_1Q_1 : Q_1D_1 = C_2Q_2 : Q_2D_2$$
$$E_1R_1 : R_1F_1 = E_2R_2 : R_2F_2$$

and

$$P_1Q_1 : Q_1R_1 = P_2Q_2 : Q_2R_2.$$

This is, in fact, not difficult to prove directly analytically but there is no obvious synthetic proof.

In order to determine the lines l_r it is necessary to determine the points L, M, N. For this we use the following:

Given the projections of six points A, B, C, D, E, F *on the planes* π_1 *and* π_2, *to construct the projections of the point* N *in which the* [3] *defined by the points* A, B, C, D *meets the line* EF.

Suppose A_1B_1, A_1C_1, A_1D_1 meet E_1F_1 in points G_1, H_1, J_1 respectively. Construct G_2 on A_2B_2 so that $A_1G_1 : G_1B_1 = A_2G_2 : G_2B_2$ and let H_2, J_2 be constructed similarly. Then G, H, J lie on the lines AB, AC, AD, and are therefore in the given [3].

Let H_2J_2, J_2G_2, G_2H_2 meet E_2F_2 in P_2, Q_2, R_2 respectively. Construct P_1, Q_1, R_1 on E_1F_1 so that $H_2R_2 : R_2J_2 = H_1R_1 : R_1J_1$ etc. Then the points P, Q, R lie in the plane in [4] determined by the lines (E_1F_1) and (E_2F_2) and also in the solid ABCD. Hence they are collinear (proving, by the way, that $P_1Q_1 : Q_1R_1 = P_2Q_2 : Q_2R_2$).

Choosing any two of these points (say Q, R), it is only necessary now to find points N_1, N_2 on E_1F_1, E_2F_2 respectively so that,

$$Q_1N_1 : N_1R_1 = Q_2N_2 : N_2R_2$$

and

$$E_1N_1 : N_1F_1 = E_2N_2 : N_2F_2$$

and this is easily done by elementary geometry.

If we construct L, M similarly, the unique transversals l_r of theorem 4 may be drawn.

Similarly, all incidence theorems in [4] lead to theorems in the plane. As a final example, the properties of the configuration of 15 points and 15 lines in [4] (see H. F. Baker, *Principles of Geometry*, Vol. IV, Chapter V) become quite complicated two-dimensional theorems.

■ ■ ■

Lines to a Don

If only that man with the duster and chalk
Would state all his facts without quite so much talk
And whenever he sketched a geometrical figure
Would make all the sides just a little bit bigger,
And, of course, I suppose that I really can't grumble
If, owing to age, his voice is a mumble.
I know that I always turn up to him late,
But that's just the distance from Trinity gate,
And to start all his lectures so promptly at nine
May be his intention; it's clearly not mine.
I note what he says just as well as I'm able
Though the desk where I'm sitting is not all that stable,
And my conscience compels me to fill in the gaps
Where I think that I know what his words were—perhaps.
At the end of his lecture I must dash away
To start the important affairs of the day,
So when he goes on until well past the hour
To find out in which term he's left out the power
It really gets hard—try as hard as I can—
To follow with pleasure that odd little man.

"SUCRA."

Experimental Arithmetic

By Professor D. R. Hartree

Two large automatic calculating machines have recently been designed and constructed in the United States, the I.B.M. Automatic Sequence-Controlled Calculator (refs. 1, 2) at Harvard University and the ENIAC (refs. 3, 4, 5) at the University of Pennsylvania. Others are under construction or projected, both in Britain and in the United States. These machines are digital machines, in the sense that they handle numbers directly in their numerical digital form. They can carry out automatically extended sequences of computing operations, and their organisation allows the possibility of control of the computing sequences by the results of the calculations themselves. They are designed in such a way that the computing sequences can be readily changed from those required for one calculation to those required for another.

The ENIAC and some of the machines now under development use the technique of electronic circuits, which enable individual arithmetical operations to be carried out very rapidly. The ENIAC, for example, does a multiplication by a 10-figure number in less than 3 milliseconds, or at the rate of over a million multiplications an hour.

The speed and facilities of such machines will make it possible to undertake numerical calculations of kinds, and on a scale, which have hitherto been almost or quite impracticable. This situation will call for new numerical methods, not only for the new kinds of problem, but also for handling most effectively problems of familiar kinds on the new scale; for example, schoolroom methods of solving simultaneous algebraical linear equations, which are suitable for systems of 2 or 3 equations, are probably not the most effective way of dealing with 100 or 200 equations for a corresponding number of unknowns.

Development of methods suitable for such machines requires both algebraical analysis and numerical experiment, both of which can be done without actual access to such a machine, though, of course, the work must be done with the capabilities and limitations of such machines always in mind. Useful numerical experiments can be done by pencil-and-paper methods, supplemented perhaps by an ordinary desk calculating machine, by trying simple cases of proposed general methods. Such experiments may often show up in an elementary way practical difficulties which would not appear from a purely algebraical analysis, and might be difficult to diagnose if first met in the course of work on the full scale made possible by the large automatic machines. [I believe that there is real scope for experimental arithmetic in this connection, in the development of methods of using these large machines on work of a kind which they now make practicable for the first time.]

The numerical examples must, of course, be regarded as simple illustrative examples of a general method. In themselves they may often appear trivial, and the method not the most suitable for dealing with them. But this is not the point; they are studied not for themselves but for the light they can throw on the general method.

For example, one method of solving a set of simultaneous (not necessarily linear) equations

$$f_j(x_1, x_2 \ldots, x_n) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

is to minimise the quantity S defined by

$$2S = \Sigma_j f_j^2 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

S is clearly zero for any set $(x_1, x_2 \ldots, x_n)$ which *is* a solution of the equations (1), and positive for any set which *is not* a solution. Non-zero stationary values of S (if any) can easily be distinguished from the zero values which represent solutions of equations (1).

We might explore the possibilities of a method of solving the equations by direct minimising of S in a simple case such as

$$xy = 3, \quad 3x + 2y = 9 \quad \ldots \quad \ldots \quad (3)$$

which will illustrate some, at least, of the features arising from non-linearity when there are not equal roots, and

$$xy = 1, \quad x + y = 2 \quad \ldots \quad \ldots \quad (4)$$

which will illustrate some of the peculiarities (if there are any) which arise when there are equal roots. For these particular equations, such a method would have no advantages over the classroom method. But its application in these simple cases may, and does, illustrate some points which are relevant to the general case, and the simple form of the equations makes the numerical work easy. It is important, of course, to be careful that one does not take liberties with the method which depend on the simple form of the equations and would not apply in the general case.

As an example I shall consider one possible method of trying to minimise S.

At any point P $(x_1, x_2, \ldots, x_n)$, other than one at which S is stationary, there is a direction of greatest rate of decrease of S; this is the inward normal to the locus S = const. through P, or the directions of the vector (−grad S) at P, with components

$$-\left(\frac{\partial S}{\partial x_Q}\right)_P = -\Sigma_j\left(f_j \frac{\partial f_j}{\partial x_Q}\right)_P.$$

It should be noted that the evaluation of *each component* of (grad S) will usually involve about as much work as the evaluation of one value of S itself, so that evaluation of (grad S) is not a calculation to be made more often than necessary. We may, therefore, consider starting at any point P, going along the normal at P to the neighbourhood of the minimum of P on that line, calculating a new value of (grad S) at some point Q in that neighbourhood, and so on. To avoid a purely arbitary choice of points along the normal at P at which to calculate S, we would like an estimate of how far to go along it. That is, if from P we make a displacement ($\delta x_1, \delta x_2, \ldots, \delta x_n$) to Q, where

$$\delta x_j = -c\,(\partial S/\partial x_j)_P \quad \ldots \quad (5)$$

(c being independent of j), we want an estimate of the best value of c.

Now approximately

$$S_Q - S_P = \tfrac{1}{2}\Sigma_j\left[\left(\frac{\partial S}{\partial x_j}\right)_Q + \left(\frac{\partial S}{\partial x_j}\right)_P\right]\delta x_j; \quad \ldots \quad (6)$$

and if we write R for the magnitude of (grad S), that is

$$R^2 = \Sigma_j\,(\partial S/\partial x_j)^2$$

and γ_{PQ} for the angle between the directions of (grad S) at P and Q, that is

$$\cos\gamma_{PQ} = \Sigma_j\left(\frac{\partial S}{\partial x_j}\right)_P\left(\frac{\partial S}{\partial x_j}\right)_Q \Big/ R_P R_Q, \quad \ldots \quad (7)$$

(5) and (6) give

$$c = 2(S_P - S_Q)/R_P(R_P + R_Q\cos\gamma_{PQ}). \quad \ldots \quad (8)$$

Now we hope S_Q will be substantially less than S_P, and since R will decrease as we approach the minimum of S, we may hope that $R_Q\cos\gamma_{PQ}$ is substantially smaller than R_P. If we make the crude approximation of neglecting S_Q and R_Q in (8), it gives

$$c = 2S_P/R_P^2. \quad \ldots \quad (9)$$

This is likely to give an overestimate of c, since the numerator in (8) is less than S_P and the denominator probably greater than R_P^2; on the other hand, the denominator should be less than $2R_P^2$, so that if there is any substantial decrease in S along the normal at P, c will be greater than $\tfrac{1}{2}\cdot 2S_P/R_P^2$. If a simple general working rule is required, the value

$$c = \tfrac{3}{4}\cdot 2S_P/R_P^2 \quad \ldots \quad (10)$$

seems suitable, so that

$$\delta x_j = -\tfrac{3}{4}\,2\,S_P\left(\frac{\partial S}{\partial x_j}\right)_P \Big/ \Sigma_j\left(\frac{\partial S}{\partial x_j}\right)_P^2. \quad \ldots \quad (10)$$

The purpose of this process is to get to the point at which S = 0; intermediate points P, Q . . . reached on the way are of no interest. So there is no need to adhere strictly to the numerical values given by (10) if the choice of slightly different values would simplify the numerical work. With an automatic machine it is easier to draw up the programme of operating instructions on the basis of a strict application of the formula than to draw up the programme for making modifications from it; but pencil-and-paper numerical work may be simplified by taking simple numerical values of $(x_1, \ldots, x_n)$ in the neighbourhood of those given by formula (10).

Some numerical experiments with this method have been made on equations (3); these equations were made up to have a solution $x = 1, y = 3$, but the other solution was not initially determined. Three applications of (10), starting from (0, 0) as P, lead easily to a point in the neighbourhood of (2·05, 1·45), the exact part reached depending on the departures made from a strict application of (10); the value of S has come down from 90 at (0, 0) to about 0·01, and it might seem that a good approximation to a solution has been found. But the solution in this neighbourhood is ($x = 2\cdot 0, y = 1\cdot 5$), and in terms of x and y the approximation to it is still only moderate.

From about ($x = 2\cdot 05$, $y = 1\cdot 45$), the process of working along successive normals to loci S = const. is found to be no longer satisfactory. In the region between here and the solution ($x = 2\cdot 0$, $y = 1\cdot 5$), the direction of the normal at a point P is extremely sensitive to the position of P, and unless a very fortunate choice of P is made, the minimum of S on the normal at P is not much smaller than S_P, so that, firstly, equation (9) considerably overestimates c, and secondly, not much improvement in the solution can be made by going along the normal at P. The "curves of steepest descent" are sharply curved in this region, so that the direction of — (grad S) gives only a very local indication of the direction in which to go in order to decrease S.

The reason for this can be seen from the form of the curves S = const., which in this region are approximately ellipses of axial ratio about 6 : 1, with their major axes nearly along $(x - 2) - (y - 1\cdot 5) = 0$. If these are regarded as contour lines of a surface, this surface has a steep-sided valley with its axis in this direction, the floor of the valley sloping gently towards ($x = 2$, $y = 1\cdot 5$). When one is high up on the side of the valley, as at the starting point ($x = 0$, $y = 0$) taken, application of (10) brings one quickly and effectively to the floor of the valley; but then, unless one happens to choose a point very near the major axis from which to continue the process, subsequent steps along successive normals are represented by a path bouncing from side to side across the valley instead of running down its axis.

There seems scope for both algebraical and numerical experiment here, in devising a method, capable of being translated into practicable numerical procedure, for damping out these oscillations. The method should be capable of dealing with equations like (4) which have double (or multiple) roots, for which the axis of the valley is curved. Further, in application to a system of n equations, the number of individual arithmetical operations must increase as some power of n, say as n^2 or n^3, not exponentially, as 2^n or 3^n; a process in which the number of operations increases exponentially with n would become impracticably long for $n = 50$ or 100, even with the aid of high-speed automatic machines.

This experiment, and slight extensions of it, show several features of the proposed method for minimising S:—

(i) The difficulties found in applying this process in regions where the curves of steepest descent are sharply curved do not depend on the non-linearity of the equations; in the case considered above, they occur in a region in which non-linear terms in $(x - 2)$ and $(y - 1{\cdot}5)$ are of minor importance. Therefore, means of avoiding these difficulties may usefully be explored in the simpler case of linear equations.

(ii) On the other hand, these difficulties may occur in a region so far from a root that non-linear terms cannot be neglected. Hence no method of avoiding them which is restricted to linear equations will be satisfactory.

(iii) These difficulties occur already for quite well-conditioned equations. They will be very much more severe for badly-conditioned equations, for which the loci S = const. will be much more elongated, with much greater curvature near the end of the major axis.

I have considered here one rather simple example of numerical arithmetic, and this not fully. Several more complicated examples arose during the war in the quantitative treatment of various technical problems relating to the national war effort. One of the most complex which I met was concerned with the numerical solution of a non-linear integro-differential equation occurring in the theory of the transient behaviour of an electronic valve with space-charge, in which the motion of each electron depends on the field acting on it at each instant, which field is a space-integral over the instantaneous space-charge distribution of all the other electrons. In this case, not only the experimental numerical work on methods handling this situation, but the whole evaluation of solutions, had to be carried out by pencil-and-paper methods. With the advent of the large automatic machines this will no longer be necessary. But pencil-and-paper experimental work on simple numerical examples will still, I believe, be required in designing methods for using the big machines.

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■ ■ ■

A Tennis Problem

By J. S. R. Chisholm

The Westland Lawn Tennis Club was holding a knock-out singles tournament, and the organisers, after much discussion, decided to arrange the play for the 100 entrants as follows:

1st round: 8 byes.
2nd round: 6 byes.
3rd round: no byes.
4th round: 1 bye.
5th round: no byes.
Semi-final.
Final.

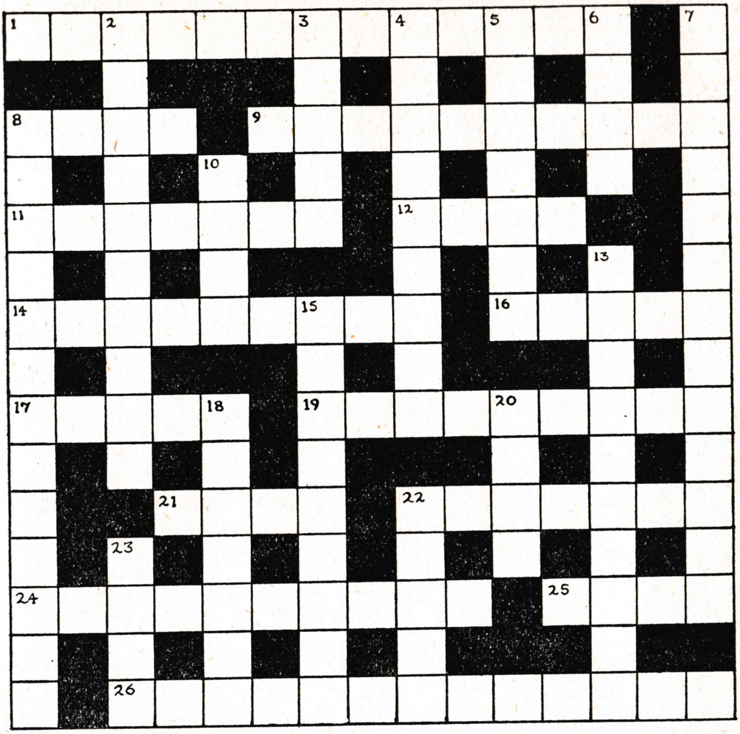
In the whole tournament, how many matches were played?

■ ■ ■

In summer-time the flies go round
In vortices above the ground;
In winter-time they fly much slower
Because their circulation's lower.

"Sucra."

Crossword



Where mathematical symbols or formulae are given as clues, the solution is a word describing them.

ACROSS

1. $5x - 3y = 4$. (13.)
8. An ace is one. (4.)
9. Mathematical term which is bad if out of focus. (10.)
11. The Ace is one. (7.)
12. ∇ of a scalar field. (4.)
14. "Two, two, the —— —— boys,' (English folk song). (4, 5.)
16. As ten (anagram). (5.)
17. Xe. (5.)
19. Least sets (anagram). (9).
21. R. (4.)
22. Keep within limits. (7.)
24. $(1 - 1)$ and algebraic. (10.)
25. Often applied to grids. (4.)
26. Was Newton's Law of Gravitation this? (5, 2, 3, 3.)

DOWN

2. Credit Nile for a straight course. (6, 4.)
3. Tie. (5.)
4. Used for cake decoration. (5, 4.)
5. Part of an astrolabe. (7.)
6. Studied by a geophysicist upside down. (4.)
7. $S + \lambda S' = 0$, $\Sigma + \mu \Sigma' = 0$. (6, 7)
8. $x + iy$. (7, 6.)
10. Survey. (4.)
13. $x^2 + y^2 - 3xy$. (10.)
15. Insert. (9.)
18. $\dot{S} \neq 0$. (3, 4.)
20. Very long nose. (4.)
22. Spurious substitute. (5.)
23. $\iint dx\, dy$. (4.)

The Deltoid

By A. M. Macbeath

The Deltoid or Tricuspidal Hypocycloid was first investigated by Steiner (*Werke* II, p. 639–647), who obtained many beautiful properties of the curve, which he defined as the envelope of Simson's line. A different definition is used here, which leads to some new properties of the curve and of circles connected with the complete quadrilateral. The methods are elementary, and very little knowledge beyond Euclid's elements is assumed.

1. Definition.

We have a circle S, centre O, radius r. P is a variable point on S and p a line through P. As P moves round S, p rotates in the opposite sense with *half* the angular velocity of OP (or, with the *same* absolute value of angular velocity as PX, where X is a fixed point of S). P is called the *central point* of p. The angle between p and OP increases one and a half times as fast as the angle between OP and a fixed line; so there are three points P for which p passes through O, and three points P for which p touches S; each of these triads will form an equilateral triangle. The envelope of p is called a *deltoid*. Clearly the deltoid touches S at the three last named points. S is called the *incircle* of the deltoid.

The following notation will be used:

Δ denotes the deltoid.

$p, p_1, p_2 \ldots$ will be tangents of Δ.

P, P_1, $P_2 \ldots$ will be the central points of $p, p_1, p_2 \ldots$; but each p will have a second intersection with S, which is denoted by P′ (or P'_1, $P'_2 \ldots$).

Q_{ij} will be the intersection of p_i and p_j.

Q, Q_1, $Q_2 \ldots$ will be the contacts of $p, p_1, p_2 \ldots$ with Δ.

2. Properties of the Curve.

T. 1: The circle $P_1P_2Q_{12}$ has radius r.

Proof: By the definition, the angle subtended at the circumference of S by P_1P_2 is equal to the angle between p_1 and p_2; or, the angle subtended by P_1P_2 at the circumference of $P_1P_2Q_{12}$; so the two circles are equal.

Cor: In the limit the circle through Q touching S at P has radius r; and so PP′ = PQ.

T. 2: The deltoid is a hypocycloid generated by the motion of a point on the circumference of a circle radius r rolling inside a circle radius $3r$ concentric with S.

Proof: Let p_o be one of the three tangents to Δ through O. Let p be another tangent, where, without loss of generality, the arc $PP_o \leqslant \frac{\pi}{3}$.

Let p, p_o meet at L.

Let ψ be the circle radius r touching S at P, and passing (T. 1 Cor.) through Q.

Let C be the centre of ψ.

Draw the circle Σ centre O radius $3r$.

Σ will touch ψ at R, say, where OPCR are collinear.

Denote angle POP_o by θ; QCR by ϕ.

Then $PLO = \frac{1}{2} POP_o = \frac{1}{2}\theta$

$$\phi = 2QPC = 2(PLO + POP_o) = 3\theta;$$

or: $\phi : \theta = 3 =$ radius of Σ : radius of ψ.

Hence result.

T. 3: The three circles $P_2P_3Q_{23}$, $P_3P_1Q_{31}$, $P_1P_2Q_{12}$ meet at the circumcentre of the triangle $p_1p_2p_3$.

Proof: It is known, by the "Point O Theorem," that the circles have a common point, which we call H, say.

Now the last two circles are equal, and so:

angle $HQ_{31}P_1 =$ angle $P_1Q_{13}H$ on the chord P_1H

$\therefore HQ_{12} = HQ_{13} = HQ_{23}$

i.e. H is the circumcentre.

Cor. 1: If p_1, p_2 are fixed and p varies touching Δ, H describes the circle $P_1P_2Q_{12}$ which we call the *central locus* of p_1, p_2 with respect to Δ.

Cor. 2: If a, b are two lines meeting at X and C is a circle through X, then the envelope E of lines l, such that the circumcentre of abl is on C, is a deltoid. [For if a, b meet C at A, B, and S is the reflexion of C in the line AB, then clearly E is identical with the deltoid whose incircle is S, and has a as a tangent with central point A.]

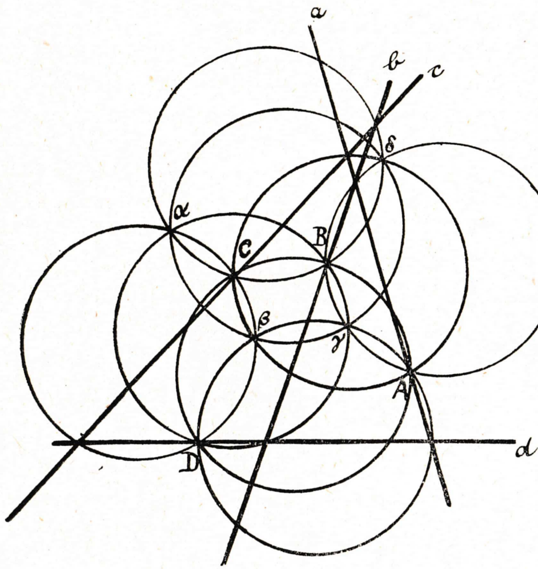
Thus a deltoid is uniquely defined by two tangents and their central locus. We deduce:

T. 4: There is a unique deltoid touching 4 lines a, b, c, d which do not all meet at a point and of which no two are parallel.

Proof: Assume a does not pass through an intersection of b, c, d. Let γ, δ be the circumcentres of abd, abc respectively. Clearly the central locus of a, b must be the circle through γ, δ and the intersection (ab); and, by T. 3, Cor. 2, a, b and this central locus define a deltoid with the required property.

3. The Complete Quadrilateral and the Associated System of Circles.

From now on we assume further that no three of a, b, c, d are concurrent. Δ is the deltoid touching them; α, β, γ, δ are the circumcentres of bcd, cda, dab, abc respectively.



Then the circle $(ab)\,\gamma\delta$ is the central locus of a, b and has (T. 1) the same radius r as the incircle of Δ. It also passes through the central points, A, B, say, of a, b. Applying this to all the pairs of lines we get part (i) of the following theorem:

T. 5 (i): The six circles $(ab)\gamma\delta$; $(ac)\beta\delta$; $(ad)\beta\gamma$; $(bc)\alpha\delta$; $(cd)\alpha\beta$ $(bd)\alpha\gamma$ have the same radius r.

The first three circles meet at a point A of a and corresponding triads meet at points B, C, D of b, c, d.

A, B, C, D lie on a seventh circle of radius r.

(ii) α, β, γ, δ lie on an eighth circle of radius r.

(iii) $\left.\begin{array}{l}\alpha \text{ is the orthocentre of BCD, } B\gamma\delta,\ \beta C\delta,\ \beta\gamma D \\ \text{A is the orthocentre of } \beta\gamma\delta,\ \beta CD,\ B\gamma D,\ BC\delta\end{array}\right\}\text{etc.}$

(iv) A, B, C, D, α, β, γ, δ lie on a rectangular hyperbola.

(v) Aα, Bβ, Cγ, Dδ have the same midpoint.

To prove the rest of T. 5 we need the following lemma:

If three equal circles have a common point H and L, M, N are the other intersections of pairs, then the circle LMN is also equal to them and H is the orthocentre of LMN.

Proof: Let X be a point on the circle LMN. Then, since the circles are equal

$$\begin{array}{lrcll} \text{angle} & \mathrm{MXH} & = & \mathrm{MNH} \text{ on the chord} & \mathrm{MH} \\ & \mathrm{LXH} & = & \mathrm{LNH} \quad ,, \quad ,, \quad ,, & \mathrm{LH} \\ \text{so} & \mathrm{LXM} & = & \mathrm{LNM} & \end{array}$$

Hence the circles LXM, LNM are equal.

Now, if H′ is the orthocentre of LMN it is easy to show that angle $\mathrm{LH'N} = \pi - \mathrm{LMN}$; so H′ lies on the circle LHN. Similarly it lies on each of the other two; so it coincides with H, and the lemma is proved.

Returning to T. 5, consider the triad of circles $(ab)\gamma\delta$; $(ac)\beta\delta$; $(ad)\beta\gamma$; which by (i) are equal and have a common point A. From the lemma the circle $\beta\gamma\delta$ has radius r and is thus the reflexion of circle $(ad)\beta\gamma$ in the line $\beta\gamma$. By a similar argument the circle $\alpha\beta\gamma$ is this reflexion; so α, β, γ, δ lie on a circle radius r.

This proves (ii); (iii) now follows at once from the lemma.

Proof of (iv): Since D is not the orthocentre of ABC, there is a unique rectangular hyperbola through A, B, C, D. This, by a well-known theorem, passes through α, the orthocentre (by (iii)) of BCD, and similarly through β, γ, δ.

Proof of (v): α, the orthocentre of BCD, is external centre of similitude of the circumcircle and nine-points circle of this triangle. Since A is on the circumcircle, the midpoint of αA is on the *n.p.c.*

In the same way the midpoint of αA lies on the *n.p.c.* of each of $\mathrm{B}\gamma\delta$, $\beta\mathrm{C}\delta$, $\beta\gamma\mathrm{D}$.

Now these *n.p.c.*'s, by the eleven-point conic theorem, all pass through the centre of the hyperbola (iv); and they have only one common point, being all of radius $\frac{1}{2}r$; for there are not more than two equal circles in a coaxal system.

Thus, $\mathrm{A}\alpha$, $\mathrm{B}\beta$, $\mathrm{C}\gamma$, $\mathrm{D}\delta$ have the same midpoint, namely the centre of the hyperbola (iv).

This completes the proof of T. 5.

(It is hoped to continue this article in the next issue.)

Solutions to Problems in Eureka No. 9

A THREE-COLOUR PROBLEM

If we are given a set Σ of seven points A, B, C, . . G, we call the following operation *heptagonizing* Σ. We bring in seven new points A′, B′, . . ., G′ joined in a circuit A′B′, B′C′, . . ., G′A′ and make the joins AA′, BB′, . . . GG′. As is easily verified we cannot colour the points of the resulting network in three colours with A, B, . . . G all the same colour and with no two points of the same colour joined.

We now take a set M of at least nineteen points and heptagonize every subset of seven points of M. It is clear that the resulting network contains no circuit of less than six lines. Furthermore, if we colour the points of M in three colours, at least one set of seven must all have the same colour, and so the heptagonized network is not colourable.

HOUSE NUMBERS

The whole problem hinges on the Diophantine equation

$$x^3 + y^3 = u^3 + v^3.$$

There are an infinity of solutions of this equation in integral x, y, u and v (see Hardy and Wright, *Theory of Numbers*, § 13.7), but the only solutions for which x, y, u and v take distinct positive values less than 24 (or for which $x^3 + y^3 < 10{,}000$) are

$$1729 = 12^3 + 1^3 = 10^3 + 9^3 \text{ and } 4104 = 16^3 + 2^3 = 15^3 + 9^3.$$

Let S, B and R be the house numbers of Smith, Brown and Robinson, and let s_1, s_2; b_1, b_2; r_1, r_2 be the ages of their children, the first number of each pair being the greater. We can at once discard the second solution, since it is clear that S and B differ by 2, and hence, if 4104 is the number of the Rolls Royce, both S and B would be expressible as the sum of two squares, which is impossible, since a square can only be of the forms $4m$ and $4m + 1$.

Hence 1729 is the number of the Rolls Royce, and

$$s_1 = 12,\ s_2 = 1,\ b_1 = 10,\ b_2 = 9, \text{ or } s_1 = 10,\ s_2 = 9,\ b_1 = 12,\ b_2 = 1.$$

In either case one of b_1, b_2 is divisible by 2 and one by 3. It follows that S is not divisible by 2 or 3. Let x be the greater of S and B. Then, since $S - B = 2$, x is odd and is of the form $4k + 3$ (two semi-detached houses can have numbers of the forms $4k + 2$, $4k + 4$ or $4k + 1$, $4k + 3$ only) and therefore is not the sum of two squares. The second digit of 1729 is 7, and it follows that $x - 2$ and $x/7$ ($= y$, say) are each expressible as the sum of two squares. Accordingly x must be of the form $28n + 7$.

Now Robinson's house is second last in the street and therefore R is of the form $4m + 3$. The largest factor of 1729 of this form is 247 (the others are 7, 19 and 91) so that the number of houses in the street does not exceed 248. Hence x, $x - 2$ and y can only take the following values:

x	7	35	63	91	119	147	175	203	231
$x - 2$	5	*33*	61	89	117	145	173	*201*	229
y	*1*	5	9	13	17	*21*	*25*	29	*33*

We can exclude $y = 1, 9, 21, 25$ and 33, since each of these numbers is a square or not expressible as the sum of two squares. Also we cannot have $x - 2 = 33$ or 201, since neither is the sum of two squares. This leaves $x = 91$ or 119, i.e. $S = 89, 91, 117$ or 119. Of these 91 and 119 have digits in common with both s_2 and b_2, and so can be excluded, and we cannot have $S = 117$ as it is divisible by 3. Accordingly $S = 89 = 5^2 + 8^2$, and from this we can easily obtain the remaining house numbers and ages: Smith's house number is 89, and his children are 10, 9; Brown's house number is 91 and his children are 12, 1; Robinson's house number is 247 and his children are 3, 2.

THE UMBRELLA PROBLEM

The simple fact is relevant, that any permutation p of the numbers $1, 2, \ldots, n$ is expressible as the product of *cycles*,

$$p = (\alpha, \beta, \gamma, \ldots, \zeta)\,(\eta, \theta, \ldots) \ldots (\ldots, \omega).$$

Here, $\alpha, \ldots, \omega$ represent the numbers from 1 to n in some arrangement, each number appearing only once. It is implied that p sends α into β, β into $\gamma, \ldots, \zeta$ into α, and η into θ, and so forth. The number of terms enclosed between a pair of brackets is the *order* of the corresponding cycle. Cycles of order 1 may of course be omitted.

In the umbrella problem, a permutation p of A, B, C, D, E, F is defined by the stipulation that the umbrella belonging to x was borrowed by px, where x runs through the letters A, . . ., F. The data are now: (i) p contains no cycle of order 1; (ii) $A = p^2B$; (iii) $C = p^3D$; (iv) $p^2E \neq F$. It follows from (i) that p must be of one of four forms, either (*, *) (*, *) (*, *) or (*, *) (*, *, *, *) or (*, *, *) (*, *, *) or (*, *, *, *, *, *). The first form is incompatible with (ii), which reduces the alternatives to (*, *) (B, *, A, *) and (B, *, A) (*, *, *) and (B, *, A, *, *, *). Using (iii) similarly, we obtain the alternatives (C, D) (B, *, A, *) and (B, D, A, *, C, *) and (B, C, A, *, D, *). Finally, (iv) shows that p must be either (B, D, A, F, C, E) or (B, C, A, F, D, E). Thus, $pA = F$; the borrower of A's umbrella was F.

FERMAT'S LAST THEOREM

The statements of 2 and 3 are true, those of 1, 4 and 5 are false.

CROSSWORD PUZZLE

Across:—1. Impulsive Action. 9. Rank. 10. All bounded. 11. Violate. 12. Axioms. 14. Plane Sets. 17. Agram. 19. Regal. 20. Six Geoids. 23. Lamina. 25. Annulus. 26. Number Pair. 27. Meet. 28. General Solution.

Down:—2. Mean Value. 3. Unknown Variable. 4. Scalars. 5. Valse. 6. Apolar. 7. Tending to a Limit. 8. Oleum. 13. Sum. 15. Ers. 16. Six. 18. And is Zero. 19. Rul. 21. General. 22. Daorba. 24. Acute. 25. Adams.

Book Reviews

A Chapter in the Theory of Numbers. By L. J. Mordell, F.R.S. (Cambridge University Press.) 1/6.

Professor Mordell's inaugural lecture was devoted to a study of the integer and rational solutions of the equation $y^2 = x^3 + k$, where k is a given integer. The choice of this special topic is amply justified by its history and associations. Professor Mordell traces the development of the problem from the days of Bachet and Euler through the centuries to his own fundamental researches extending over more than thirty years.

The idea of reducing the question of integer solutions to a problem of factorisation, by writing the equation in the form

$$(y + \sqrt{k})(y - \sqrt{k}) = x^3,$$

goes back to Euler. But a difficulty not perceived by early workers, whose arguments were to that extent fallacious, arises from the fact that "integers" of the form $u + v\sqrt{k}$ (where u and v are ordinary integers) do not obey the law of *unique* factorisation into "primes" of the same form, except for some special values of k (for which, in any case, a proof is required). Professor Mordell explains how the arithmetical theory of algebraic numbers, created by Kummer in an attempt to meet a similar difficulty with the equation $x^n + y^n = z^n$ of "Fermat's Last Theorem," may be used to repair the omission and to extend the scope of the method. The main general conclusion is that, for given $k \neq 0$, the equation has at most a finite number of integer solutions. By way of contrast it may be recalled that certain classes of indeterminate equation of the *second* degree, typified by the "Pellian equation" $x^2 - ky^2 = 1$ (where k is a given positive integer, not a square), have an infinity of integer solutions.

For rational solutions there is an old process, conveniently described in geometrical terms, for successively enlarging a given stock of solutions. Starting from a set of rational points $P_1, P_2, \ldots$, on the curve $y^2 = x^3 + k$, we can find further rational points (in general) as the residual intersections with the cubic of the tangents at these points and the chords joining them in pairs. Mordell's "finite basis theorem" now asserts that, though the equation may have an infinity of rational solutions, they can all be derived by this process from a *finite* number of them.

Much of the discussion extends to the more general equation—$Ey^2 = Ax^3 + Bx^2 + Cx + D$, and there is a close connection with the important theorem of Thue that if $f(x, y)$ is an irreducible binary form of degree at least 3 with integer coefficients, the equation $f(x, y) = m$ has at most a finite number of integer solutions for a given integer m.

A. E. I.

Calculating Machines—Recent and Prospective Developments. By D. R. Hartree, F.R.S. (Cambridge University Press.) 1/6.

Professor Hartree's inaugural lecture, which forms the text of this small book, will be remembered as having given a wide survey of present-day advances made possible by electronics in the techniques of digital calculating machines, that is to say, those machines which operate directly on numbers as such and do not treat them as measures of some physical quantity. It forms a good introduction to this subject for those who have not yet come into contact with it. Its later pages, in particular, stimulate the mathematician to regard his subject and methods from a new angle and forecast those new approaches necessary for use in electronic computers. It can only be regretted that time did not allow Professor Hartree to go into more detail, and that the answers to the questions asked at the end of the lecture could not have been printed in an appendix.

V. W. D. H.

Methods of Algebraic Geometry, Volume I. By W. V. D. HODGE, M.A., F.R.S., and D. PEDOE, B.A., Ph.D. (Cambridge University Press.) 30/-.

This volume, to quote the authors, is "the first part of a work designed to provide a convenient account of the foundations and methods of modern algebraic geometry." Recent work has made it clear that any precise account of the foundations of algebraic geometry must be based, to a greater or lesser extent, on the modern and precise algebraic theory of fields. Accordingly, the first four chapters of the present work are purely algebraic, and give in clear and concise form the fundamental notions of rings and fields, matrices, and the theory of algebraic equations.

The main applications of these ideas will be found in a second volume, to be published in due course. The major geometrical content of this first volume is a detailed account, occupying two chapters, of the foundations of projective geometry. It is shown how this science can be built up either on an algebraic or a purely axiomatic basis, and the relations between the methods are studied in great detail. It has long been known that in the synthetic development the assumption of Pappus's theorem (or an equivalent axiom) is necessary to ensure the commutative nature of the field of coordinates. The implications of this fact are given unusual prominence in the present treatment, since the coordinates are introduced into the geometry before, and not after, this assumption has been made. These two chapters will repay careful reading.

The remaining chapters of the book deal with Grassmann coordinates and with the algebraic theory of collineations and correlations, where the canonical forms are completely worked out.

Although the geometrical content of the book does not involve advanced notions, and many of the ideas will be familiar to all those who take Part II of the Tripos, the volume is intended primarily for those whose interest in geometry extends to the Part III level. For such a specialist it will be indispensable, and will reward the effort necessary to come to grips with the more involved portions. The appearance of the second volume will be eagerly awaited.

As usual, the Cambridge University Press has produced a volume of distinguished appearance, which may mislead the reader into thinking that the technical job of setting up a work of this sort in type is an easy one. Those with a little more experience of the difficulties involved will congratulate the Press on maintaining their traditionally high standard.

J. A. T.

An Introduction to Analytical Geometry, Volume II. By A. ROBSON. (Cambridge University Press.) 10/6.

This book was written as a text-book for the higher divisions in schools rather than as a university text-book, and amply succeeds in its aim; nevertheless, several of the chapters in this second volume will, no doubt, be useful to candidates for Part I of the Tripos. It approaches the subject of analytical geometry in the manner of most school text-books. Starting from metrical ideas of coordinates, it makes a transition to the various "particular" homogeneous coordinate systems such as areals and trilinears, and then to general homogeneous coordinates. However, it is superior in this respect to most existing text-books since at each stage it clearly points out which geometry is in use, and so does not leave the reader with muddled ideas on the respective rôles of various coordinate systems in metrical and projective geometries.

The idea of "symbols" for points in proving incidence relations is introduced; in addition it covers all the usual topics, for instance, homography, involution, duality, reciprocation, and systems of conics, and thus will be a valuable book for those who are preparing for Cambridge Scholarships.

The printing is excellent, and the University Press must be congratulated on producing a very attractive little volume.

G. C. S.

Projective and Analytical Geometry. By J. A. TODD, Ph.D. (Pitman.) 25/-.

Mathematical Tripos candidates owe a great debt of gratitude to Dr. Todd for this book. The monogram on the front cover is misleading—the subject covered is a branch of mathematics and is treated as such; in fact, in some paragraphs the very rigour of the development of the theory tends to obscure the geometrical picture and to make the reading difficult for the average Preliminary or Part II candidate. It is unfortunate, too, that so much information and such an abundance of powerful methods have been packed into 280 pages—two equations on one line (as in 4.6.7.) and the paucity of diagrams present a deterring sight on a casual glance. However, once this initial inertia has been overcome, the elegance of the presentation becomes readily apparent, and the examination value of the book is evident; indeed, it has evolved from Dr. Todd's courses of lectures given in this University.

In particular, the treatment of line coordinates by suffix notation is both lucid and welcome—examiners please note! The last chapter develops some theory of the invariants of conics and quadrics. It will be noticed that the problems which required special treatment earlier in the book now are immediate as geometrical interpretations of algebraic theorems on invariants. The methods of this chapter are not to be found in other text-books usually available to the undergraduate. Many of the examples which occupy the closing pages of the book will be recognised from Tripos Papers, and they will certainly afford ample opportunity for using the methods of this book. The author is most exacting in his demand for rigour; the special cases of theorems such as Pascal's are not glossed over as in more elementary treatises.

Dr. Todd has succeeded in his aim to present projective geometry as an independent branch of pure mathematics. The elegance of the methods and the aesthetic appeal of the results will, I am sure, maintain this book in its position as the first of its kind.

E. S. P.

The following book has been received:—

The Solar System Analysed. By F. C. Attwood. (Dawson Printing Company, Ltd., Auckland, New Zealand).

Copies may be obtained from the office of the High Commissioner for New Zealand, New Zealand House, London.

A review of this book will appear in our next issue.

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The Mathematical Association

President: Sir Harold Spencer Jones, F.R.S.

The Mathematical Association, which was founded in 1871 as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

The subscription to the Association is 15s. per annum: to encourage students, and those who have recently completed their training, the rules of the Association provide for junior membership for a limited period at an annual subscription of 5s. Full particulars can be had from The Mathematical Association, Gordon House, Gordon Square, London, W.C.1.

The *Mathematical Gazette* is the journal of the Association. It is published five times a year and deals with mathematical topics of general interest.

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Analysis Amoris, or Love Confounded

How shall I, love, thy estimate compute
Or signify the measure of thy worth?
What measures are there, that perfection suit?
Others' excess, 'gainst thee, is but a dearth.
Means have I none to stablish thy location,
So far from null art thou; yet plainly see
Thou hast none, or little deviation
From thy true value, thy sure constancy.
What likelihood that I should any find
By random chance or critical selection
Equal to thee? What test was e'er designed
Could prove an error there to need correction?
    What expectation could foreshadow thee,
    That art beyond all probability?

V. R. C.